Equivalent Electrical Network for the Transversely Vibrating Uniform Bar

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1. Introduction

A treatment by network theory on the linear mechanical vibrating system is very useful for not only the analysis but also the synthesis of electro-mechanical and electro-acoustical instruments.

Recently, many papers on analyses and treatments of the transversely vibrating bar by the analogy of impedance or mobility have been reported.^{(1) ~ (8)} However, the technical intercommunication between the mechanical engineering and the electrical engineering does not seem to be satisfactory.

By equivalent networks, the authors have studied on analyses of the transversely vibrating uniform bars, and presented some papers on the driving-point immittances and the applications to mechanical filter.^{(2) ~ (4)}

This paper presents the equivalent networks for the transversely vibrating bars under various end-conditions. And the value of each element of the networks is decided, so as to be convenient for practical use.

Furthermore, the frequency-characteristics of each element in the impedance or admittance matrix on the bar are shown in the numeral tables calculated by the electronic computer.

Then, the equivalent networks for the bars, whose distributed mass is neglected, are shown; and, by employing the equivalent networks, the possibility of analyzing easily some simple problems on the "strength of materials" is pointed out.

The paper, moreover, explains that the resonating bars can be easily analyzed in the simpler circuits by using the normalized function on vibration modes.

These results are very useful for analyzing not only a unit mechanical system but also a composite one combined with unit systems.

2. Transversely Vibrating Uniform Bar

2.1 Expression by Open-Circuit Impedance Matrix $[Z]$

In this chapter, the relation between each element of the matrix $[Z]$ and the short-circuit admittance matrix $[Y]$ is explained by means of the H or F

function defined newly.

And, the proof of expression by matrix $[3]$ or $[9]$ and the frequency-characteristics of common elements in these matrices are shown in the appendix.

The equivalent network of the transversely vibrating bar shown in Fig.1 (a) is expressed as Fig.1 (b), and can be more simply shown as Fig.1 (c) by Eqs.(1).

Fig. 1 (a) The transversely vibrating uniform bar
 (b) the equivalent four-terminal-pair network
 (c) the equivalent two-terminal-pair network

$$\begin{aligned} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} &= \begin{bmatrix} P_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix}, \quad \begin{bmatrix} Q_2 \\ Q_1 \end{bmatrix} = \begin{bmatrix} P_2 \\ M_2 \end{bmatrix} = - \begin{bmatrix} \bar{Q}_2 \\ \bar{Q}_1 \end{bmatrix}, \\ \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} &= \begin{bmatrix} v_1 \\ \dot{\theta}_1 \end{bmatrix}, \quad \begin{bmatrix} \dot{q}_2 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} v_2 \\ \dot{\theta}_2 \end{bmatrix}, \end{aligned} \quad (1)$$

where P , M , v ($= j\omega\eta$) and $\dot{\theta}$ ($\equiv \partial v / \partial x$) are the impressed force, the impressed bending moment, the velocity of displacement and the velocity of angular displacement, respectively; η is the displacement and θ is the angular displacement. Q is the general impressed force, $\bar{Q}$ is the general terminal force and $\dot{q}$ is the general terminal velocity.

The fundamental equation is written as Eq. (2), and each element of matrix $[3]$ is concisely and systematically expressed in Eq. (3) by employing the H and the F functions defined in Eqs. (4) and (5).

$$\begin{aligned} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} &= \begin{bmatrix} 3_{11} & 3_{12} \\ 3_{21} & 3_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \\ \begin{bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{bmatrix} &= \begin{bmatrix} z_{11} & Z_{11} & z_{12} & Z_{12} \\ \bar{z}_{11} & \bar{Z}_{11} & \bar{z}_{12} & \bar{Z}_{12} \\ z_{21} & Z_{21} & z_{22} & Z_{22} \\ \bar{z}_{21} & \bar{Z}_{21} & \bar{z}_{22} & \bar{Z}_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{bmatrix} \end{aligned} \quad (2)$$

$$\begin{aligned}
 &= \frac{-K\alpha^2}{j\omega l^2} \begin{bmatrix} \left(\frac{\alpha H_7}{H_3}\right)/l & \left(\frac{H_1}{H_3}\right) & -\left(\frac{\alpha H_7}{H_3}\right)/l & -\left(\frac{H_{10}}{H_3}\right) \\ \left(\frac{H_1}{H_3}\right) & -\left(\frac{H_5}{\alpha H_3}\right)l & \left(\frac{H_{10}}{H_3}\right) & -\left(\frac{H_8}{\alpha H_3}\right)l \\ -\left(\frac{\alpha H_7}{H_3}\right)/l & \left(\frac{H_{10}}{H_3}\right) & \left(\frac{\alpha H_6}{H_3}\right)/l & -\left(\frac{H_1}{H_3}\right) \\ -\left(\frac{H_{10}}{H_3}\right) & -\left(\frac{H_8}{\alpha H_3}\right)l & -\left(\frac{H_1}{H_3}\right) & -\left(\frac{H_5}{\alpha H_3}\right)l \end{bmatrix} \begin{bmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{bmatrix} \\
 &= j\sqrt{\rho SK} \begin{bmatrix} F_a/l & F_b & F_c/l & -F_d \\ F_b & -F_e \cdot l & F_d & F_f \cdot l \\ F_c/l & F_d & F_a/l & -F_b \\ -F_d & F_f \cdot l & -F_b & -F_e \cdot l \end{bmatrix} \begin{bmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{bmatrix}, \quad (3)
 \end{aligned}$$

or

$$\begin{aligned}
 \begin{bmatrix} P_1 \\ M_1/l \\ P_2 \\ M_2/l \end{bmatrix} &= \frac{j\sqrt{\rho SK}}{l} \begin{bmatrix} z_{11} & Z_{11}/l & z_{12} & Z_{12}/l \\ \bar{Z}_{11}/l & Z_{11}/l^2 & \bar{Z}_{12}/l & Z_{12}/l^2 \\ z_{21} & Z_{21}/l & z_{22} & Z_{22}/l \\ \bar{Z}_{21}/l & Z_{21}/l^2 & \bar{Z}_{22}/l & Z_{22}/l^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \dot{\theta}_1 l \\ v_2 \\ \dot{\theta}_2 l \end{bmatrix} \\
 &= \frac{j\sqrt{\rho SK}}{l} \begin{bmatrix} F_a & F_b & F_c & -F_d \\ F_b & -F_e & F_d & F_f \\ F_c & F_d & F_a & -F_b \\ -F_d & F_f & -F_b & -F_e \end{bmatrix} \begin{bmatrix} v_1 \\ \dot{\theta}_1 l \\ v_2 \\ \dot{\theta}_2 l \end{bmatrix}, \quad (3)'
 \end{aligned}$$

where

$$\begin{cases} H_1 = \sinh\alpha \cdot \sin\alpha \equiv S \cdot s, & H_6 = \sinh\alpha \cdot \cos\alpha - \cosh\alpha \cdot \sin\alpha \\ H_2 = \cosh\alpha \cdot \cos\alpha \equiv C \cdot c, & \equiv S \cdot c - C \cdot s, \\ H_3 = \cosh\alpha \cdot \cos\alpha - 1 \equiv C \cdot c - 1, & H_7 = \sin\alpha + \sinh\alpha \equiv s + S, \\ H_4 = \cosh\alpha \cdot \cos\alpha + 1 \equiv C \cdot c + 1, & H_8 = \sin\alpha - \sinh\alpha \equiv s - S, \\ H_5 = \sinh\alpha \cdot \cos\alpha - \cosh\alpha \cdot \sin\alpha & H_9 = \cos\alpha + \cosh\alpha \equiv c + C, \\ \equiv S \cdot c - C \cdot s, & H_{10} = \cos\alpha - \cosh\alpha \equiv c - C, \\ S : \sinh\alpha, & C : \cosh\alpha, \quad s : \sin\alpha, \quad c : \cos\alpha; \end{cases} \quad (4)$$

$$\left\{ \begin{array}{l} F_a = \alpha H_6/H_3 = \alpha(S \cdot c + C \cdot s)/(C \cdot c - 1) (\equiv -F'_a/\alpha^2), \\ F_b = H_1/H_3 = S \cdot s/(C \cdot c - 1) (\equiv -F'_b/\alpha^2), \\ F_c = -\alpha H_7/H_3 = -\alpha(s + S)/(C \cdot c - 1) (\equiv -F'_c/\alpha^2), \\ F_d = H_{10}/H_3 = (c - C)/(C \cdot c - 1) (\equiv F'_d/\alpha^2) \\ F_e = H_8/\alpha H_3 = (S \cdot c - C \cdot s)/\alpha(C \cdot c - 1) (\equiv F'_e/\alpha^2), \\ F_f = -H_9/\alpha H_3 = -(s - S)/\alpha(C \cdot c - 1) (\equiv -F'_f/\alpha^2); \end{array} \right. \quad (5)$$

$$\alpha^4 = \frac{\rho S}{K} \omega^2 l^4, \quad \frac{K \alpha^2}{j \omega l^2} = \frac{\sqrt{\rho S K}}{j}. \quad (6)$$

In Eqs. (6), $K(=EI)$ expresses the bending strength (E : Young's modulus, I : the second moment of area of the cross-section about the neutral axis), and ρ , S and l indicate the density, the sectional area and the length of the bar, respectively.

2.2 Expression by Short-Circuit Admittance Matrix $[\mathcal{Y}]$

Fundamental equations by matrix $[\mathcal{Y}]$ are shown in Eqs. (7) and (8), that is,

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \quad (7)$$

$$\begin{aligned} \begin{pmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix} &= \begin{pmatrix} \mathcal{Y}_{11} & \overline{\mathcal{Y}}_{11} & \mathcal{Y}_{12} & \overline{\mathcal{Y}}_{12} \\ \mathcal{Y}_{11} & \mathcal{Y}_{11} & \mathcal{Y}_{12} & \mathcal{Y}_{12} \\ \mathcal{Y}_{21} & \overline{\mathcal{Y}}_{21} & \mathcal{Y}_{22} & \overline{\mathcal{Y}}_{22} \\ \mathcal{Y}_{21} & \mathcal{Y}_{21} & \mathcal{Y}_{22} & \mathcal{Y}_{22} \end{pmatrix} \begin{pmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{pmatrix} \\ &= \frac{-j\omega l^2}{K\alpha^2} \begin{pmatrix} \left(\frac{H_6}{\alpha H_3}\right)l & \left(\frac{H_1}{H_3}\right) & -\left(\frac{H_8}{\alpha H_3}\right)l & -\left(\frac{H_{10}}{H_3}\right) \\ \left(\frac{H_1}{H_3}\right) & -\left(\frac{\alpha H_6}{H_3}\right)/l & \left(\frac{H_{10}}{H_3}\right) & -\left(\frac{\alpha H_7}{H_3}\right)/l \\ -\left(\frac{H_8}{\alpha H_3}\right)l & \left(\frac{H_{10}}{H_3}\right) & \left(\frac{H_6}{\alpha H_3}\right)l & -\left(\frac{H_1}{H_3}\right) \\ -\left(\frac{H_{10}}{H_3}\right) & -\left(\frac{\alpha H_7}{H_3}\right)/l & -\left(\frac{H_1}{H_3}\right) & -\left(\frac{\alpha H_6}{H_3}\right)/l \end{pmatrix} \begin{pmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{pmatrix} \\ &= \frac{1}{j\sqrt{\rho S K}} \begin{pmatrix} F_e \cdot l & F_b & F_f \cdot l & -F_d \\ F_b & -F_a/l & F_d & F_e/l \\ F_f \cdot l & F_d & F_e \cdot l & -F_b \\ -F_d & F_e/l & -F_b & -F_a/l \end{pmatrix} \begin{pmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{pmatrix}, \quad (8) \end{aligned}$$

or

$$\begin{aligned} \begin{pmatrix} v_1 \\ \dot{\theta}_1 l \\ v_2 \\ \dot{\theta}_2 l \end{pmatrix} &= \begin{pmatrix} y_{11} & \bar{Y}_{11} \cdot l & y_{12} & \bar{Y}_{12} \cdot l \\ Y_{11} \cdot l & Y_{11} \cdot l^2 & Y_{12} \cdot l & Y_{12} \cdot l^2 \\ y_{21} & \bar{Y}_{21} \cdot l & y_{22} & \bar{Y}_{22} \cdot l \\ Y_{21} \cdot l & Y_{21} \cdot l^2 & Y_{22} \cdot l & Y_{22} \cdot l^2 \end{pmatrix} \begin{pmatrix} P_1 \\ M_1/l \\ P_2 \\ M_2/l \end{pmatrix} \\ &= \frac{l}{j\sqrt{\rho SK}} \begin{pmatrix} F_e & F_b & F_f & -F_d \\ F_b & -F_a & F_d & F_c \\ F_f & F_d & F_e & -F_b \\ -F_a & F_c & -F_b & -F_a \end{pmatrix} \begin{pmatrix} P_1 \\ M_1/l \\ P_2 \\ M_2/l \end{pmatrix}, \quad (8)' \end{aligned}$$

2.3 Expression by Transfer Matrix $[\Gamma]$

Fundamental equations by matrix $[\Gamma]$ are shown in Eqs. (9) and (10),

that is,

$$\begin{pmatrix} \bar{Q}_1 \\ \dot{q}_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{Q}_2 \\ \dot{q}_2 \end{pmatrix}, \quad (9)$$

$$\begin{aligned} \begin{pmatrix} \bar{P}_1 \\ \bar{M}_1 \\ v_1 \\ \dot{\theta}_1 \end{pmatrix} &= \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{pmatrix} \begin{pmatrix} \bar{P}_2 \\ \bar{M}_2 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} H_9 & -H_8 \frac{\alpha}{l} & -H_7 \frac{K\alpha^3}{j\omega l^3} & -H_{10} \frac{K\alpha^2}{j\omega l^2} \\ H_7 \frac{l}{\alpha} & H_9 & H_{10} \frac{K\alpha^2}{j\omega l^2} & -H_8 \frac{K\alpha}{j\omega l} \\ H_8 \frac{j\omega l^3}{K\alpha^3} & H_{10} \frac{j\omega l^2}{K\alpha^2} & H_9 & -H_7 \frac{l}{\alpha} \\ -H_{10} \frac{j\omega l^2}{K\alpha^2} & H_7 \frac{j\omega l}{K\alpha} & H_8 \frac{\alpha}{l} & H_9 \end{pmatrix} \begin{pmatrix} \bar{P}_2 \\ \bar{M}_2 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix}. \quad (10) \end{aligned}$$

3. Bar with Various End-Conditions

As to the one end of the bar, four end-conditions are considered: that is, the free end ($P = M = 0$), the simple support ($v = 0$), the sliding support ($\dot{\theta} = 0$) and the fixed end ($v = \dot{\theta} = 0$).

Table 1 shows the equivalent networks of the bars under various end-conditions and gives the value of each element of them.

Table 1 The equivalent networks of the bar with various end-conditions

Left end	Right end	Simple support	Sliding support	Fixed end
		$Z_a = j\sqrt{\rho S K} (F_a + F_b - F_d),$ $Z_b = j\sqrt{\rho S K} (-F_b),$ $Z_c = j\sqrt{\rho S K} (F_d),$ $Z_d = j\sqrt{\rho S K} (-F_f),$ $Z_e = j\sqrt{\rho S K} (-F_d - F_e + F_f),$ $Z_f = j\sqrt{\rho S K} (F_b - F_e + F_f).$	$\partial_a = j\sqrt{\rho S K} (F_a + F_b + F_c),$ $\partial_b = j\sqrt{\rho S K} (-F_b),$ $\partial_c = j\sqrt{\rho S K} (-F_c),$ $\partial_d = j\sqrt{\rho S K} (-F_d),$ $\partial_e = j\sqrt{\rho S K} (F_a + F_c + F_d),$ $\partial_f = j\sqrt{\rho S K} (F_b + F_d - F_e).$	$\partial_a = j\sqrt{\rho S K} (F_a + F_b),$ $\partial_b = j\sqrt{\rho S K} (F_b - F_e),$ $\partial_c = j\sqrt{\rho S K} (-F_b) = -\partial_c,$ $\partial_d = j\sqrt{\rho S K} (F_a - F_b),$ $\partial_e = j\sqrt{\rho S K} (-F_b - F_e).$
Free end	Simple support	$Z_a = j\sqrt{\rho S K} (-F_e + F_f),$ $Z_c = j\sqrt{\rho S K} (-F_f) = -Z_c,$ $Z_d = j\sqrt{\rho S K} (-F_e - F_f).$	$\partial_a = j\sqrt{\rho S K} (F_a + F_d),$ $\partial_b = j\sqrt{\rho S K} (F_d - F_e),$ $\partial_c = j\sqrt{\rho S K} (-F_d) = -\partial_c,$ $\partial_d = j\sqrt{\rho S K} (F_a - F_d),$ $\partial_e = j\sqrt{\rho S K} (-F_d - F_e).$	$Z_1 = j\sqrt{\rho S K} (-F_e).$
Sliding support	Notes:	$F_a = \frac{\alpha H_6}{H_3} = \alpha \frac{S \cdot c + C \cdot s}{C \cdot c - 1},$ $F_b = \frac{H_1}{H_3} = \frac{S \cdot s}{C \cdot c - 1},$ $F_c = \frac{-\alpha H_7}{H_3} = -\alpha \frac{s + S}{C \cdot c - 1},$ $F_d = \frac{H_{10}}{H_3} = \frac{c - C}{C \cdot c - 1},$ $F_e = \frac{H_5}{\alpha H_3} = \frac{1}{\alpha} \frac{S \cdot c - C \cdot s}{C \cdot c - 1},$ $F_f = \frac{-H_8}{\alpha H_3} = \frac{-1}{\alpha} \frac{s - S}{C \cdot c - 1},$ $\alpha^4 = \frac{\rho S}{K} \omega^2 l^4, \quad \frac{K \alpha^2}{j \omega l^2} = \frac{\sqrt{\rho S K}}{j}.$	$\partial_a = j\sqrt{\rho S K} (F_a + F_c),$ $\partial_c = j\sqrt{\rho S K} (-F_c) = -\partial_c,$ $\partial_d = j\sqrt{\rho S K} (F_a - F_c).$	$\partial_1 = j\sqrt{\rho S K} (F_a),$ <p>where, $K = EI,$ E; Young's modulus, I; moment of inertia, ρ; density, S; sectional area, l; length.</p>

Moreover, the values of functions ($F_a \sim F_f$) and ($F'_a \sim F'_f$) defined in Eqs. (5) for frequency-constant α are shown by the numeral tables in the end of this paper (See Tables 3 and 4), and the physical meaning of these functions is given in Eqs. (3)' and (8)'.

In the appendix, an equivalent four-terminal-pair network on the unit bar and the values of their elements are shown for reference.

4. Bar of Pure Stiffness

4.1 Fundamental Equation

The fundamental equations for a stiffness bar such as its distributed mass is neglected are simply shown as follows.

$$\begin{pmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{pmatrix} = \begin{pmatrix} z_{11} & Z_{11} & z_{12} & Z_{12} \\ \bar{Z}_{11} & Z_{11} & \bar{Z}_{12} & Z_{12} \\ z_{21} & Z_{21} & z_{22} & \bar{Y}_{22} \\ \bar{Z}_{21} & Z_{21} & \bar{Z}_{22} & Z_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix} = \frac{K}{j\omega} \begin{pmatrix} 12/l^3 & 6/l^2 & -12/l^3 & 6/l^2 \\ 6/l^2 & 4/l & -6/l^2 & 2/l \\ -12/l^3 & -6/l^2 & 12/l^3 & -6/l^2 \\ 6/l^2 & 2/l & -6/l^2 & 4/l \end{pmatrix} \begin{pmatrix} v_1 \\ \dot{\theta}_1 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix}, \quad (11)$$

or

$$\begin{pmatrix} P_1 \\ M_1/l \\ P_2 \\ M_2/l \end{pmatrix} = \begin{pmatrix} z_{11} & Z_{11}/l & z_{12} & Z_{12}/l \\ \bar{Z}_{11}/l & Z_{11}/l^2 & \bar{Z}_{12}/l & Z_{12}/l^2 \\ z_{21} & Z_{21}/l & z_{22} & Z_{22}/l \\ \bar{Z}_{21}/l & Z_{21}/l^2 & \bar{Z}_{22}/l & Z_{22}/l^2 \end{pmatrix} \begin{pmatrix} v_1 \\ \dot{\theta}_1 l \\ v_2 \\ \dot{\theta}_2 l \end{pmatrix} \\ = \frac{K}{j\omega l^3} \begin{pmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ \dot{\theta}_1 l \\ v_2 \\ \dot{\theta}_2 l \end{pmatrix}, \quad (11)'$$

$$\begin{pmatrix} \bar{P}_1 \\ \bar{M}_1 \\ v_1 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l & 1 & 0 & 0 \\ \frac{-j\omega l^3}{6K} & \frac{-j\omega l^2}{2K} & 1 & -l \\ \frac{j\omega l^2}{2K} & \frac{j\omega l}{K} & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{P}_2 \\ \bar{M}_2 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix}. \quad (11)''$$

4.2 Bar with Various End-Conditions

Table 2 shows the equivalent networks for the stiffness-bars restricted variously at the ends.

Table 2 The equivalent networks of the stiffness-bar with various end-conditions

Left end	Right end			
	Simple support	Sliding support	Fixed end	
Free end				
Sliding support	<p>Notes:</p> $\delta_o \equiv \frac{K}{l^3}$ $\delta_o \equiv \frac{K}{l}$			

It is found that some problems on the beam treated in the "strength of materials" can be considered as a problem of the network theory.

Like Table 1, this table will be also useful for a construction of the composite mechanical filter.

5. Equivalent Networks by the Normalized Function ⁽⁶⁾

Fig. 2 The uniform bar impressed with forces and bending moments at any points

5.1 Bar Impressed with Forces and Bending Moments

If the forces (P_{x1} , P_{x2}) and the bending moments (M_{x1} , M_{x2}) are impressed at any points (x_1 , x_2) of the uniform bar and the bar is vibrated transversely as shown in Fig. 2, then the following equations are given.

$$\begin{pmatrix} v(x_i) \\ \dot{\theta}(x_i) \\ v(x_j) \\ \dot{\theta}(x_j) \end{pmatrix} = \begin{pmatrix} Y_{x_i x_i} & \bar{Y}_{x_i x_i} \\ Y_{x_i x_j} & \bar{Y}_{x_i x_j} \\ Y_{x_j x_i} & \bar{Y}_{x_j x_i} \\ Y_{x_j x_j} & \bar{Y}_{x_j x_j} \end{pmatrix} \begin{pmatrix} P_{x_i} \\ M_{x_i} \\ P_{x_j} \\ M_{x_j} \end{pmatrix}, \quad (12)$$

$$\begin{pmatrix} Y_{x_i x_j} & \bar{Y}_{x_i x_j} \\ Y_{x_j x_i} & \bar{Y}_{x_j x_i} \end{pmatrix} = \frac{1}{Z_{0m}} \begin{pmatrix} \sum_m \mathcal{E}_m(x_i) \cdot \mathcal{E}_m(x_j), & \frac{1}{l} \sum_m \mathcal{E}_m(x_i) \cdot \mathcal{E}'_m(x_j) \\ \frac{1}{l} \sum_m \mathcal{E}'_m(x_i) \cdot \mathcal{E}_m(x_j), & \frac{1}{l^2} \sum_m \mathcal{E}'_m(x_i) \cdot \mathcal{E}'_m(x_j) \end{pmatrix} \\ \equiv \sum_m [\mathcal{Y}_m], \quad (13)$$

where

$$X = \frac{x}{l}, \quad \frac{1}{Z_{0m}} = \frac{j\omega}{(\omega_m^2 - \omega^2)(\rho S l)}, \\ \omega_m = \frac{\alpha_m^2}{l^2} \sqrt{\frac{K}{\rho S}}, \quad \mathcal{E}'_m(x) = \frac{\partial \mathcal{E}_m(x)}{\partial X}. \quad (14)$$

Fig. 3 The general expression by the equivalent four-terminal-pair network on the bar shown in Fig. 2

α_m is the frequency constant at the m -th mode.

And $\mathcal{E}_m(x)$ is the normalized function of the m -th mode of the vibrating bar, and $m = 0, 1, 2, \dots, \infty$.

The sum of the short admittance matrices, $\sum_m [\mathcal{Y}_m]$, means that m networks are to be connected in parallel one another, and therefore the resultant equivalent network is shown as Fig. 3.

Fig. 4 The equivalent network of only the m -th mode of vibration on the bar in Fig. 2

Fig. 5 The equivalent $2 \times n$ -terminal-pair network of only the m -th mode of vibration on the bar impressed with forces and bending moments at any points X_j ($j = 1, 2, \dots, n$)

Fig. 4 shows the equivalent network only for the m -th vibration-mode of the bar. And this simple equivalent circuit is very convenient for practical use, because the vibrating bars are frequently used in such a certain vibration-mode as the m -th.

Another example is shown in Fig. 5.

5.2 Bar with Loads z_{Xj} and Z_{Xj}

In this case, the (shearing force) load z_{Xj} and the bending moment load Z_{Xj} are to be put by the circuit theory as follows; $P_{Xj} = -z_{Xj} \cdot v_{(Xj)}$, $M_{Xj} = -Z_{Xj} \cdot \theta_{(Xj)}$.

A few examples are shown in Fig. 6 and Fig. 7.

Fig. 6 (a) The bar impressed with a force P_{X1} and loaded with z_{Xj} and Z_{Xj} , (b) the equivalent one-terminal-pair network (c) another expression by the equivalent one-terminal-pair network

$$\begin{aligned} n_1 &= \frac{\Xi_m(X_1)}{\Xi_m(X_2)}, \\ n_2 &= \frac{\Xi'_m(X_1)}{\Xi'_m(X_2)}, \\ n'_1 &= \frac{\Xi_m(X_1)}{\Xi_m(X_2) \cdot l}, \\ n'_2 &= \frac{\Xi'_m(X_1)}{\Xi'_m(X_2) \cdot l}. \end{aligned}$$

Fig. 7. (a) An example of the composite mechanical filter consisting of resonators and couplers, (b) the equivalent network

6. Conclusions

In this paper, the fundamental equations on the transversely vibrating uniform bar are systematically presented and the equivalent networks of the bar under various end-conditions are given, and then the value of each element of the networks is decided. Some consideration on the bar of which the distributed masses are neglected is also shown.

Moreover, an outline on the analysis by the method of normalized function is described by citing a few examples.

General considerations in the first half of this paper are available for an analysis of a composite mechanical system, while the method of normalized function in the second is very convenient for design of a resonator of mechanical filter and the like.

Although considerations only on a unit mechanical system are explained in this paper, these results are amplified to analysis and synthesis of a composite system combined with unit ones.

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Appendix

A-1. Analysis by the Network Theory on the Transversely Vibrating-Bar

This chapter describes on a proof of the expression by matrix $[F]$. And matrices $[3]$ and $[y]$ are deduced from matrix $[F]$.

The general terminal force $\bar{Q}$ and the general terminal velocity $\dot{q}$ shown in Fig. 1(c) are written as Eqs. (A-1).

$$\begin{bmatrix} \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{\eta} \\ \dot{\theta} \end{bmatrix}, \quad \begin{bmatrix} \bar{Q} \end{bmatrix} = \begin{bmatrix} \bar{P} \\ \bar{M} \end{bmatrix} = \begin{bmatrix} \frac{K}{j\omega} \cdot \frac{\partial^3 \dot{\eta}_{(x)}}{\partial x^3} \\ -\frac{K}{j\omega} \cdot \frac{\partial^2 \dot{\eta}_{(x)}}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \frac{K}{j\omega l^3} \cdot \frac{\partial^3 \dot{\eta}_{(x)}}{\partial X^3} \\ -\frac{K}{j\omega l^2} \cdot \frac{\partial^2 \dot{\eta}_{(x)}}{\partial X^2} \end{bmatrix}, \quad (A-1)$$

where $X = x/l$, $\theta \equiv \partial\eta/\partial x$, $\dot{q} = \partial q/\partial t$; and the notations η , θ , K and so on are indicated in Ch. 2.

When the displacement η is considered as $\eta \cdot e^{j\omega t}$, the differential equation of the transverse vibration of the bar is written as follows.

$$K \frac{\partial^4 \eta_{(x)}}{\partial x^4} + \rho S \frac{\partial^2 \eta_{(x)}}{\partial t^2} = 0, \quad \text{or} \quad \frac{\partial^4 \eta_{(x)}}{\partial X^4} - \alpha^4 \eta_{(x)} = 0, \quad (A-2)$$

where

$$\alpha^4 = \frac{\rho S}{K} \omega^2 l^4. \quad (A-3)$$

Then, the solution of Eqs. (A-2) is given as

$$\begin{aligned} \eta_{(x)} &= a_1 \cos \alpha x + a_2 \sin \alpha x + a_3 \cosh \alpha x + a_4 \sinh \alpha x \\ &\equiv a_1 c_x + a_2 s_x + a_3 C_x + a_4 S_x. \end{aligned} \quad (A-4)$$

where a_1 , a_2 , a_3 , and a_4 are arbitrary constants.

And $c_x \equiv \cos \alpha x$, $s_x \equiv \sin \alpha x$, $C_x \equiv \cosh \alpha x$, $S_x \equiv \sinh \alpha x$.

From Eqs. (A-1) and Eq. (A-4), the following equations are obtained.

$$\begin{bmatrix} \bar{P}_x \\ \bar{M}_x \\ v_x \\ \dot{\theta}_x \end{bmatrix} = \begin{bmatrix} H_x \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \bar{Q}_x \\ \dot{q}_x \end{bmatrix} = \begin{bmatrix} H_x \end{bmatrix} \begin{bmatrix} a_n \\ a_m \end{bmatrix}, \quad (A-5)$$

where

$$\begin{bmatrix} H_x \end{bmatrix} = \begin{bmatrix} \alpha^3 s_x & -\alpha^3 c_x & \alpha^3 S_x & \alpha^3 C_x \\ \alpha^2 c_x & \alpha^2 s_x & -\alpha^2 C_x & -\alpha^2 S_x \\ c_x & s_x & C_x & S_x \\ -\alpha' s_x & \alpha' c_x & \alpha' S_x & \alpha' C_x \end{bmatrix}, \quad (A-6)$$

$$\alpha^3 = \frac{K}{j\omega} \cdot \frac{\alpha^3}{l^3}, \quad \alpha^2 = \frac{K}{j\omega} \cdot \frac{\alpha^2}{l^2}, \quad \alpha = \frac{K}{j\omega} \cdot \frac{\alpha}{l}, \quad \alpha' = \frac{\alpha}{l}. \quad (A-7)$$

If the input terminal ($X=0$) and the output terminal ($X=1$) are expressed with the suffix 1 and 2 respectively, then the following equations are deduced from Eqs. (A-5).

$$\begin{pmatrix} \bar{Q}_x \\ \dot{q}_x \end{pmatrix} = [H_x] [H_1]^{-1} \begin{pmatrix} \bar{Q}_1 \\ \dot{q}_1 \end{pmatrix}, \quad (\text{A-8})$$

and accordingly,

$$\begin{pmatrix} \bar{Q}_2 \\ \dot{q}_2 \end{pmatrix} = [F_{2,1}] \begin{pmatrix} \bar{Q}_1 \\ \dot{q}_1 \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} \bar{Q}_1 \\ \dot{q}_1 \end{pmatrix}, \quad (\text{A-9})$$

or

$$\begin{pmatrix} \bar{Q}_1 \\ \dot{q}_1 \end{pmatrix} = [F_{1,2}] \begin{pmatrix} \bar{Q}_2 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{Q}_2 \\ \dot{q}_2 \end{pmatrix}. \quad (\text{A-10})$$

where $[F]$ is the transfer matrix, and

$$[F_{1,2}] = [H_1][H_2]^{-1} = [[H_2][H_1]^{-1}]^{-1} = [F_{2,1}]^{-1}, \quad (\text{A-11})$$

$$[F_{1,2}] = \frac{1}{2} \begin{pmatrix} (c+C) & -(s-S)\alpha' & -(s+S)\alpha^3 & -(c-C)\alpha^2 \\ (s+S)/\alpha' & (c+C) & (c-C)\alpha^2 & -(s-S)\alpha \\ (s-S)/\alpha^3 & (c-C)/\alpha^2 & (c+C) & -(s+S)\alpha' \\ -(c-C)/\alpha^2 & (s+S)/\alpha & (s-S)\alpha' & (c+C) \end{pmatrix}, \quad (\text{A-12})$$

Consequently, the matrices $[\mathfrak{Z}]$ and $[\mathfrak{Y}]$ are deduced from Eq.(A-12) and written as Eq. (3) and Eq. (8) in Ch. 2.

A-2. Analysis by the Normalized Function $\mathcal{E}_{m(x)}$ on Vibration Modes

An analysis by the normalized function on the vibration mode of the transversely vibrating bar was already studied by T. HAYASAKA⁽⁶⁾ and developed by T. YASUDA⁽⁷⁾.

Thereafter, the authors amplified this analyzing method to a treatment by polyterminal networks.

When the forces P_{x_j} are impressed at any points x_j ($j = 1, 2, \dots, n$) of the uniform bar as shown in Fig. 2, the velocity of displacement $v(x_i)$ at the point x_i is given as

$$\begin{aligned} v(x_i) &= \sum_m \frac{j\omega}{(\omega_m^2 - \omega^2)(\rho S l)} \mathcal{E}_{m(x_i)} \sum_{j=1}^n \mathcal{E}_{m(x_j)} \cdot P_{x_j} \\ &\equiv \sum_m \frac{1}{Z_{om}} \mathcal{E}_{m(x_i)} \sum_{j=1}^n \mathcal{E}_{m(x_j)} \cdot P_{x_j} \end{aligned} \quad (\text{A-13})$$

$$\equiv (y_{x_i x_j}) P_{x_j}. \quad (\text{A-14})$$

And then, the short admittances $(y_{x_i x_i})$ and $(y_{x_i x_j})$ are defined as the coefficient of P_{x_j} in Eq. (A-13).

For example, if only two forces P_{x_1} and P_{x_2} are impressed, the short admittances are shown as

$$\begin{aligned}(y_{x_i x_i}) &= \sum_m \frac{1}{Z_{0m}} \bar{\varepsilon}_{m(x_i)} \cdot \bar{\varepsilon}_{m(x_i)} \equiv \sum_m (y_{m x_i x_i}), \\(y_{x_i x_j}) &= \sum_m \frac{1}{Z_{0m}} \bar{\varepsilon}_{m(x_i)} \cdot \bar{\varepsilon}_{m(x_j)} \equiv \sum_m (y_{m x_i x_j}).\end{aligned}\quad (\text{A-15})$$

By differentiating Eq. (A-13) with x , the velocity of angular displacement $\dot{\theta}_{(x_i)}$ ($\partial v_{(x)}/\partial x \cdot l$) is given as

$$\begin{aligned}\dot{\theta}_{(x_i)} &= \sum_m \frac{1}{Z_{0m} \cdot l} \bar{\varepsilon}'_{m(x_i)} \sum_{j=1}^n \bar{\varepsilon}_{m(x_j)} \cdot P_{x_j} \\&\equiv (Y_{x_i x_j}) P_{x_j},\end{aligned}\quad (\text{A-16})$$

where

$$\bar{\varepsilon}'_{m(x)} = \partial \bar{\varepsilon}_{m(x)} / \partial X \quad (\text{A-17})$$

And similarly, the short admittances $(Y_{x_i x_i})$ and $(Y_{x_i x_j})$ are defined as the coefficient of P_{x_j} in Eq. (A-16).

Next, the bending moment M_{x_j} at the point X_j of the bar is substituted or

$$M_{x_j} = \lim_{\delta x \rightarrow 0} P_{x_j} \cdot l \cdot \delta X. \quad (\text{A-18})$$

The equation (A-18) means that the bending moment M_{x_j} is equal to the impression of two forces having such opposite signs as $+P_{x_j}$ and $-P_{x_j}$ at two points $(X_j + \frac{1}{2} \delta X)$ and $(X_j - \frac{1}{2} \delta X)$ respectively.

Accordingly, Eq. (A-19) is obtained.

$$\begin{aligned}&\sum_{j=1}^n \left[\bar{\varepsilon}_{m(X_j + \frac{1}{2} \delta X)} \cdot P_{x_j} - \bar{\varepsilon}_{m(X_j - \frac{1}{2} \delta X)} \cdot P_{x_j} \right] \\&= \sum_{j=1}^n (P_{x_j} \cdot \delta X) \bar{\varepsilon}'_{m(x_j)} = \sum_{j=1}^n \frac{M_{x_j}}{l} \cdot \bar{\varepsilon}'_{m(x_j)}.\end{aligned}\quad (\text{A-19})$$

Consequently, when the bending moments M_{x_j} are impressed, the velocities $v_{(x_i)}$ and $\dot{\theta}_{(x_i)}$ are written as follows;

$$v_{(x_i)} = \sum_m \frac{1}{Z_{0m} \cdot l} \bar{\varepsilon}_{m(x_i)} \sum_{j=1}^n \bar{\varepsilon}'_{m(x_j)} \cdot M_{x_j} \equiv (\bar{Y}_{x_i x_j}) M_{x_j}, \quad (\text{A-20})$$

$$\dot{\theta}_{(x_i)} = \sum_m \frac{1}{Z_{0m} \cdot l} \bar{\varepsilon}'_{m(x_i)} \sum_{j=1}^n \bar{\varepsilon}'_{m(x_j)} \cdot M_{x_j} \equiv (Y_{x_i x_j}) M_{x_j}. \quad (\text{A-21})$$

A-3. Expression by Equivalent Four-Terminal-Pair Network

Fig. A-1 The equivalent network of Fig. 1

An equivalent network on the vibrating bar expressed in Eq. 3 of Ch.2 will be generally shown as Fig. A-1.

The value of each element of the network is given as Eqs. (A-22).

$$\begin{cases} z_1 = \frac{j\sqrt{\rho SK}}{l} (F_a + F_b + F_c + F_d), & z_m = \frac{j\sqrt{\rho SK}}{l} (-F_d), \\ z_2 = \frac{j\sqrt{\rho SK}}{l} (-F_b), & z_4 = \frac{j\sqrt{\rho SK}}{l} (-F_c), \\ z_3 = \frac{j\sqrt{\rho SK}}{l} (F_b + F_d - F_e - F_f), & z_5 = \frac{j\sqrt{\rho SK}}{l} (F_f). \end{cases} \quad (\text{A-22})$$

From this network, the equivalent circuits of the bars under various end-conditions can be introduced. (cf. Table 1)

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Table 3 The frequency-characteristics of functions $F_a \sim F_f$

α	F_a	F_b	F_c	F_d	F_e	F_f
0.15	-533.02154	-266.51351	533.03280	266.51537	177.69993	-88.815614
0.20	-299.85720	-149.93379	299.87720	149.93742	99.963955	-49.972044
0.25	-191.92625	-95.971364	191.95750	95.976723	63.985436	-31.989124
0.30	-133.28082	-66.652343	133.32582	66.659959	44.438643	-22.218235
0.35	-97.902067	-48.967319	97.963320	48.977574	32.648720	-16.324720
0.40	-74.934300	-37.488448	75.014304	37.501817	24.996774	-12.499676
0.45	-59.180207	-29.617077	59.281466	29.633975	19.750104	-9.8770908
0.50	-47.904670	-23.985648	48.029685	24.005498	15.996934	-8.0012060
0.55	-39.555531	-19.818082	39.706800	19.843310	13.219848	-6.6133579
0.60	-33.198450	-16.647215	33.378485	16.677230	11.107359	-5.5578518
0.65	-28.244636	-14.178646	28.455940	14.213874	9.4632147	-4.7365557
0.70	-24.307207	-12.218938	24.552288	12.259796	8.1584443	-4.0849975
0.75	-21.123926	-10.636972	21.405301	10.683880	7.1056342	-3.5594716
0.80	-18.511879	-9.3412921	18.832062	9.3946691	6.2438115	-3.1294961
0.85	-16.340270	-8.2665004	16.701783	8.3267697	5.5293793	-2.7732717
0.90	-14.513585	-7.3648427	14.918955	7.4324253	4.9304964	-2.4748829
0.95	-12.960772	-6.6007917	13.412534	6.6761129	4.4234832	-2.2224859
1.00	-11.628095	-5.9474816	12.128791	6.0309669	3.9904264	-2.0071283
1.05	-10.474273	-5.3842739	11.026457	5.4763518	3.6175683	-1.8219300
1.10	-9.4672074	-4.8351209	10.073442	4.9962211	3.2942102	-1.6615434
1.15	-8.5816061	-4.4673777	9.2444681	4.5779340	3.0119237	-1.521737
1.20	-7.7973354	-4.0909816	8.5194181	4.2114295	2.7639996	-1.3992080
1.25	-7.0982066	-3.7578390	7.8821191	3.8886197	2.5450441	-1.2912095
1.30	-6.4710755	-3.4613817	7.3194471	3.6029392	2.3506792	-1.1955700
1.35	-5.9051855	-3.1962345	6.8206672	3.3490182	2.1773222	-1.1104991
1.40	-5.3916600	-2.9579632	6.3769292	3.1224274	2.0220186	-1.0345203
1.45	-4.9231248	-2.7428850	5.9808863	2.9194904	1.8823142	-0.96640772
1.50	-4.4934081	-2.5479187	5.6263986	2.7371324	1.7561565	-0.90513600
1.55	-4.0973123	-2.3704707	5.3083038	2.5727682	1.6418185	-0.84984290
1.60	-3.7304338	-2.2083457	5.0222381	2.4242100	1.5378384	-0.79979894
1.65	-3.3890229	-2.0596749	4.7644951	2.2895991	1.4429726	-0.75438382
1.70	-3.0698646	-1.9228586	4.5319091	2.1673459	1.3561570	-0.71306711
1.75	-2.7701912	-1.7965210	4.3217657	2.0560862	1.2764771	-0.67539354
1.80	-2.4876076	-1.6794735	4.1317294	1.9546442	1.2031436	-0.64097039
1.85	-2.2200300	-1.5706842	3.9597820	1.8620019	1.1354715	-0.60945757
1.90	-1.9656347	-1.4692535	3.8041753	1.7772755	1.0728652	-0.58055953
1.95	-1.7228306	-1.3743939	3.6633830	1.6996943	1.0148032	-0.55401851
2.00	-1.4901986	-1.2854131	3.5361000	1.6285349	0.96082865	-0.52960890
2.05	-1.2664902	-1.2017000	3.4211538	1.5633569	0.91053883	-0.50713273
2.10	-1.0505905	-1.1227132	3.3175409	1.5034914	0.86357805	-0.48641586
2.15	-0.84150121	-1.0479709	3.2243795	1.4485319	0.81963098	-0.46730461
2.20	-0.63832322	-0.97704300	3.1408980	1.3980753	0.77841717	-0.44966304
2.25	-0.44024256	-0.90954336	3.0664226	1.3517657	0.73968630	-0.43337089
2.30	-0.24651773	-0.84512449	3.0003659	1.3092886	0.70321426	-0.41832142
2.35	-0.056469515	-0.78347184	2.9422172	1.2703659	0.66879982	-0.40441977
2.40	0.13053010	-0.72429982	2.8915358	1.2347516	0.63626184	-0.39158188
2.45	0.31506356	-0.66734725	2.8479425	1.2022286	0.60543633	-0.37973269
2.50	0.49767790	-0.61237456	2.8111160	1.1726055	0.57617476	-0.36880577
2.55	0.67888908	-0.55916046	2.7807878	1.1457145	0.54834192	-0.35874202
2.60	0.85918997	-0.50749944	2.7567387	1.1214084	0.52181438	-0.34948914
2.65	1.0390547	-0.45719924	2.7387965	1.0995601	0.49647894	-0.34100097
2.70	1.2189449	-0.40807878	2.7268340	1.0800599	0.47223137	-0.33323692
2.75	1.3993140	-0.35996608	2.7207674	1.0628155	0.44897534	-0.32616157
2.80	1.5806129	-0.31269641	2.7205562	1.0477501	0.42662136	-0.31974434
2.85	1.7632943	-0.26611042	2.7262024	1.0348023	0.40508572	-0.31395906
2.90	1.9478201	-0.22005267	2.7377543	1.0239259	0.38428983	-0.30878408
2.95	2.1346643	-0.17436963	2.7553048	1.0150892	0.36415942	-0.30420181
3.00	2.3243194	-0.12890796	2.7789951	1.0082749	0.34462343	-0.30019865

α	F_a	F_b	F_c	F_d	F_e	F_f
3.05	52.173047	-0.083512851	2.8090195	1.0034818	0.32561391	-0.29676523
3.10	2.7141712	-0.038025980	2.8456276	1.0007233	0.30706466	-0.29383620
3.15	2.9155117	-0.007716544	2.8891325	1.0000303	0.28891094	-0.29159048
3.20	3.1219694	0.053886537	2.9399168	1.0014514	0.27108866	-0.28985150
3.25	3.3342491	0.10066637	2.9984418	1.0050548	0.25353346	-0.28868742
3.30	3.5531288	0.14825199	3.0652587	1.0109302	0.23617999	-0.28811159
3.35	3.7794790	0.19685657	3.1410238	1.0191928	0.21896090	-0.28814317
3.40	4.0142780	0.24671470	3.2265149	1.0299852	0.20180593	-0.28880783
3.45	4.2586358	0.29808754	3.3226533	1.0434834	0.18464038	-0.29013861
3.50	4.5138258	0.35126919	3.4305317	1.0599017	0.16738390	-0.29217729
3.55	4.7813154	0.40659439	3.5514484	1.0795002	0.14994861	-0.29497577
3.60	5.0628143	0.46444835	3.6839515	1.1025943	0.13223682	-0.29859814
3.65	5.3603312	0.52527923	3.8388748	1.1295663	0.11413838	-0.30312318
3.70	5.6762473	0.58361402	4.0035109	1.1608819	0.095527103	-0.30864784
3.75	6.0134111	0.65807935	4.2015071	1.1971093	0.076256241	-0.31529142
3.80	6.3752680	0.73142882	4.4181912	1.2383473	0.056152510	-0.32320147
3.85	6.7660285	0.81057985	4.6636401	1.2872615	0.035008156	-0.33256171
3.90	7.1909000	0.83666344	4.9429325	1.3431337	0.012570122	-0.34350249
3.95	7.6534067	0.93103297	5.2624673	1.4079306	-0.011474804	-0.35661565
4.00	8.1708440	1.0356620	5.6304144	1.4834011	-0.037520928	-0.37197553
4.05	8.7449261	1.2126839	6.0573375	1.5718162	-0.066073975	-0.39016889
4.10	9.3927486	1.3451988	6.5573030	1.6761756	-0.097795498	-0.41183939
4.15	10.133247	1.4972876	7.1490365	1.8005210	-0.13357053	-0.43785519
4.20	10.992479	1.6745329	7.8585070	1.9504276	-0.17461323	-0.46941443
4.25	12.007332	1.8819674	8.7224815	2.1338018	-0.22263872	-0.50821706
4.30	13.231825	2.1401288	9.7948672	2.3622365	-0.28015472	-0.55675633
4.35	14.748369	2.4577810	11.157947	2.6534346	-0.35098311	-0.61884049
4.40	16.683280	2.8564081	12.943920	3.0358381	-0.44125607	-0.70058860
4.45	19.281327	3.4147996	15.379431	3.5582132	-0.56147869	-0.81249323
4.50	22.948385	4.1941698	18.888230	4.3117334	-0.73128201	-0.97417357
4.55	28.585116	5.3970328	24.364850	5.4888998	-0.99209528	-1.22701474
4.60	38.462057	7.5119637	34.079700	7.5782393	-1.4490536	-1.6762390
4.65	60.523005	12.248308	55.976449	12.289074	-2.4699772	-2.6835585
4.70	155.59776	32.695410	150.88481	32.710735	-6.8714337	-7.0835648
4.75	-225.02909	-49.219235	-229.91134	-49.229441	10.753402	10.548600
4.80	-61.396776	-14.020608	-66.450307	-14.056239	3.1779206	2.9802902
4.85	-34.059290	-8.1502296	-39.286970	-8.2113571	1.9133863	1.7228117
4.90	-22.725861	-5.7239513	-28.130598	-5.8106533	1.3900493	1.2064173
4.95	-16.481235	-4.3930334	-22.066101	-4.5054171	1.1025062	0.92571233
5.00	-12.496688	-3.5487943	-18.264929	-3.6870002	0.91977916	0.74972746
5.05	-9.7116065	-2.9630178	-15.666670	-3.1272185	0.79275886	0.62936209
5.10	-7.6386826	-2.5308501	-13.784219	-2.7212531	0.69888149	0.54206092
5.15	-6.0226237	-2.1973642	-12.362504	-2.4142125	0.62632671	0.47601206
5.20	-4.7167274	-1.9309995	-11.255054	-2.1745734	0.56830217	0.42443174
5.25	-3.6306614	-1.7123271	-10.371787	-1.9829455	0.52062636	0.38314699
5.30	-2.7057378	-1.5287277	-9.6542818	-1.8267501	0.48058326	0.34945035
5.35	-1.9021462	-1.3716448	-9.0630102	-1.6974734	0.44633122	0.32150890
5.40	-1.1919046	-1.2350689	-8.5702990	-1.5891510	0.41657697	0.29803817
5.45	-0.55474754	-1.1146519	-8.1562117	-1.4974826	0.39038488	0.27811171
5.50	0.024384902	-1.0071671	-7.8060455	-1.4192922	0.36706092	0.26104467
5.55	0.55693934	-0.91016667	-7.5087371	-1.3521862	0.34607888	0.24632052
5.60	1.0517941	-0.82175719	-7.2558254	-1.2943295	0.32703209	0.23351257
5.65	1.5159627	-0.74044831	-7.040750	-1.2442933	0.30600014	0.22240131
5.70	1.9550820	-0.66504754	-6.8583705	-1.2009545	0.29353013	0.21265334
5.75	2.3737570	-0.59458687	-6.7046173	-1.1634160	0.27861306	0.20410266
5.80	2.7758103	-0.52826890	-6.5762575	-1.1309603	0.26468020	0.19659224
5.85	3.1644651	-0.46542791	-6.4707154	-1.1030078	0.25159068	0.18999432
5.90	3.5424836	-0.40550028	-6.3859393	-1.0790891	0.23922606	0.18420443
5.95	3.9122729	-0.34800233	-6.3203072	-1.0588241	0.22748555	0.17913679
6.00	4.2759668	-0.29751292	-6.2725524	-1.0419052	0.21628237	0.17472097

α	F_a	F_b	F_c	F_d	F_e	F_f
6.05	4.6354902	-0.23865996	-6.2417118	-1.0280862	0.20554081	0.17089900
6.10	4.9926155	-0.18610943	-6.2270837	-1.0171722	0.19519414	0.16762346
6.15	5.3490048	-0.13455865	-6.2282003	-1.0090134	0.18518249	0.16485591
6.20	5.7062485	-0.083718018	-6.2448068	-1.0034995	0.17545163	0.16256556
6.25	6.0659022	-0.03325995	-6.2768513	-1.0005565	0.16595149	0.16072858
6.30	6.4295167	0.016878154	-6.3244761	-1.0001437	0.15663519	0.15932718
6.35	6.7986713	0.067148992	-6.3880206	-1.0022533	0.14745797	0.15834935
6.40	7.1750054	0.11774225	-6.4680262	-1.0069091	0.13837637	0.15778853
6.45	7.5602501	0.16892007	-6.5652492	-1.0141680	0.12934728	0.15764344
6.50	7.9562693	0.22095652	-6.6806825	-1.0241214	0.12032714	0.15791823
6.55	8.3651006	0.27414351	-6.8155309	-1.0358981	0.11127102	0.15862268
6.60	8.7890035	0.32873822	-6.9715001	-1.0526687	0.10213157	0.15977253
6.65	9.2305205	0.38527003	-7.1503453	-1.0716511	0.092857976	0.16139033
6.70	9.6925503	0.44395133	-7.3544359	-1.0941189	0.083394611	0.16350613
6.75	10.178433	0.50528880	-7.5865876	-1.1104108	0.073679454	0.16615888
6.80	10.692090	0.56379350	-7.8502233	-1.1509449	0.063642116	0.16939795
6.85	11.233125	0.63303456	-8.1495128	-1.1862363	0.053201369	0.17328552
6.90	11.82.054	0.71036620	-8.4895661	-1.2269210	0.042261939	0.17789945
6.95	12.450589	0.78900319	-8.8766804	-1.2737879	0.030710256	0.18333736
7.00	13.131892	0.87355979	-9.3186898	-1.3278222	0.018408871	0.18972214
7.05	13.876142	0.96583227	-9.8254285	-1.3902654	0.005185365	0.19720946
7.10	14.696159	1.0674651	-10.409400	-1.4626988	-0.009162149	0.20599853
7.15	15.608350	1.1805546	-11.086710	-1.5471640	-0.024912758	0.21634720
7.20	16.634092	1.3078296	-11.878453	-1.6453373	-0.042412707	0.22859438
7.25	17.801791	1.4529118	-12.812764	-1.7637920	-0.062125097	0.24319316
7.30	19.150074	1.6207223	-13.928005	-1.9044033	-0.084678069	0.26076250
7.35	20.732854	1.8181274	-15.277840	-2.0749940	-0.11094709	0.28216929
7.40	22.627776	2.0550113	-16.939666	-2.2854075	-0.14219164	0.30866481
7.45	24.950933	2.3461504	-19.029338	-2.5503303	-0.18029336	0.34212309
7.50	27.884081	2.7146749	-21.728380	-2.8933066	-0.22819679	0.38548151
7.55	31.728786	3.1989593	-25.333125	-3.3516228	-0.29078592	0.44361733
7.60	37.024704	3.8676708	-30.398000	-3.9948926	-0.37679918	0.52526242
7.65	44.842533	4.8570450	-37.978474	-4.9589275	-0.50357188	0.64774603
7.70	57.650533	6.4811394	-50.547585	-6.5578433	-0.71106530	0.85102357
7.75	82.699987	9.6625929	-75.356395	-9.7142160	-1.1166745	1.2524848
7.80	154.49973	18.792230	-146.91357	-18.818847	-2.2790787	2.4108045
7.85	2461.2403	312.29031	-2453.4134	-312.29243	-39.623821	99.751577
7.90	-161.05346	-21.370644	169.13247	21.394063	2.8297199	-2.7060103
7.95	-74.047289	-10.306818	82.376310	10.355234	1.4213229	-1.3015421
8.00	-46.184728	-6.7684995	54.766845	6.8419834	0.97048884	-0.85459660
8.05	-32.384132	-5.0196066	41.222612	5.1182552	0.74737153	-0.63533098
8.10	-24.097361	-3.9724954	33.195744	4.0964346	0.61358141	-0.50536009
8.15	-18.536464	-3.2724362	27.898563	3.4218238	0.52398295	-0.41955303
8.20	-14.521228	-2.7692638	24.151161	2.9442914	0.45946915	-0.35880746
8.25	-11.466228	-2.3884925	21.368422	2.5393359	0.41056242	-0.31365038
8.30	-9.0480093	-2.0889705	19.227223	2.3159913	0.37202646	-0.27885016
8.35	-7.0732557	-1.8461037	17.534593	2.0995510	0.34073191	-0.25128211
8.40	-5.4192781	-1.6442866	16.168216	1.9244977	0.31469301	-0.22896531
8.45	-4.0044278	-1.4731350	15.046836	1.7804881	0.29258860	-0.21058346
8.50	-2.7722237	-1.3254683	14.114394	1.6603843	0.27350504	-0.19522801
8.55	-1.6823213	-1.1961592	13.330990	1.5591041	0.25679014	-0.18225192
8.60	-0.70509706	-1.0814457	12.667485	1.4729333	0.24196552	-0.17118221
8.65	0.18171581	-0.97850237	12.102129	1.3990976	0.22867197	-0.16166528
8.70	0.99521084	-0.88516298	11.618386	1.3354849	0.21663427	-0.15343183
8.75	1.7487580	-0.79973612	11.203490	1.2804623	0.20563775	-0.14627337
8.80	2.4529950	-0.72087891	10.847458	1.2327498	0.19551220	-0.14002625
8.85	3.1165218	-0.64750902	10.542399	1.1913325	0.18612032	-0.13456046
8.90	3.7463985	-0.5784117	10.282027	1.1553987	0.17735125	-0.12977171
8.95	4.3485080	-0.51384150	10.061305	1.1242944	0.16711170	-0.12557559
9.00	4.9278235	-0.45219318	9.8761887	1.0974895	0.16132470	-0.12190345

α	F_a	F_b	F_c	F_d	F_e	F_f
9.05	5.4886148	-0.39327097	9.7234250	1.0745540	0.15392472	-0.11869911
9.10	6.0346027	-0.33662122	9.6004154	1.0551389	0.14685556	-0.11591653
9.15	6.5690829	-0.28184672	9.5051051	1.0389618	0.14003842	-0.11351798
9.20	7.0950249	-0.22859435	9.4358981	1.0257969	0.13352033	-0.11147268
9.25	7.6151516	-0.17654521	9.3916009	1.0154665	0.12717298	-0.10975571
9.30	8.1320057	-0.12540651	9.3713756	1.0078346	0.12099163	-0.10834719
9.35	8.6480116	-0.074904611	9.3747074	1.0028034	0.11494440	-0.10723171
9.40	9.1655222	-0.024778952	9.4013865	1.0003089	0.10900143	-0.10639780
9.45	9.6868737	0.025223398	9.4514997	1.0033200	0.10313431	-0.10533774
9.50	10.214427	0.075352903	9.5254220	1.0028370	0.097315474	-0.10554722
9.55	10.750617	0.12586260	9.6238339	1.0078916	0.091517684	-0.10552535
9.60	11.298007	0.17701324	9.7477306	1.0155480	0.085713463	-0.10577465
9.65	11.859335	0.22907874	9.8984566	1.0259051	0.079874515	-0.10630112
9.70	12.437583	0.28235237	10.390992	1.0390992	0.073971162	-0.10711456
9.75	13.036046	0.33715351	10.287742	1.0553089	0.067971635	-0.10822889
9.80	13.658426	0.39333606	10.531144	1.0747611	0.061841300	-0.10966265
9.85	14.308922	0.45279821	10.811220	1.0977390	0.055541720	-0.11143972
9.90	14.992381	0.51449480	11.131969	1.1245933	0.049029528	-0.11359035
9.95	15.714457	0.57945271	11.498263	1.1557556	0.042254974	-0.11615230
10.00	16.481824	0.64829069	11.916081	1.1916081	0.035160117	-0.11917259
10.05	17.302470	0.72174524	12.392737	1.2332569	0.027676448	-0.12270973
10.10	18.186057	0.80070496	12.937292	1.2810680	0.019721757	-0.12683679
10.15	19.144449	0.88625699	13.561042	1.3362106	0.011195918	-0.13164549
10.20	20.192388	0.97975003	14.278201	1.3999703	0.019750833	-0.13725189
10.25	21.348493	1.0828840	15.106889	1.4739901	0.0080964138	-0.14380449
10.30	22.636633	1.1978333	16.070511	1.5603917	-0.019217606	-0.15149563
10.35	24.087998	1.3274491	17.199812	1.6619666	-0.031647869	-0.16057850
10.40	25.744153	1.4755078	18.535938	1.7824524	-0.045732334	-0.17139244
10.45	27.661813	1.6471749	20.135201	1.9269668	-0.061941290	-0.18440233
10.50	29.920484	1.8496857	22.076724	2.1027021	-0.080933476	-0.20026180
10.55	32.635265	2.0935116	24.475232	2.3200892	-0.10366191	-0.21991910
10.60	35.979333	2.3944060	27.503540	2.5948427	-0.13155993	-0.24480305
10.65	40.225966	2.7772259	31.434580	2.9517829	-0.16688881	-0.27717062
10.70	45.833210	3.2836325	36.726053	3.4325345	-0.21343807	-0.32080725
10.75	53.631947	3.9891947	44.208502	4.1126328	-0.27808104	-0.38258242
10.80	65.302404	5.0467289	55.561827	5.1448603	-0.37471657	-0.47639125
10.85	84.832459	6.8189272	74.773581	6.8918776	-0.53633131	-0.63521690
10.90	124.55176	10.427153	114.17311	10.475018	-0.86491193	-0.96104275
10.95	250.85732	21.910115	240.15723	21.932972	-1.9096699	-2.0030782
11.00	-2493.2787	-227.66930	-2504.3088	-227.67201	20.788809	20.698150
11.05	-191.89062	-18.366252	-203.24204	-18.393497	1.7526549	1.6646255
11.10	-94.836505	-9.5441194	-106.51702	-9.5963863	0.94994685	0.86456803
11.15	-60.489448	-6.4252496	-72.502437	-6.5026166	0.66595844	0.58321346
11.20	-42.828072	-4.8240698	-55.177172	-4.9266378	0.52001826	0.4339260
11.25	-32.012298	-3.8456410	-44.701502	-3.9735408	0.43073283	0.35321484
11.30	-24.667290	-3.1830286	-37.700975	-3.3364235	0.37018683	0.29526788
11.35	-19.322670	-2.7025057	-32.705620	-2.8815925	0.32621794	0.25389248
11.40	-15.235146	-2.3364732	-28.972568	-2.5414832	0.29267808	0.22294367
11.45	-11.988471	-2.0470766	-26.086017	-2.2782771	0.26612446	0.19898184
11.50	-9.3313467	-1.8114634	-23.795141	-2.0691591	0.24447872	0.17993184
11.55	-7.1030254	-1.6150172	-21.939687	-1.8995519	0.22641159	0.16446773
11.60	-5.1958494	-1.4479503	-20.412526	-1.7597037	0.21103298	0.15170290
11.65	-3.5349646	-1.3034591	-19.139362	-1.6428690	0.19772427	0.14102224
11.70	-2.0666659	-1.1766637	-18.067087	-1.5441985	0.18604183	0.13198586
11.75	-0.75138181	-1.0639704	-17.156772	-1.4601518	0.17565902	0.12427101
11.80	0.44072683	-0.9267110	-16.379260	-1.3880724	0.16632982	0.11763575
11.85	1.5326254	-0.87068343	-15.7212326	-1.3259329	0.15786517	0.11189533
11.90	2.5423162	-0.78637725	-15.138766	-1.2721623	0.15011719	0.10690644
11.95	3.4841494	-0.70845522	-14.645091	-1.2255269	0.14296829	0.10255639
12.00	4.3697444	-0.63586921	-14.220618	-1.1850471	0.13632365	0.098755600

Table 4 The frequency-characteristics of functions $F'_a \sim F'_f$

α	F'_a	F'_b	F'_c	F'_d	F'_e	F'_f
0.00	12.000000	6.0000000	-12.000000	6.0000000	4.0000000	2.0000000
0.15	11.992985	5.9965540	-11.993238	5.9966183	3.9982484	1.9983513
0.20	11.994288	5.9973516	-11.995088	5.9974968	3.9985582	1.9988818
0.25	11.995391	5.9982103	-11.997344	5.9985452	3.9990898	1.9993203
0.30	11.995274	5.9987109	-11.999324	5.9993963	3.9994779	1.9996412
0.35	11.993003	5.9984966	-12.000507	5.9997528	3.9994682	1.9997782
0.40	11.989488	5.9981517	-12.002289	6.0002907	3.9994838	1.9999482
0.45	11.983992	5.9974581	-12.004497	6.0008799	3.9993961	2.0001109
0.50	11.976168	5.9964100	-12.007421	6.0016245	3.9992335	2.0003015
0.55	11.965548	5.9949698	-12.011307	6.0026013	3.9990040	2.0005408
0.60	11.951442	5.9929974	-12.016255	6.0038028	3.9986492	2.0008266
0.65	11.933359	5.9904779	-12.022635	6.0053618	3.9982082	2.0011948
0.70	11.910531	5.9872796	-12.030621	6.0073000	3.9976377	2.0016488
0.75	11.882208	5.9832968	-12.040482	6.0096825	3.9969192	2.0022028
0.80	11.847603	5.9784769	-12.052520	6.0125882	3.9960394	2.0028775
0.85	11.805845	5.9725465	-12.067038	6.0160911	3.9949765	2.0036888
0.90	11.756004	5.9655226	-12.084354	6.0202645	3.9937021	2.0046551
0.95	11.697097	5.9572145	-12.104812	6.0251919	3.9921936	2.0057935
1.00	11.628095	5.9474816	-12.128791	6.0309669	3.9904264	2.0071283
1.05	11.547886	5.9361620	-12.156669	6.0376779	3.9883691	2.0086778
1.10	11.455321	5.9230963	-12.188865	6.0454275	3.9859943	2.0104675
1.15	11.349174	5.9081070	-12.225809	6.0543177	3.9832691	2.0125193
1.20	11.228163	5.8910135	-12.267962	6.0644585	3.9801594	2.0148595
1.25	11.090948	5.8716234	-12.315811	6.0759683	3.9766314	2.0175118
1.30	10.936118	5.8497351	-12.369866	6.0889672	3.9726478	2.0205133
1.35	10.762201	5.8251374	-12.430666	6.1035857	3.9681697	2.0238846
1.40	10.567654	5.7976079	-12.498781	6.1193577	3.9631565	2.0276598
1.45	10.350870	5.7669157	-12.574813	6.1382286	3.9575656	2.0318722
1.50	10.110168	5.7328171	-12.659397	6.1585479	3.9513521	2.0365560
1.55	9.8437928	5.6950559	-12.753200	6.1810756	3.9444689	2.0417476
1.60	9.5499105	5.6533650	-12.856930	6.2059776	3.9368663	2.0474853
1.65	9.2266148	5.6074649	-12.971338	6.2334335	3.9284929	2.0539099
1.70	8.8719087	5.5570614	-13.097217	6.2636297	3.9192937	2.0607639
1.75	8.4837106	5.5018456	-13.235407	6.2967640	3.9092111	2.0683927
1.80	8.0598486	5.4414945	-13.386803	6.3330472	3.8981853	2.0767441
1.85	7.5980527	5.3756667	-13.552354	6.3727015	3.8861512	2.0858685
1.90	7.0959510	5.3040051	-13.733073	6.4159646	3.8730434	2.0958199
1.95	6.5510634	5.2261328	-13.930037	6.4630876	3.8587892	2.1066554
2.00	5.9607944	5.1416524	-14.144400	6.5143396	3.8433146	2.1184356
2.05	5.3224251	5.0501443	-14.377399	6.5700074	3.8265394	2.1312256
2.10	4.6331045	4.9511652	-14.630355	6.6303971	3.8083792	2.1450939
2.15	3.8898393	4.8442455	-14.904694	6.6958387	3.7887442	2.1601156
2.20	3.0894844	4.7288881	-15.201946	6.7666845	3.7675391	2.1763691
2.25	2.2287280	4.6045633	-15.523764	6.8433139	3.7446619	2.1939401
2.30	1.3040788	4.4707086	-15.871936	6.9261367	3.7200034	2.2129203
2.35	0.31185290	4.3267232	-16.248394	7.0155957	3.6934470	2.2334082
2.40	-0.75185338	4.1719670	-16.655246	7.1121692	3.6648682	2.2555116
2.45	-1.8911690	4.0057519	-17.094775	7.2163772	3.6341316	2.2793455
2.50	-3.1104869	3.8273410	-17.569475	7.3287844	3.6010923	2.3050361
2.55	-4.4144762	3.6359409	-18.082073	7.4500085	3.5655953	2.3327200
2.60	-5.8081242	3.4306962	-18.635554	7.5807208	3.5274652	2.3625466
2.65	-7.2967616	3.2106817	-19.233198	7.7216608	3.4865234	2.3946793
2.70	-8.8861083	2.9748943	-19.878620	7.8736367	3.4425667	2.4292971
2.75	-10.582312	2.7222435	-20.575803	8.0375422	3.3953760	2.4665969
2.80	-12.392005	2.4515339	-21.329161	8.2143608	3.3447115	2.5067956
2.85	-14.322358	2.1614819	-22.143579	8.4051817	3.2903088	2.5501325
2.90	-16.381167	1.8506430	-23.024514	8.6112168	3.2318775	2.5968741
2.95	-18.576916	1.5174517	-23.978040	8.8338138	3.1690974	2.6473163
3.00	-20.918875	1.1601716	-25.010956	9.0744741	3.1016109	2.7017879

α	F_a'	F_b'	F_c'	F_d'	F_e'	F_f'
3.05	-23.417227	0.77687830	-26.130904	9.3348894	3.0290234	2.7606586
3.10	-26.083185	-0.36542967	-27.346481	9.6169509	2.9503914	2.8243425
3.15	-28.929165	-0.076567415	-28.667417	9.9228007	2.8667188	2.8933065
3.20	-31.968967	-0.55179814	-30.104748	10.254862	2.7759479	2.9680794
3.25	-35.218006	-1.0632885	-31.671042	10.615891	2.6779472	3.0492609
3.30	-38.693573	-1.6144642	-33.380667	11.009030	2.5720001	3.1375352
3.35	-42.415203	-2.2092229	-35.250140	11.437891	2.4572887	3.2336867
3.40	-46.405054	-2.8520219	-37.298512	11.906629	2.3328766	3.3386185
3.45	-50.688413	-3.5479869	-39.547881	12.420061	2.1976821	3.4533748
3.50	-55.294366	-4.3030476	-42.024013	12.983796	2.0504528	3.5791718
3.55	-60.256527	-5.1241058	-44.757128	13.604401	1.8897274	3.7174321
3.60	-65.614073	-6.0192506	-47.782891	14.289622	1.7137892	3.8698319
3.65	-71.413012	-6.9980325	-51.143676	15.048647	1.5206086	4.0383586
3.70	-77.707826	-8.0718159	-54.890204	15.892473	1.3077660	4.2253889
3.75	-84.563594	-9.2542409	-59.083694	16.834350	1.0723534	4.4337856
3.80	-92.058870	-10.561832	-63.798681	17.890399	0.81084224	4.6670292
3.85	-100.28946	-12.014820	-69.126805	19.080434	0.51890839	4.9293959
3.90	-109.37359	-13.638251	-75.182003	20.429064	0.19119156	5.2261939
3.95	-119.45909	-15.463528	-82.107646	21.967237	-0.17903563	5.5640957
4.00	-130.73350	-17.530592	-90.086630	23.734418	-0.60033485	5.9516085
4.05	-143.43865	-19.891048	-99.355970	25.781715	-1.0837784	6.3997452
4.10	-157.89210	-22.612792	-110.22826	28.176512	-1.6439423	6.9230201
4.15	-174.51985	-25.787036	-123.12428	31.009473	-2.3004185	7.5409610
4.20	-193.90733	-29.539290	-138.62406	34.405543	-3.0801774	8.2804705
4.25	-216.88243	-34.047224	-157.54988	38.541795	-4.0214119	9.1796706
4.30	-244.65644	-39.570982	-181.10709	43.677753	-5.1800608	10.294425
4.35	-279.07601	-46.507418	-211.13625	50.209616	-6.6414779	11.710009
4.40	-323.10446	-55.493661	-250.59429	58.773826	-8.5427175	13.563395
4.45	-381.81848	-67.621569	-304.55118	70.461517	-11.118682	16.089397
4.50	-464.70480	-84.931938	-382.48666	87.312723	-14.808461	19.727015
4.55	-591.78336	-111.73207	-504.41331	113.63395	-20.538853	25.402949
4.60	-813.85713	-158.95315	-721.12645	160.35554	-30.661974	35.469217
4.65	-1308.6587	-264.83904	-1210.3508	265.72050	-53.407082	58.154979
4.70	-3437.1545	-722.24161	-3333.0455	722.58014	-151.78997	156.47595
4.75	5077.2188	1110.5090	5187.3746	-1110.7393	242.62363	-238.00279
4.80	1414.5817	323.03481	1531.0151	-323.85575	73.219291	-68.665886
4.85	801.15965	191.71378	924.12775	-193.15165	45.007629	-40.524838
4.90	545.64792	137.43207	675.41566	-139.51379	33.375084	-28.966079
4.95	403.83146	107.64030	540.67464	-110.39398	27.014158	-22.682266
5.00	312.41720	88.719858	456.62323	-92.175005	22.994479	-18.743187
5.05	247.67024	75.564361	399.53925	-79.751890	20.217333	-16.050307
5.10	198.68213	65.827411	358.52754	-70.779793	18.177908	-14.099005
5.15	159.73504	58.279592	327.88451	-64.030951	16.611750	-12.625030
5.20	127.54031	52.214226	304.33666	-58.800465	15.366891	-11.476634
5.25	100.07010	47.196016	285.87238	-54.654935	14.349764	-10.560489
5.30	76.004175	42.941961	271.18878	-51.313410	13.499584	-9.8160603
5.35	54.444180	39.259903	259.40601	-48.585932	12.775115	-9.2023885
5.40	34.755938	36.014609	249.90992	-46.339643	12.147384	-8.6907930
5.45	16.477389	33.107948	242.25988	-44.478977	11.595407	-8.2606131
5.50	-0.73764329	30.466805	236.13288	-42.933589	11.103593	-7.8966013
5.55	-17.155124	28.035409	231.28787	-41.650715	10.660095	-7.5872878
5.60	-32.984263	25.770305	227.54268	-40.590173	10.255726	-7.3238950
5.65	-48.393319	23.636961	224.75841	-39.720972	9.8832360	-7.0996154
5.70	-63.520614	21.607395	222.82846	-39.019012	9.5367939	-6.9091070
5.75	-78.482341	19.658528	221.67141	-38.465442	9.2116443	-6.7481442
5.80	-93.378258	17.770966	221.22530	-38.045504	8.9038419	-6.6133630
5.85	-108.29591	15.928107	221.44406	-37.747684	8.6100620	-6.5020806
5.90	-123.31385	14.115465	222.29455	-37.563092	8.3274591	-6.4121562
5.95	-138.50424	12.320152	223.75468	-37.485020	8.0535572	-6.3418902
6.00	-153.93480	10.530465	225.81189	-37.508587	7.7861653	-6.2899549

α	F_a'	F_b'	F_c'	F_d'	F_e'	F_f'
6.05	-169.67053	8.7355512	228.46226	-37.630525	7.5233075	-6.2553306
6.10	-185.77522	6.9251319	231.70978	-37.848978	7.2631739	-6.2372689
6.15	-202.31273	5.0892636	235.56611	-38.163409	7.0040647	-6.2352627
6.20	-219.34819	3.2181206	240.05037	-38.574521	6.7443607	-6.2490201
6.25	-236.94930	1.3017967	245.18950	-39.084238	6.4824801	-6.2784602
6.30	-255.18752	-0.66989393	251.01846	-39.695703	6.2168507	-6.3236958
6.35	-274.13942	-2.7076152	257.58096	-40.413359	5.9458740	-6.3850417
6.40	-293.88822	-4.8227226	264.93035	-41.242997	5.6678961	-6.4630182
6.45	-314.52530	-7.0274972	273.13078	-42.191924	5.3811702	-6.5583612
6.50	-336.15240	-9.3354130	282.25884	-43.269129	5.0838217	-6.6720452
6.55	-358.88373	-11.761446	292.40546	-44.485521	4.7738049	-6.8053095
6.60	-382.84899	-14.322450	303.67854	-45.854249	4.4488512	-6.9596914
6.65	-408.19669	-17.037604	316.20615	-47.391091	4.1064118	-7.1370839
6.70	-435.09858	-19.928975	330.14063	-49.114997	3.7435841	-7.3397902
6.75	-463.75508	-23.022221	345.66390	-51.048717	3.3570201	-7.5706140
6.80	-494.40224	-26.347483	362.99433	-53.219692	2.9428114	-7.8329612
6.85	-527.32092	-29.940542	382.39551	-55.661173	2.4963412	-8.1309898
6.90	-562.84847	-33.844340	404.18824	-58.413709	2.0120909	-8.4697928
6.95	-601.39458	-38.110971	428.76586	-61.527140	1.4833821	-8.8556528
7.00	-643.464271	-42.804430	456.61580	-65.063288	0.90203468	-9.2963849
7.05	-689.67895	-48.004278	488.34836	-69.099666	0.25788572	-9.8018032
7.10	-740.83338	-53.810916	524.73785	-73.734647	-0.46186395	-10.384386
7.15	-797.93787	-60.352903	566.78033	-79.094892	-1.2736025	-11.060210
7.20	-862.31133	-67.797886	615.77900	-85.346126	-2.1986747	-11.850333
7.25	-935.70664	-76.368676	673.47091	-92.709317	-3.2654504	-12.782840
7.30	-1020.5074	-86.368291	742.22339	-101.48565	-4.5124943	-13.896034
7.35	-1120.0406	-98.219787	825.34711	-112.09636	-5.9936392	-15.243490
7.40	-1239.0970	-112.53242	927.61611	-125.14891	-7.7864142	-16.902485
7.45	-1384.8392	-130.21721	1056.1758	-141.55251	-10.066732	-18.988687
7.50	-1568.4796	-152.70046	1222.2214	-162.73162	-12.836069	-21.683335
7.55	-1808.6201	-182.34868	1444.3365	-191.05088	-16.575524	-25.287297
7.60	-2138.5469	-223.39667	1755.7885	-230.74326	-21.763921	-30.339157
7.65	-2624.2971	-284.24642	2222.5952	-290.20883	-29.470285	-37.907717
7.70	-3418.1001	-384.26676	2996.9663	-388.81453	-42.159062	-50.457187
7.75	-4967.1680	-580.35949	4526.0935	-583.46010	-67.070262	-75.227368
7.80	-9399.7636	-1143.33193	8938.2216	-1144.9387	-138.65915	-146.67335
7.85	-151667.80	-19244.110	151185.47	-19244.240	-2441.7189	-2449.5916
7.90	10051.346	1333.7419	-10555.557	1335.2035	176.660282	168.88210
7.95	4679.9738	651.41666	-5206.3887	654.47668	89.831161	82.260715
8.00	2955.8226	433.18397	-3505.0781	437.88694	62.111286	54.694182
8.05	2098.5727	325.28306	-2671.3283	331.67573	48.431544	41.171036
8.10	1581.0279	260.63542	-2177.9728	268.76707	40.257076	33.156676
8.15	1231.2383	217.36339	-1853.0923	227.28609	34.804257	27.867761
8.20	976.40737	186.20530	-1623.9241	197.97415	30.894706	24.126214
8.25	780.42014	162.56677	-1454.3882	176.24008	27.943905	21.347829
8.30	623.31736	143.90918	-1324.5634	159.54864	25.628903	19.209988
8.35	493.16507	128.71497	-1222.5557	146.38594	23.756681	17.520017
8.40	382.38426	116.02086	-1140.8293	135.79256	22.204739	16.155792
8.45	285.92616	105.18552	-1074.3817	127.13130	20.891558	15.036186
8.50	200.29316	95.765085	-1019.7650	119.96277	19.760739	14.105224
8.55	122.98189	87.442228	-974.52870	113.97441	18.772001	13.323071
8.60	52.148979	79.983724	-936.88719	108.93815	17.895770	12.660636
8.65	-13.596431	73.213994	-905.51155	104.68398	17.109808	12.096200
8.70	-75.327508	66.997986	-879.39564	101.08285	16.397048	11.613255
8.75	-133.88928	61.229797	-857.76720	98.035395	15.744140	11.199055
8.80	-189.95993	55.824863	-840.02715	95.464145	15.140465	10.843633
8.85	-244.09378	50.714525	-825.70705	93.308140	14.577432	10.539112
8.90	-296.75223	45.842088	-814.43936	91.519131	14.047993	10.279217
8.95	-348.32636	41.159989	-805.93568	90.058792	13.546270	10.058919
9.00	-399.15370	36.627648	-799.97128	88.896650	13.067301	9.8741795

α	F_a'	F_b'	F_c'	F_d'	F_e'	F_f'
9.05	-449.53127	32.209876	-796.37282	88.008659	12.606819	9.7217539
9.10	-499.72545	27.875603	-975.01040	87.376052	12.161109	9.5990478
9.15	-549.98004	23.596912	-795.79116	86.984479	11.726878	9.5040091
9.20	-600.52291	19.348226	-798.65442	86.823450	11.301161	9.4350476
9.25	-651.57141	15.105650	-803.56885	86.885852	10.881238	9.3909729
9.30	-703.33717	10.846409	-810.53028	87.167615	10.464566	9.3709485
9.35	-756.03079	6.5483484	-819.56036	87.667580	10.048727	9.3744642
9.40	-809.86554	2.1894682	-830.70651	88.387294	9.6313664	9.4013096
9.45	-865.06204	-2.2525125	-844.04255	89.331077	9.2101517	9.4515748
9.50	-921.85204	-6.8005995	-859.66934	90.506039	8.7827215	9.5256366
9.55	-980.48315	-11.478984	-877.71771	91.922234	8.3466416	9.6241757
9.60	-1041.2243	-16.313540	-898.35035	93.592904	7.8993528	9.7481917
9.65	-1104.3709	-21.332385	-921.76902	95.534848	7.4381145	9.8990260
9.70	-1170.2522	-26.566534	-948.21437	97.768844	6.9599466	10.078409
9.75	-1239.2391	-32.050656	-977.97847	100.32030	6.4615536	10.288509
9.80	-1311.7552	-37.824015	-1011.4111	103.22006	5.9392385	10.532001
9.85	-1388.2874	-43.931614	-1048.9316	106.50538	5.3887965	10.812160
9.90	-1469.4033	-50.425635	-1091.0443	110.22139	4.8053840	11.132990
9.95	-1555.7705	-57.367267	-1138.3574	114.42269	4.1833481	11.499368
10.00	-1648.1824	-64.829035	-1191.6081	119.17578	3.5160117	11.917259
10.05	-1747.5927	-72.893074	-1251.6974	124.56203	2.7953904	12.393990
10.10	-1855.1597	-81.679913	-1319.7332	130.68175	2.0118164	12.938621
10.15	-1972.3090	-91.304411	-1397.0924	137.65976	1.1534315	13.562447
10.20	-2100.8160	-101.93319	-1485.5040	145.65296	0.20548767	14.279687
10.25	-2242.9260	-113.77050	-1587.1675	154.86108	-0.85062947	15.108459
10.30	-2401.5204	-127.07845	-1704.9205	165.54196	-2.0387958	16.072171
10.35	-2580.3666	-142.19967	-1842.4869	178.03402	-3.3901988	17.201570
10.40	-2784.4876	-159.59092	-2004.8471	192.79005	-4.9464092	18.537806
10.45	-3020.7391	-179.87562	-2198.8143	210.42959	-6.7641437	20.137195
10.50	-3298.7334	-203.92785	-2433.9588	231.82291	-8.9229157	22.078863
10.55	-3632.3866	-233.01307	-2724.1545	258.23173	-11.537830	24.477546
10.60	-4042.6379	-269.03546	-3090.2978	291.55653	-14.782074	27.506071
10.65	-4562.5296	-314.99990	-3565.3887	334.79860	-18.928946	31.437385
10.70	-5247.4442	-375.94308	-4204.7658	392.99087	-24.436525	36.729222
10.75	-6197.8419	-461.00131	-5108.8450	475.26613	-32.135740	44.212181
10.80	-7616.8724	-588.65046	-6480.7315	600.09651	-43.706941	55.566275
10.85	-9936.6892	-802.74116	-8802.5329	811.32906	-63.138263	74.779322
10.90	-14797.995	-1238.8500	-13564.907	1244.5369	-102.76019	114.18149
10.95	-30078.420	-2627.0776	-28795.452	2629.8182	-228.97420	240.17408
11.00	301686.72	27547.985	303021.36	-27548.313	2515.4459	-2504.4762
11.05	23430.324	2242.5653	24816.361	-2245.8920	214.00354	-203.25494
11.10	11684.806	1175.9310	13123.962	-1182.3708	117.04295	-106.52343
11.15	7520.1994	798.80309	9013.6842	-808.42155	82.793618	-72.506555
11.20	5372.3534	605.13132	6921.4245	-617.99745	65.231091	-55.180128
11.25	4051.5565	486.71394	5657.5338	-502.90126	54.514624	-44.703753
11.30	3149.7663	406.44092	4814.0375	-426.02792	47.269156	-37.702756
11.35	2489.1947	348.14354	4213.2197	-371.21395	42.024211	-32.707064
11.40	1979.9596	303.64806	3765.2749	-330.29116	38.036443	-28.973759
11.45	1571.7185	268.37686	3419.9420	-298.68782	34.889582	-26.087017
11.50	1234.0706	239.56603	3146.9074	-273.64629	32.332311	-23.795986
11.55	947.56135	215.44733	2926.8091	-253.40497	30.203872	-21.940406
11.60	699.15350	194.83619	2746.7095	-236.78640	28.396598	-20.413142
11.65	479.77423	176.90373	2597.0421	-222.97429	26.835632	-19.133831
11.70	282.90590	161.07349	2473.2035	-211.38533	25.467266	-18.067544
11.75	103.73765	146.89441	2368.7068	-201.59221	24.251923	-17.157166
11.80	-61.366804	134.04232	2280.6482	-193.27520	23.159764	-16.379602
11.85	-215.21509	122.26354	2206.3641	-189.19081	22.167822	-15.712622
11.90	-360.01740	111.35388	2143.8007	-180.15090	21.258095	-15.139021
11.95	-497.54524	101.16918	2091.3556	-175.00831	20.416229	-14.645309
12.00	-629.24319	91.565166	2047.7690	-170.64678	19.630606	-14.220806

横振動細棒の等価回路網について

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本文は、均一な横振動細棒について、その基本マトリクスを整理表示し、各種の端条件を与えた場合の等価回路表示を行なって各素子の値を決定し、実用に便利なものにしたものである。

さらに棒の分布質量を無視した場合についても等価回路を与え、材料力学における簡単な諸問題も回路の問題におきかえて容易に取扱われることを示した。

また、振動子がとくに固有の共振状態で使用される場合には規準函数展開法を用いることによってさらに簡単な等価回路表示が得られることを例証した。

これらの成果は単に単位系の解析にとどまるだけでなく、これら単位系を組合わせた複合機械系の解析ならびに合成にも役だつ。

なお、付録に横振動細棒のインピーダンスあるいはアドミタンス・マトリクス表示の誘導・証明について簡単に述べ、さらに規準函数展開法を用いた多端子等価回路考察の原理について略述した。また、上記マトリクスの各要素の周波数特性を電子計算機によって計算した図表も付記し実用に便ならしめた。