On some Applications of Laplace-Transformation to the Heat Transfer Problems. (Report 3)

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1. Temperature distribution in the spherical sulphur smelting kettle.

In this case, we consider the spherical kettle which is applied to the refining of sulphureous ores with the object of finding the temperature distribution in the kettle.

As shown in Fig. 11 $R_i$ and $R_o$ represent the inner and outer radii of the kettle respectively, and the thickness $\omega = R_o - R_i$.

The kettle is exposed to the high temperature gas ($\vartheta_o$ °C), and the heat converted into the inner layer by the conduction that fulfilled with sulphureous ore and air, and the heat convection is neglected.

Therefore, using the spherical co-ordinate, the fundamental equations of heat conduction for the kettle and the mixed layer are as follows:

$$\frac{\partial \vartheta}{\partial t} = \kappa \left( \frac{\partial^2 \vartheta}{\partial r^2} + \frac{2}{r} \frac{\partial \vartheta}{\partial r} \right) \quad \text{for} \quad R_i \leq r \leq R_o \quad (59)$$

and,

$$\frac{\partial \vartheta_i}{\partial t} = \kappa_i \left( \frac{\partial^2 \vartheta_i}{\partial r^2} + \frac{2}{r} \frac{\partial \vartheta_0}{\partial r} \right) \quad \text{for} \quad 0 \leq r \leq R_i \quad (60)$$

and the initial and boundary conditions now considered are as follows for each equations (59) and (60):

\[
\begin{align*}
    t = 0 & ; \quad \vartheta = 0 \\
    r = R_i & ; \quad \alpha \vartheta = \lambda \frac{\partial \vartheta}{\partial r} \\
    r = R_o & ; \quad \vartheta = \vartheta_o \quad \text{gas temperature, say} \quad (61)
\end{align*}
\]

and,

\[
\begin{align*}
    t = 0 & ; \quad \vartheta_i = 0 \\
    r = R_i & ; \quad \vartheta = \vartheta_i \\
    r = 0 & ; \quad \vartheta_i = \text{finite} \quad (61)
\end{align*}
\]

Where,

$\lambda$ : thermal conductivity of the kettle. (usually, cast iron)
$\vartheta$ : temperature of the kettle.
\( \kappa \): thermal diffusivity of the kettle.
\( \lambda_t \): thermal conductivity of mixed layer in the kettle.
\( \theta_t \): temperature of mixed layer.
\( \kappa_t \): thermal diffusivity of mixed layer.

and, \( \alpha \): coefficient of heat transfer by convection between the kettle and the layer.

In equations (59) and (60), using the operator \( p = \partial / \partial t \), the transformed equations of \( \theta \) and \( \theta_t \) are become next equations, where \( \bar{\theta} (p, r) \) and \( \bar{\theta}_t (p, r) \) are the transformed quantities of \( \theta (t, r) \) and \( \theta_t (t, r) \).

\[
\bar{\theta}(p, r) = \frac{R_t \partial_t \left( \frac{\alpha}{\lambda_t} + \frac{1}{R_t} \right) \sinh \sqrt{\frac{\alpha}{R_t}} (R_t - R_t) + \frac{\sqrt{p/R_t}}{p/R_t} \cosh \sqrt{\frac{p}{R_t}} (R_t - R_t)}{p R_t \left( \frac{\alpha}{\lambda_t} + \frac{1}{R_t} \right) \sinh \sqrt{\frac{p}{R_t}} (R_t - R_t) + \frac{\sqrt{p/R_t}}{p/R_t} \cosh \sqrt{\frac{p}{R_t}} (R_t - R_t)}
\]

and

\[
\bar{\theta}_t(p, r) = \frac{R_t \partial_t \sinh \sqrt{\frac{p}{R_t}} \cdot \sqrt{p/R_t}}{p R_t \sinh \sqrt{\frac{p}{R_t}} \cdot \sqrt{p/R_t}} \sinh \sqrt{\frac{p}{R_t}} (R_t - R_t) + \frac{\sqrt{p/R_t}}{p/R_t} \cosh \sqrt{\frac{p}{R_t}} (R_t - R_t)}
\]

(63)

Similarly as the previous paper\(^{\text{11}}\), (33) and (34) in Report 2, we obtained inverse transformations of \( \bar{\theta} (p, r) \) and \( \bar{\theta}_t (p, r) \)

\[
\bar{\theta}(r, t) = L^{-1} \bar{\theta}(p, r) = \frac{R_t \partial_t \{a(r - R_t) + 1\}}{r \{a(R_t - R_t) + 1\}} - \]

\[
\frac{4R_t \partial_t}{r} \sum_{\delta_m} \frac{\sin \frac{\delta_m}{\sqrt{\kappa}} (R_t - R_t)}{\sin \frac{\delta_m}{\sqrt{\kappa}} (R_t - R_t) + \frac{\delta_m}{\kappa a} (R_t - R_t)} e^{-\delta_m t}
\]

(65)

Accordingly, the temperature inside the kettle

\[
\bar{\theta}(r, t) = \frac{R_t \partial_t}{R_t \{a(R_t - R_t) + 1\}}
\]

\[
- \frac{4R_t \partial_t}{R_t} \sum_{\delta_m} \frac{\cos \frac{\delta_m}{\sqrt{\kappa}} (R_t - R_t) \{a(R_t - R_t) + 1 + \frac{\delta_m}{\kappa a} (R_t - R_t)\}}{\cos \frac{\delta_m}{\sqrt{\kappa}} (R_t - R_t) + \frac{\delta_m}{\kappa a} (R_t - R_t)} e^{-\delta_m t}
\]

(66)

where, \( a = \frac{\alpha}{\lambda_t} + \frac{1}{R_t} \) and \( \delta_m \)'s are the roots of the equation

\[
\tan \frac{\delta_m}{\sqrt{\kappa}} (R_t - R_t) = -\frac{\delta_m}{\sqrt{\kappa}} a
\]

Applying the method of convolution to the eq. (64), we have the inverse transformation as follows:

\[
\bar{\theta}_t(r, t) = L^{-1} \bar{\theta}_t(p, r)
\]

\[
= \frac{R_t \partial_t}{r} \left( \frac{r}{R_t} + \frac{2}{\pi} \sum_{n} (-1)^n \frac{e^{-n \pi R_t / R_t}}{n \pi R_t} \sin \frac{n \pi r}{R_t} \right) \ast \left( -\frac{2}{\sqrt{\pi}} \times \right)
\]

\[
\sum_{\delta_m} \frac{\cos \frac{\delta_m}{\sqrt{\kappa}} (R_t - R_t) \{a(R_t - R_t) + \frac{\delta_m}{\kappa a} (R_t - R_t)\} + 1}{\cos \frac{\delta_m}{\sqrt{\kappa}} (R_t - R_t) + \frac{\delta_m}{\kappa a} (R_t - R_t)} e^{-\delta_m t}
\]

(64)
Finally if we rewrite above equations with non-dimensional coefficients, the temperature inside the kettle is

$$\left. \frac{\partial \theta}{\partial r} \right|_{r=R_t} = \frac{1}{1 + \frac{1}{\varepsilon}(1-\varepsilon)Nu}$$

$$- \frac{2}{\xi} \sum \frac{e^{-\beta_m^2}}{\cos \beta_m(1-\varepsilon)/\sqrt{\beta_m} \cdot \left(1 + Nu(1-\varepsilon) + \beta_m^2 \frac{1-\varepsilon}{F_s(Nu + \frac{1}{\varepsilon})}\right)}$$

and the center termperature of the layer becomes

$$\left. \frac{\partial \theta}{\partial r} \right|_{r=0} = \frac{2}{\xi \pi} \sum \frac{1-e^{-\beta_m^2}}{\cos \beta_m(1-\varepsilon)/\sqrt{\beta_m} \cdot \left(1 + Nu(1-\varepsilon) + \beta_m^2 \frac{1-\varepsilon}{F_s(Nu + \frac{1}{\varepsilon})}\right)}$$

$$- \frac{2}{\xi} \sum \sum (-1)^n \frac{\beta_m^2 \left(1 + \frac{1}{\varepsilon} + \beta_m^2 \frac{1-\varepsilon}{F_s(Nu + \frac{1}{\varepsilon})}\right)}{\left[Nu(1-\varepsilon) + \frac{1}{\varepsilon} + \beta_m^2 \frac{1-\varepsilon}{F_s(Nu + \frac{1}{\varepsilon})}\right] \left[\beta_m^2 - n^2 \pi^4 F_s \left(\frac{1}{\varepsilon}\right)^2\right]}$$

where, $\beta_m$'s are the roots of the equation $\tan \beta_m(1-\varepsilon)/\sqrt{\beta_m} = -\beta_m \frac{1}{Nu + \frac{1}{\varepsilon}}$

$\sqrt{\beta_m}$, and $F_{s,i} = \kappa / R_t$, $Nu = \alpha R_t / \lambda$ and $\varepsilon = R_t / R_s$.

2. Temperature distribution in the kettle of conituous type.

Recently, continuous type sulphur sempling kettles are constructed aiming at the superior performance, and these heating method are equivalent to that of the cylindrical boilers. (e.g. Cornish boiler)

Nowday, some kinds of these kettle, for example MAEDA type kettle, are widely used.

In this case, we can consider theoretically the heating method is treated as like as that of the regenerating heating furnaces mentioned above already.

 Thickness of the kettle, compared with the other quantities, is neglisable small for which we can understand by the numerical example of the former problems 8.

If the diameter of the kettle is very large scale, it may be assumed that the
flow of heat into the kettle is one-dimensional one.

In Fig. 12, c, d and l are the thickness of mixed layer of sulphur ore and air,

the thickness of the kettle and the actual length of the kettle, and high temperature gas flow direction is shown in the figure.

A) Temperature distribution of surface of the kettle.

If we assume the heat due to the furnace gas transfer into the kettle and the inner layer by pure heat conduction, the fundamental equation of heat conduction and initial and boundary conditions are as follows:

\[ \frac{\partial \theta'}{\partial t} = \kappa' \frac{\partial^2 \theta'}{\partial y^2} \]  

(70)

and,

\[ t = 0 ; \quad \theta' = 0 \]

\[ y = 0 ; \quad \alpha (\Theta, - \theta') = - \lambda' \frac{\partial \theta'}{\partial y} = - W' \frac{\Theta}{\partial x} \]

\[ y = 1 ; \quad \alpha \theta' = - \lambda' \frac{\partial \theta'}{\partial y} \]

\[ x = 0 ; \quad \Theta_1 = \text{gas temperature, } \Theta_p \text{ say.} \]

where,

\( \lambda \): thermal conductivity of mixed layer in the kettle.

\( \theta \): temperature of mixed layer.

\( \kappa \): thermal diffusivity of the mixed layer.

\( \lambda' \): thermal conductivity of the kettle.

\( \theta' \): temperature of the kettle.

\( \kappa' \): thermal diffusivity of the kettle.

\( \alpha_{\text{ct}} \): coefficients of heat transfer by convection between the kettle and upper and lower side flowing gases.

\( W' \): water values of upper and lower side flowing gases.

\( \Theta_{\text{up}} \): temperatures of upper and flowing gases.

Similarly as the former case 1, applying in (70) to each operators \( s = \partial / \partial t \),

\( p = \partial / \partial x \), and \( q = \partial / \partial y \), then transformed temperature is

\[ \overline{\theta'}(p, q, s) = \frac{\theta - dX/dy}{\kappa' - s/\kappa'} \]  

(72)

where,

\[ \overline{\theta'}(p, q, s) = X. \]

If we perform the inverse transformation about \( q \),

\[ \overline{\theta'}(p, y, s) = L_y^{-1} \overline{\theta'}(p, q, s) = X \cosh \sqrt{s/\kappa'} y + \frac{dX}{dy} \frac{\sinh \sqrt{s/\kappa'} y}{\sqrt{s/\kappa'}} \]  

(73)
and in the above equation X can be determined from the boundary condition (71)

\[
X = \frac{1}{s} \cdot \frac{w \Theta_0 (\sinh \sqrt{s/\kappa'} + h_2 \sqrt{s/\kappa'} \cosh \sqrt{s/\kappa'})}{p(wh_1 \sqrt{s/\kappa'} \cosh \sqrt{s/\kappa'} + h_2 \sqrt{s/\kappa'} + wh_1 \sinh \sqrt{s/\kappa'} + wh_1 \sqrt{s/\kappa'} \cosh \sqrt{s/\kappa'}) + w \sin \alpha \sqrt{s/\kappa'}} + h_2 \sqrt{s/\kappa'} h_1 \sinh \sqrt{s/\kappa'} + h_2 \sqrt{s/\kappa'} h_2 \sinh \sqrt{s/\kappa'}
\]

where, \( h_{1,2} = \lambda/\alpha_{1,2} \), \( w = W_z/\alpha_{1,2} \), or \( GC/F \), \( x = x/l \), and \( y = y/c \).

Now the thickness of the kettle \( d \) is nearly to zero, then we have

\[
\overline{\varphi}(x, y, s) \bigg|_{y=0} = X
\]

\[
= \frac{1}{s} \frac{w \Theta_0 (1 + h_2)}{p(wh_1 + wh_1 h_2 \sqrt{s/\kappa'} + wh_1 + w) + h_1 + h_1 h_2 \sqrt{s/\kappa'}}
\]

(75)

Next, the inverse transformation about \( p \) equals to

\[
\varphi(x, y, s) = \frac{1}{s} \Theta_0 (1 + h_2) \exp \left( \frac{h_1 + h_1 h_2 \sqrt{s/\kappa'}}{1 + h_1 + h_1 h_2 \sqrt{s/\kappa'} + h_2} \right) \left( \frac{h_2 + h_1 h_2 \sqrt{s/\kappa'}}{h_2 + h_1 h_2 \sqrt{s/\kappa'} + w} \right) x
\]

(76)

and as we have following formula, generally

\[
L^{-1} \frac{b}{s(\sqrt{s} + a)} = \frac{b}{a} \left[ 1 - e^{\frac{a^2 t}{s}} + \text{erf} \left( \frac{a}{2\sqrt{s}} \right) \right]
\]

If we rewriting these results by non-dimensional coefficients

\[
\varphi/\Theta_0 = -A e^{-\frac{1}{s},x} \left( 1 - e^{\frac{c^2 F_x}{s}} - c \sqrt{F_x} e^{\frac{c^2 F_x}{s}} \right)
\]

(77)

Similarly,

\[
\Theta_0/\Theta_0 = e^{-\frac{1}{s},x} - e^{-A B x} \left( 1 - e^{-e^{\frac{c^2 F_x}{s}} + c \sqrt{F_x} e^{\frac{c^2 F_x}{s}}} \right)
\]

(78)

where, \( A = (1 + h_1)/(1 + h_1 + h_2) \), \( B = h_1/(1 + h_2) \), \( C = (1 + h_1 + h_2)/(1 + h_1) \), and \( F_x = \kappa' \sqrt{s/\kappa} \).

B) Temperature distribution in the kettle.

If the longitudinal temperature distribution of the kettle is determined for the upper and lower sides, we can consider that the temperature distribution in the layer is deduced from the following equation

\[
\partial \varphi/\partial t = \kappa \partial^2 \varphi/\partial y^2
\]

(79)

and initial and boundary conditions are:

\[
t = 0 ; \quad \varphi = 0
\]

\[
y = 0 ; \quad \varphi = \Theta_1
\]

\[
y = l ; \quad \varphi = \Theta_2
\]

(80)

Taking the operator \( s = \partial/\partial t \), the transformed temperature of the above equation is

\[
\overline{\varphi}(s, y) = \frac{1}{s \sinh \sqrt{s/\kappa}} \left( \Theta_1 \sinh \sqrt{s/\kappa} (1-x) + \Theta_2 \sinh \sqrt{s/\kappa} x \right)
\]

(81)

and as the formula of inverse transformation established is
the solution of the temperature distribution is finally

\[
\frac{\partial}{\partial z} = \frac{\Theta_1}{\Theta_0} \left( 1 - x + \frac{2}{\pi} \sum_n \frac{(-1)^n}{n} e^{-n^2\pi^2 F_0 \frac{c}{c}} \sin (1-x) n \pi \right)
\]

\[
+ \frac{\Theta_1}{\Theta_0} \left( x + \frac{2}{\pi} \sum_n \frac{(-1)^n}{n} e^{-n^2\pi^2 F_0 \frac{c}{c}} \sin x n \pi \right) \tag{82}
\]

where, \( x/l = \) and \( F_0 c = \kappa l/c^2 \).

3. Conclusion.

In the present paper, we find the temperature distribution in the sulphur smelting kettles theoretically of which type are cylindrical and continuous ones, and the application of Laplace-Transformation is efficient and appropriate as same as the cases of previous projects 1 and 2.

For the numerical calculations, this method show good convergence in the cases of 8 and 9, and these results indicate adequate temperature distributions in the sulphur smelting kettles.

In case of 8, for example, we take the constants as follows;

\( \lambda_s = 0.4 \) Kcal/mh°C, \( \alpha_c = 10 \) Kcal/mh°C, \( \kappa_i = 0.001 \) m²/h, \( \varepsilon = R_i/R_s = 0.91 \) and \( \lambda = 55 \) Kcal/mh°C.

![Graph showing temperature distribution](image)
then, temperatures of the kettle and the centre of mixed layer are shown by graphically in Fig. 13 versus the Fourier's number $Fo = \alpha t / \kappa l^2$.

Experimental data are the result for the actual kettle at some sulphur smelting refinery.

It seems that the differences between actual and theoretical values are caused by heat convection and change of state, etc., and inside temperature of the kettle rises to the necessary point (say, sulphur point) rapidly in the short period, therefore we take into no consideration about real operation in the refinery for the kettle's thickness.

Fig. 14 shows the surface temperature of continuous type, and the temperature change takes nearly straight line along the longitudinal direction.

![Fig. 14](image)

In this case, we select the constants as follows:

$$Nu = 70, \quad \alpha_i = 36.8 \text{ Kcal/m}^2\text{h}^°\text{C}, \quad C = 30\text{cm}, \quad F = 6\text{m}^2.$$  
$G =$ weight flow $= 2700\text{kg/h}$ (for $v = 8\text{m/s}$)  
$\quad = 5400\text{kg/h}$ (for $v = 6\text{m/s}$)  
and, $w = 3.057$ (for $v = 3\text{m/s}$), $w = 3.520$ (for $v = 6\text{m/s}$)  
$C_p = 0.25 \text{ kg/m for gas.}$

and the other dimensions are shown with non-dimensional numbers.

Temperature distribution $\theta/\Theta_o$ are represented versus the ordinate $y/c$ and these results indicated for $x/l = 0.1, 0.2, 0.4, 0.6, 0.8$, and $1.0$ respectively as shown in Fig. 15, 16, 17, 18, 19 and 20.

In the conclusion, we wished to express hearty thanks to Prof. Dr. Y. Tanasawa of Tohoku University for his valuable suggestions.
Fig. 16
Fig. 18
Reference


伝熱問題におけるラプラス変換の応用例（その8）

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前回についてといまでは熱伝達のうち、特に伝熱と連続式結熟釜との場合について理論的に解き、この場合の数値例を示し伝熱現象を明かさせた。

この結果は「伝熱釜における伝熱問題について」と題し、昭和29年6月13日 日本機械学会盛岡地方講演会で発表したものである。

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