Flow Formation in Couette Motion in Magnetohydrodynamics

Masakazu KATAGIRI

Department of Electrical Engineering, Faculty of Engineering

1. Introduction

Recently, boundary layer flows of an electrically conducting, viscous and incompressible fluid in the presence of the externally applied magnetic field have been investigated by many authors in view of important practical problems in aero-space science. In studying boundary layer flows we usually use the boundary layer approximation, in which the viscous effects can be considered to be confined in a narrow region near the body. In the presence of the external magnetic field, however, we must take account of the magnetic viscous effects due to the electrically conducting property of the fluid. These effects form a thin layer near the body in which the magnetic viscous effects have to be taken account of. This layer is called the magnetic boundary layer and has generally a large thickness compared with the one of the viscous boundary layer. The theoretical treatment is intricate by this double structure of the boundary layer.

When the external magnetic field is applied parallel to the flow direction, the double structure of the boundary layer can be considered to degenerate into the single boundary layer and the theoretical treatment becomes to be rather simple. In this case, Tamada & Sone[1], and Greenspan & Carrier[2] discussed the boundary layer flow on a flat plate, independently. When the external magnetic field is applied transverse to the flow direction this degeneration does not appear and the theoretical treatment is still kept intricate. Among the boundary layer type flows only Rayleigh's problem has been discussed exactly by Hashimoto[3], Kakutani[4],[5], and Chang & Yen[6]. Regarding the treatment of boundary layer flows the determinable method does not exist.

To avoid the intricateness in the theoretical treatment there is the assumption proposed by Rossow[7]. This assumption is that the induced magnetic field can be neglected compared with the external magnetic field and the electric field is zero. By this assumption the magnetic boundary layer can be considered to extend infinitely in the flow field. Under this assumption Rossow discussed the boundary layer flow of a flat plate, and Ong & Nicholls[8], and Gupta[9] did on Rayleigh's problem.

In this paper we will discuss the flow formation in Couette motion in the presence of the transverse magnetic field under the assumption proposed by

Rossow as the preliminary study of unsteady boundary layer flows. The velocity profiles and skin friction are calculated. Finally the knowledge for the treatment of unsteady boundary layer flows is obtained.

2. Equations and Their Solutions

The configuration for Couette motion is as follows. The fluid is between two infinite parallel flat plates at a distance h from each other. One of them, for instance, the upper one is fixed and the lower one is moving parallel to the upper one. The fluid between parallel plate move with the linear velocity distribution. Here we will investigate flow formation in this motion in the presence of the external transverse magnetic field. At time $t < 0$ both plates are in the state of rest, and at time $t = 0$ the lower one starts to move parallel to the upper one with the velocity u_0 impulsively.

Investigating this motion, we take the x-axis coinciding with the lower plate and the y-axis perpendicular to it. Since the plates are infinite all physical variables does not depend on x and z . By the configuration of this motion we can assume for the flow velocity $\boldsymbol{V}$ and the magnetic flux density $\boldsymbol{B}$ as

$$\boldsymbol{V} = (u,\ v,\ 0)\ ,$$

$$\boldsymbol{B} = (B_x,\ B_y,\ 0)\ .$$

The equation of continuity (See section 2. in reference 10.) gives

$$\frac{\partial v}{\partial y} = 0\,, \quad \text{or} \quad v = const.\ in\ space.$$

Since v vanishes at the lower plate, we obtain $v = 0$. By the assumption that the induced magnetic field can be neglected compared with the external magnetic field and the electric field $\boldsymbol{E}$ is zero, equations (3), (6) and (7) in reference 10 give

$$B_y = const. = B_0\ ,$$

$$j_x = j_y = 0\,,$$

$$j_z = \sigma u B_y\ ,$$

where B_0 is the external magnetic flux density applied perpendicular to the plates, and j_x , j_y and j_z the x , y and z component of the current density. Here σ is the electric conductivity which is assumed to be constant in the incompressible case. Considering these results the modified Navier-Stokes' equation (the equation (2) in referefce 10) becomes

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0{}^2}{\rho} u\ , \tag{2.1}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.2}$$

where ρ and ν are the density and the kinematic viscosity of the fluid, respectively. and p the pressure of the fluid. The boundary conditions for this motion are

$$\begin{aligned} 0 \leq y \leq h\ ; \quad & u = 0 \quad && \text{at} \quad t \leq 0, \\ y = 0\ ; \quad & u = u_0 && \\ y = h\ ; \quad & u = 0 && \text{at} \quad t > 0. \end{aligned} \tag{2.3}$$

(2.2) immediately gives

$$p = const. \tag{2.4}$$

(2.1) subjected to (2.3) can be solved by the Laplace transform with respect to t ,

$$F^*(s) = \int_0^\infty e^{-st} F(t)\, dt \ . \tag{2.5}$$

(2.1) and (2.3) give

$$su^* = \nu\frac{d^2u^*}{dy^2} - \frac{\sigma B_0{}^2}{\rho} u^*, \tag{2.6}$$

$$\begin{aligned} y = 0\ ; \quad & u^* = u_0{}^*, \\ y = h\ ; \quad & u^* = 0\ . \end{aligned} \tag{2.7}$$

The solution of (2.6) subjected to (2.7) is given by

$$u^* = A\, e^{y\sqrt{\frac{m+s}{\nu}}} + B\, e^{-y\sqrt{\frac{s+m}{\nu}}}, \tag{2.8}$$

where

$$m = \frac{\sigma B_0{}^2}{\rho}\ . \tag{2.9}$$

An arbitrary constants A and B are

$$A = \frac{u_0{}^*\, e^{-h\sqrt{\frac{m+s}{\nu}}}}{e^{-h\sqrt{\frac{m+s}{\nu}}} - e^{h\sqrt{\frac{m+s}{\nu}}}},$$

$$B = -\frac{u_0{}^*\, e^{-h\sqrt{\frac{m+s}{\nu}}}}{e^{-h\sqrt{\frac{m+s}{\nu}}} - e^{h\sqrt{\frac{m+s}{\nu}}}}\ . \tag{2.10}$$

Applying the inverse Laplace transform to (2.8), we expand the denominator of (2.10) in infinite series as

$$\frac{1}{e^{h\sqrt{\frac{m+s}{\nu}}}-e^{-h\sqrt{\frac{m+s}{\nu}}}}=\sum_{n=0}^{\infty} e^{-(2n+1)h\sqrt{\frac{m+s}{\nu}}},$$

(2.8) becomes

$$u^* = u_0^* \sum_{n=0}^{\infty}\left[e^{-(2nh+y)\sqrt{\frac{m+s}{\nu}}} - e^{-\{(2n+2)h-y\}\sqrt{\frac{m+s}{\nu}}}\right]. \quad (2.11)$$

Calculating the inverse Laplace transform

$$u = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, u^*\, ds\ , \quad (2.12)$$

we put

$$\frac{s+m}{\nu} = \frac{r}{\nu}\ .$$

The term except u_0^* of the right hand side in (2.11) can be obtained making use of a formula 6 in Appendix V of reference 11

$$L^{-1}\left\| \sum_{n=0}^{\infty}\left[e^{-(2nh+y)\sqrt{\frac{m+s}{\nu}}} - e^{-\left\{(2n+2)h-y\right\}\sqrt{\frac{m+s}{\nu}}}\right]\right\|$$

$$= \frac{1}{2\pi i}\int_{c+m-i\infty}^{c+m+i\infty} e^{-mt}\, e^{rt} \sum_{n=0}^{\infty}\left[e^{(2nh+y)\sqrt{\frac{r}{\nu}}} - e^{-\left\{(2n+2)h-y\right\}\sqrt{\frac{r}{\nu}}}\right] dr \quad (2.13)$$

$$= \frac{e^{-mt}}{2\sqrt{\pi\nu t^3}}\sum_{n=0}^{\infty}\left[(2nh+y)e^{-\frac{(2nh+h^2)}{4\nu t}} - \left\{(2n+2)h-y\right\} e^{-\frac{\{(2n+2)h-y\}^2}{4\nu t}}\right].$$

Making use of the convolution theorem we obtain from (2.11) and (2.13)

$$u = \frac{1}{2\sqrt{\pi\nu}}\int_0^t \frac{u_0(t-\tau)e^{-m\tau}}{\tau^{\frac{3}{2}}}\sum_{n=0}^{\infty}\left[(2nh+y)\, e^{-\frac{(2nh+y)^2}{4\nu\tau}} - \left\{(2n+2)h-y\right\} e^{-\frac{\{(2n+2)h-y\}^2}{4\nu\tau}}\right] d\tau. \quad (2.14)$$

When the motion is impulsive we can put

$$u_0^* = \frac{u_0}{s},$$

where u_0 is the uniform velocity of the lower plate. (2.14) gives

$$\begin{aligned} u = \frac{u_0}{2} \Bigg[& \sum_{n=0}^{\infty} \left\{ e^{-(2nh+y)\sqrt{\frac{m}{\nu}}} \operatorname{erfc}\left(\frac{2nh+y}{2\sqrt{\nu t}} - \sqrt{mt} \right) \right. \\ & \left. + e^{(2nh+y)\sqrt{\frac{m}{\nu}}} \operatorname{erfc}\left(\frac{2nh+y}{2\sqrt{\nu t}} + \sqrt{mt} \right) \right\} \\ & - \sum_{n=0}^{\infty} \left[e^{-\{(2n+2)h-y\}\sqrt{\frac{m}{\nu}}} \operatorname{erfc}\left\{ \frac{(2n+2)h-y}{2\sqrt{\nu t}} - \sqrt{mt} \right\} \right. \\ & \left. + e^{\{(2n+2)h-y\}\sqrt{\frac{m}{\nu}}} \operatorname{erfc}\left\{ \frac{(2n+2)h-y}{2\sqrt{\nu t}} + \sqrt{mt} \right\} \right] \Bigg]. \end{aligned} \tag{2.15}$$

Skin friction is calculated from (2.15) as

$$\begin{aligned} \tau &= -\rho\nu \left(\frac{\partial u}{\partial y} \right)_{y=0} \\ &= \frac{\rho\nu u_0}{\sqrt{\pi\nu t}} e^{mt} + \rho\nu\, u_0 \sqrt{\frac{m}{\nu}} \Bigg[\operatorname{erfc}\left(\sqrt{mt} \right) \\ &\quad + \sum_{n=1}^{\infty} \left\{ e^{-2nh\sqrt{\frac{m}{\nu}}} \operatorname{erfc}\left(\frac{nh}{\sqrt{\nu t}} - \sqrt{mt} \right) + e^{2nh\sqrt{\frac{m}{\nu}}} \operatorname{erfc}\left(\frac{nh}{\sqrt{\nu t}} + \sqrt{mt} \right) \right\} \Bigg]. \end{aligned} \tag{2.16}$$

The velocity profile and skin friction given by (2.15) and (2.16), respectively, tend to agree with ordinary hydrodynamical results by the limiting process $m \to 0$.

3. Numerical Results and Conclusions

Numerical computations are performed for different values of non-dimensional parameters η_0 and M, and of non-dimensional variable η, which are defined by

$$\eta_0 = \frac{h}{\sqrt{\nu t}}, \quad M = B_0 h \sqrt{\frac{\sigma}{\rho\nu}}, \quad \text{and} \quad \eta = \frac{y}{h},$$

where M is called Hartmann's number. The velocity profiles and skin friction are shown in Fig. and Table, respectively.

The effect of the magnetic field gives the following conclusions: 1) the velocity of the fluid is retarded, and 2) skin friction increases as the strength of the magnetic field increases. Time dependence of Couette motion can be concluded: at the commencement of motion from rest the effect of the magnetic field scarcely contributes to it and as time elapses the contribution to it increases. From this conclusion we find it possible to approximate to these velocity profiles at the commencement of motion by making use of Rayleigh's velocity profile for a motion of a flat plate. In treating unsteady boundary layer flows of magnetohydrodynamics in the external transverse magnetic field at the commencement of motion from rest under the assumption proposed by Rossow, an expansion of the velocity profile in power series of time t is possible in the basis of Rayleigh's profile as in ordinary hydrodynamics.

It is well known that in Rossow's investigation of Rayleigh's problem the magnetic lines of force fixed relative, 1) to the plate, and 2) to the fluid correspond to the plate when it is an electrically perfect insulator, and an electrically perfect conductor, respectively. The discussion in this paper concerns the former case. Although there is a mistake in the conception that the magnetic lines of force are fixed relative to the fluid because Rossow does not distinguish the coordinate system of the fundamental equation, the latter case can be considered to be the case when the constant electric field in space is involved. However, this case is physically unrealizable because the neccessity of applying the external electric field impulsively occurs due to an impulsive motion of the plate.

(Received July 13, 1961)

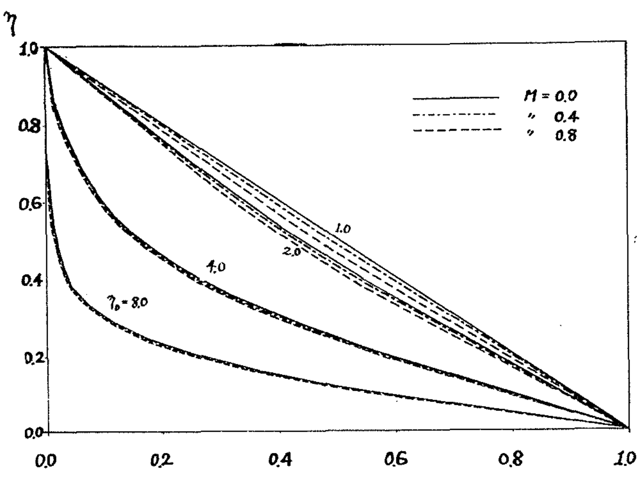

Fig. The profiles of velocity.

Table. Skin friction $\tau_w^* = \tau_w \Big/ \dfrac{u_0}{\sqrt{\pi \nu t}}$.

M \ η_0	8.0	4.0	2.0	1.0
0.0	1.000	1.000	1.000	1.000
0.4	1.009	1.050	1.198	1.815
0.8	1.050	1.198	1.757	3.833

References

1) K. Tamada & Y. Sone: Preprints for the 14th Annual Meetings of the Physical Society of Japan No. 7 , 21 (1959).
2) H. P. Greenspan & G. F. Carrier: J. Fluid Mech. **6**, 77 (1959).
3) H. Hashimoto: Preprints for the 13th Annual Meetings of the Physical Society of Japan C-51 (1958).
4) T. Kakutani: J. Phys. Soc. Japan **13**, 1504 (1958).
5) T. Kakutani: J. Phys. Soc Japan **15**, 1316 (1960).
6) C. C. Chang & J. T. Yen: Phys. Fluid **2**, 392 (1959).
7) V. J. Rossow: NACA TN 3791 (1957).
8) R. S. Ong & J. A. Nicholls: J. Aeron.Sci. **26**, 314 (1959).
9) A. S. Gupta: J. Phys. Soc. Japan **15**, 1894 (1960).
10) M. Katagiri: Bull. Yamagata Univ. (Engineering) **6**, 561 (1961).
11) H. S. Carslaw & J. C. Jaeger: "Conduction of Heat in Solids" 380 (1950).

山形大学紀要（工学）第７巻　第１号

電磁流体におけるクウェットの運動による流れの形成

片　桐　理　和

工学部電気工学科

クウェットの運動による流れの形成は，通常の流体力学では，非定常境界層の流れに関する研究の基礎的研究として知られている。本論文では，磁場が流れに対して垂直に与えられている場合の，電気的伝導性を有する非圧縮性粘性流体のクウェットの運動による流れの形成を調べる。流体中を流れる電流により誘起される磁場を無視するという仮定のもとで解析を行った。運動の初期の段階では流れの形成に及ぼす磁場の影響は殆んど見られないという結論が得られる。これから電磁流体力学における非定常境界層の取扱いは，運動の初期の段階では従来の流体力学におけると同様に，時間の巾での級数展開を行うことにより可能であることが見出される。