学 位 論 文

The maximal ideal cycles over two-dimensional Brieskorn complete intersection singularities

(2次元ブリスコーン完全交叉特異点の極大イデアルサイクル)

September, 2013

Graduate school of Science and Engineering
Yamagata University

Fan-Ning Meng

DOCTORAL THESIS

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Introduction

Let (X, x) be a germ of a normal complex surface singularity and $f : \tilde{X} \longrightarrow X$ a good resolution with exceptional divisor E. It is known that the topology of the singularity is determined by the weighted dual graph Γ_E of E. A divisor on $\tilde{X}$ supported in E is called a cycle. The fundamental cycle Z_E is by definition the smallest one among the cycles $F > 0$ such that $-F$ is nef, i.e., $FE_i \leq 0$ for every irreducible component E_i of E. The fundamental cycle is a topological invariant; in fact, it is determined by Γ_E. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. For a non-zero function $h \in \mathfrak{m}$, let $(h)_E$ denote the exceptional part of the zero divisor $\mathrm{div}_{\tilde{X}}(h)$. Then the smallest one among the cycles $(h)_E,\ h \in \mathfrak{m} \setminus \{x\}$, is called the maximal ideal cycle and denoted by $Z_{\mathfrak{m}}$. This cycle is an analytic invariant and cannot be determined by Γ_E in general. We have $Z_E \leq Z_{\mathfrak{m}}$ by the definition of these cycles. Therefore it is a natural question to ask whether $Z_E = Z_{\mathfrak{m}}$. This equality holds on the minimal resolution for rational singularities ([**2**]), minimally elliptic singularities ([**17**]), weakly elliptic Gorenstein singularities with rational homology sphere link ([**22**]), and for hypersurface $\{z^n = f(x, y)\}$ with certain conditions ([**5**], [**33**]). However, in general, it is difficult to identify the maximal ideal cycle (cf. [**30**], [**23**], [**26**]).

In this thesis, we consider a germ $(X, o) \subset (\mathbb{C}^m, o)$ of an isolated complete intersection singularity of Brieskorn type defined by

$$X = \{(x_i) \in \mathbb{C}^m | q_{j1}x_1^{a_1} + \cdots + q_{jm}x_m^{a_m} = 0, \quad j = 3, \ldots, m\},$$

where $a_i \geq 2$ are integers. Then (X, o) is a normal surface singularity by Serre's criterion for normality. Neumann [**24**] proved that the universal abelian cover of a weighted homogeneous normal surface singularity with rational homology sphere link is a complete intersection surface singularity of this type. It is known that the resolution graph of the minimal good resolution of a weighted homogeneous surface singularity can be recovered from the Seifert invariants of the link. The Seifert invariant of the link of (X, o) is in fact obtained in [**10**, §7] ([**27**] for hypersurface case); however the construction of the good resolution is needed for the computation of the maximal ideal cycle.

In [**13**, §2], Konno and Nagashima constructed a good resolution of the Brieskorn hypersurface singularity $\{x_0^{a_0} + x_1^{a_1} = x_2^{a_2}\}$ with $2 \leq a_0 \leq a_1 \leq a_2$ using a covering method due to Tomaru ([**34**], [**36**]) and Fujiki ([**7**]). We employ their method to construct a good resolution of (X, o) and the aim is to identify the maximal ideal cycle on the minimal good resolution of (X, o). We give concrete descriptions of the maximal ideal cycle and the fundamental cycle, a condition for the coincidence of these cycles, and a condition for the singularity to be a Kodaira singularity; every condition is expressed by the integers $a_1, \dots, a_m$. The thesis is divided into two chapters.

In Chapter 1, we introduce some basic facts about singularities, blowing up, the resolution of normal surface singularities, the fundamental cycle and the maximal ideal cycle. We also introduce the cyclic quotient singularities and their fundamental facts. In the last section, we review the main results of Konno and Nagashima, that is, the concrete descriptions of the fundamental cycle and the maximal ideal cycle over Brieskorn hypersurface singularities.

In Chapter 2, we describe our main results due to [**20**]. In Section 2.1, we give the construction of a partial resolution of (X, o) with cyclic quotient singularities. In Section 2.2, we compute the zero divisors of the pull-back of the coordinate functions $x_1, \dots, x_m$. The main results are as follows:

Theorem (Theorem 2.9 in Section 2.2)**.** *Let*

$$Z^{(i)} = \lambda_0^{(i)} E_0 + \sum_{w=1}^{m} \sum_{\nu=1}^{s_w} \sum_{\xi=1}^{\hat{g}_w} \lambda_{w,\nu,\xi}^{(i)} E_{w,\nu,\xi} \ (1 \leq i \leq m).$$

Then $\lambda_0^{(i)}$ and the sequence $\{\lambda_{w,\nu,\xi}^{(i)}\}$ are determined by the following:

$$\lambda_{w,0,\xi}^{(i)} := \lambda_0^{(i)} := e_{im},$$

$$\lambda_{w,s_w+1,\xi}^{(i)} := \begin{cases} 1 & if\ w = i \\ 0 & if\ w \neq i, \end{cases}$$

$$\lambda_{w,\nu-1,\xi}^{(i)} = \lambda_{w,\nu,\xi}^{(i)} c_{w,\nu} - \lambda_{w,\nu+1,\xi}^{(i)}.$$

The cycle $Z^{(i)}$ is the smallest one among the cycles $Z > 0$ such that $-Z$ is nef and the coefficients of E_0 in Z is e_{im}.

In Section 2.3, we give concrete description of the fundamental cycle, and compute the fundamental genus and the canonical cycle.

Assume that $a_1 \leq \cdots \leq a_m$. Then we have the following main results.

Theorem (Theorem 2.13 in Section 2.3)**.** *Let*

$$Z_E = \theta_0 E_0 + \sum_{w=1}^{m} \sum_{\nu=1}^{s_w} \sum_{\xi=1}^{\hat{g}_w} \theta_{w,\nu,\xi} E_{w,\nu,\xi}$$

be the fundamental cycle. Then θ_0 and the sequence $\{\theta_{w,\nu,\xi}\}$ are determined by the following:

$$\theta_{w,0,\xi} := \theta_0 := \min(e_{mm}, \alpha_1 \cdots \alpha_m),$$

$$\theta_{w,\nu,\xi} = \lceil \theta_{w,\nu-1,\xi} / \epsilon_{w,\nu} \rceil \ (1 \leq \nu \leq s_w).$$

Lemma (Lemma 2.15 in Section 2.3)**.** $Z_E = Z^{(m)}$ *if and only if* $e_{mm} \leq \alpha_1 \cdots \alpha_m$.

In Section 2.4, we identify the maximal ideal cycle and give a condition for the coincidence of the fundamental cycle and the maximal ideal cycle.

We keep the assumption that $a_1 \leq \cdots \leq a_m$. The main result is as follows:

Theorem (Theorem 2.18 in Section 2.4). *We have* $Z^{(m)} \leq \cdots \leq Z^{(1)}$. *Hence* $Z_{\mathfrak{m}} = Z^{(m)}$. *Furthermore, the maximal ideal cycle coincides with the fundamental cycle on the minimal good resolution space and on* $\tilde{X}$ *if and only if* $e_{mm} \leq \alpha_1 \cdots \alpha_m$.

In Section 2.5, we give a condition for the singularity (X, o) to be a Kodaira singularity following Konno and Nagashima. The main result is as follows:

Theorem (Theorem 2.25 in Section 2.5). (X, o) *is a Kodaira singularity if and only if* $d_{m-1} \leq a_m$.

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Chapter 1

Preliminaries

In this chapter, we mainly introduce some basic facts about singularities, blowing up which is a useful tool for removing the singularities. We also introduce the cyclic quotient singularities and their fundamental facts. At last, we review the main results of Konno and Nagashima, i.e., the concrete descriptions of the fundamental cycle and the maximal ideal cycle over the Brieskorn hypersurface singularities $(V_{a_0,a_1,a_2}, o) := (\{x_0^{a_0} + x_1^{a_1} = x_2^{a_2}\}, o)$, where a_i's are integers and $2 \leq a_0 \leq a_1 \leq a_2$.

1.1. Singularities

By a complex variety we mean an irreducible reduced complex analytic space defined over $\mathbb{C}$. Let $X = (X, \mathcal{O}_X)$ be a complex analytic space. Let x be a point of X. We denote by $\dim_x X$ the dimension of X at x, and denote by $\dim X$ the global dimension of X. There exists the smallest positive integer e such that a neighborhood U of x is biholomorphic to a closed complex subspace of a domain in $\mathbb{C}^e$. This integer is called the embedding dimension of X at x, and denoted by $\mathrm{embdim}_x X$. It is clear that for any point $x \in X$, there exists an open neighborhood U such that $\mathrm{embdim}_x X \geq \mathrm{embdim}_y Y$ for any $y \in U$. Hence the function defined by $x \mapsto \mathrm{embdim}_x X$ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$ the set $\{x \in X | \mathrm{embdim}_x X \geq n\}$ is closed.

We take an open neighborhood U of $x \in X$ which is a closed complex subspace of a domain $D \subset \mathbb{C}^m$ with coordinates $z_1, \ldots, z_m$. Let $f_1, \ldots, f_k$ be functions on D such that $\mathcal{O}_{X,x} = \mathcal{O}_{D,x}/(f_{1x}, \ldots, f_{kx})$, where f_{ix} denotes the germ of f_i at $x \in D$. We denote by $J_x(f_1, \ldots, f_k)$ the Jacobian matrix at x, i.e.,

$$J_x(f_1, \ldots, f_k) = \left(\frac{\partial f_i}{\partial z_j}(x) \right).$$

Theorem 1.1. *In the situation above, we have*

$$\mathrm{rank} J_x(f_1, \ldots, f_k) + \mathrm{embdim}_x X = m.$$

Proof. Let $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$. We put

$$e = \mathrm{embdim}_x X \text{ and } r = \mathrm{rank} J_x(f_1, \ldots, f_k).$$

By reordering suffices, we may assume that

$$\det \left(\frac{\partial f_i}{\partial z_j}(x) \right)_{1 \le i,j \le r} \neq 0.$$

Set $w_1 = f_1, \ldots, w_r = f_r, w_{r+1} = z_{r+1} - x_{r+1}, \ldots, w_m = z_m - x_m$. Then, by the implicit function theorem, we may regard the functions $w_1, \ldots, w_m$ as the coordinates at $x \in \mathbb{C}^m$. Hence a neighborhood of $x \in X$ is a closed complex subspace of an $(m-r)$-dimensional domain $\{w_1 = \cdots = w_r = 0\} \subset \mathbb{C}^m$. This means that $e \le m - r$.

Next we show that $e \ge m - r$. Since $\mathcal{O}_{X,x}$ is a quotient of $\mathcal{O}_{\mathbb{C}^e,x}$, there exist the functions $g_1, \ldots, g_e$ on a neighborhood of $x \in \mathbb{C}^m$ which generate the maximal ideal of $\mathcal{O}_{X,x}$. Then the functions $f_1, \ldots, f_k, g_1, \ldots, g_e$ generate the maximal ideal of $\mathcal{O}_{\mathbb{C}^m,x}$, and thus $\mathrm{rank} J_x(f_1, \ldots, f_k, g_1, \ldots, g_e) = m$. Hence we see that $r \ge m - e$. $\square$

Example 1.2. Let $f_1 = x + y^2, f_2 = x + y$ be functions on $\mathbb{C}^3$. Then the Jacobian matrix at the origin $o := (0, 0, 0)$ is

$$J_o(f_1, f_2) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(o) & \frac{\partial f_1}{\partial y}(o) & 0 \\ \frac{\partial f_2}{\partial x}(o) & \frac{\partial f_2}{\partial y}(o) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

and then $\operatorname{rank} J_o(f_1, f_2) = 2$. We may regard the functions $f_1 = x+y^2, f_2 = x+y, z$ as the coordinates at $o \in \mathbb{C}^3$. Clearly a neighborhood of $o \in X$ is a complex line $\{f_1 = f_2 = 0\}$. This means that $\operatorname{embdim}_o X = 1$. Thus $\operatorname{rank} J_o(f_1, f_2) + \operatorname{embdim}_o X = 2 + 1 = 3$.

Corollary 1.3. *Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$. Then*

$$\operatorname{embdim}_x X = \dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2.$$

Proof. In the situation above, it suffices to show that $\dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2 = m - r$. Let $\mathfrak{n}_x$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}^m,x}$ and $\mathfrak{f}$ the ideal of $\mathcal{O}_{\mathbb{C}^m,x}$ generated by $f_{1x}, \ldots, f_{kx}$. Then $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathfrak{n}_x/(\mathfrak{n}_x^2 + \mathfrak{f})$. We define a map $a : \mathcal{O}_{\mathbb{C}^m,x} \longrightarrow \mathbb{C}^m$ by

$$a(f) = \left(\frac{\partial f}{\partial z_1}(x), \ldots, \frac{\partial f}{\partial z_m}(x) \right).$$

Then it is clear that $\dim_{\mathbb{C}} a(\mathfrak{f}) = r$ and that a induces an isomorphism $a' : \mathfrak{n}_x/\mathfrak{n}_x^2 \longrightarrow \mathbb{C}^m$. Since a' induces an isomorphism $(\mathfrak{f} + \mathfrak{n}_x^2)/\mathfrak{n}_x^2 \longrightarrow a(\mathfrak{f})$, we obtain that

$$\dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim_{\mathbb{C}} \mathfrak{n}_x/\mathfrak{n}_x^2 - \dim_{\mathbb{C}} (\mathfrak{f} + \mathfrak{n}_x^2)/\mathfrak{n}_x^2 = m - r.$$

□

We denote by Ω_X^1 the sheaf of differential 1-forms on X. For any point $x \in X$, $\Omega_{X,x}^1$ is generated by df, $f \in \mathcal{O}_{X,x}$, with the properties

(1) for $f \in \mathbb{C}$, $df = 0$;

(2) for $f, g \in \mathcal{O}_{X,x}$, $d(f + g) = df + dg$ and $d(fg) = fdg + gdf$.

Lemma 1.4. $\dim_{\mathbb{C}} \Omega_{X,x}^1/\mathfrak{m}_x\Omega_{X,x}^1 = \dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2$.

Proof. The homomorphism $\Omega_{X,x}^1/\mathfrak{m}_x\Omega_{X,x}^1 \longrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$, defined by

$$(df \mod \mathfrak{m}_x\Omega_{X,x}^1) \mapsto (f - f(x) \mod \mathfrak{m}_x^2)$$

is an isomorphism. □

Corollary 1.5. *In the situation above, we have the following:*

(1) $\dim_x X \leq \mathrm{embdim}_x X$;

(2) $\dim_x X \leq \dim_{\mathbb{C}} \Omega^1_{X,x}/\mathfrak{m}_x\Omega^1_{X,x}$;

(3) $r \leq m - \dim_x X$.

If the equality holds in one of the above, then it holds in the others.

Proof. By Matsumura [**19**, p. 104, 5.14], $\dim \mathcal{O}_{X,x} \leq \dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2$. Since $\dim_x X = \dim \mathcal{O}_{X,x}$, (1) follows from Corollary 1.3. Now the rest of the assertion follows from (1), Theorem 1.1, Corollary 1.3 and Lemma 1.4. □

Definition 1.6. Let X be a complex analytic space. A point $x \in X$ is called a non-singular point if the equality $\dim_x X = \mathrm{embdim}_x X$ holds. A point $x \in X$ is called a singular point if which is not a non-singular point. We denote by $\mathrm{Sing}(X)$ the set of singular points of X, and call it the singular locus of X. A complex analytic space X is said to be non-singular if any point of X is a non-singular point, and said to be singular if it is not non-singular. A complex analytic space X is said to be normal, Gorenstein or Cohen-Macaulay if the local ring $\mathcal{O}_{X,x}$ has such a property for any $x \in X$.

A point $x \in X$ is a non-singular point if and only if $\mathcal{O}_{X,x}$ is isomorphic to a convergent power series ring. By definition, complex manifolds are non-singular complex analytic spaces. Corollary 1.5 implies that a point $x \in X$ is a non-singular point if and only if $r = m - \dim_x X$: this assertion is called the Jacobian criterion of non-singularity.

Theorem 1.7. *Let X be a complex variety. Then $\mathrm{Sing}(X)$ is a proper analytic subset of X.*

Proof. We follow the notation above. Set $n = \dim X$. A point $x \in U \subset X$ is a singular point if and only if $\mathrm{rank} J_x(f_1, \ldots, f_k) < m - n$. Hence $\mathrm{Sing}(U)$ is the analytic subset of the domain D defined by the functions $f_1, \ldots, f_k$ and the all determinants of $(m-n) \times (m-n)$ sub-matrices of the Jacobian matrix $(\partial f_i/\partial z_j)$.

If U is sufficiently small, then U is a finite branched analytic covering of a domain in $\mathbb{C}^n$. This shows that $\mathrm{Sing}(U)$ is a proper subset of U. □

Theorem 1.8. *Let X be a complex variety.*

(1) *If X is normal, then* $\dim \mathrm{Sing}(X) \leq \dim X - 2$.
(2) *If X is Cohen-Macaulay and* $\dim \mathrm{Sing}(X) \leq \dim X - 2$, *then X is normal.*
(3) *The following are equivalent:*
   (a) *X is normal;*
   (b) *for any open subset $U \subset X$, the restriction*

$$\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U \setminus \mathrm{Sing}(X), \mathcal{O}_X)$$

   is bijective.

Proof. See Fischer [**6**, p. 119-120]. □

Definition 1.9. Let (X, x) be a germ of a complex variety X at x. We simply call it a singularity. A singularity (X, x) is said to be isolated if there exists an open neighborhood U of x such that $\mathrm{Sing}(U) = \{x\}$. A singularity (X, x) is said to be normal, complete intersection, Gorenstein or Cohen-Macaulay if the local ring $\mathcal{O}_{X,x}$ has such a property. A hypersurface singularity is a complete intersection singularity with $\mathrm{embdim}_x X = \dim X + 1$. Unless stated otherwise, X denotes a Stein variety whenever we call (X, x) a singularity. We always assume that $\mathrm{Sing}(X) = \{x\}$ if (X, x) is an isolated singularity.

Remark 1.10. By Theorem 1.8, any isolated Cohen-Macaulay singularity is normal. For any singularity, we have the following implications:

hypersurface ⇒ complete intersection ⇒ Gorenstein ⇒ Cohen-Macaulay.

See Matsumura [**19**, p. 171].

Definition 1.11. Let X be a complex variety. The morphism $\phi : X_{\mathrm{norm}} \longrightarrow X$ is said to be the normalization of X if

(1) X_{norm} is normal;

(2) ϕ is finite and surjective;

(3) if $N = \{x \in X | (X, x) \text{ is not normal}\}$, then $X_{\text{norm}} \setminus \phi^{-1}(N)$ is isomorphic to $X \setminus N$.

Definition 1.12. Let $f : Y \longrightarrow X$ be a morphism of complex varieties such that

(1) f is proper and surjective;

(2) there exist proper analytic subsets $A \subset X$ and $B \subset Y$ such that the restriction $Y \setminus B \longrightarrow X \setminus A$ of f is an isomorphism.

Then we call f a modification. Suppose that A and B are the minimal subsets with the property above, and that X and Y are normal. The subset of B, which is the sum of all irreducible components B_i with $\dim B_i > \dim f(B_i)$ is called the exceptional set of f. The divisor on Y, which is the sum of all prime divisors supported in the exceptional set, is called the exceptional divisor of f. Let V be a closed complex subvariety of X such that $V \not\subseteq A$. Then the closure of $f^{-1}(V \setminus A)$ is called the strict transform of V by f, and denoted by $f_*^{-1}V$. If $D = \sum a_i D_i$ is a divisor on X with prime divisors D_i, then we denote by $f_*^{-1}D$ the divisor $\sum a_i f_*^{-1} D_i$.

Definition 1.13. Let M be a complex manifold and D a reduced divisor on M. Then D is said to have only normal crossings if at each point of D, the defining equation of D can be written as $\prod_{i=1}^k z_i$, where $\{z_1, \ldots, z_k\}$ is a part of suitable local coordinates. Moreover if each irreducible component of D is non-singular, then D is said to have only simple normal crossings.

Definition 1.14. Let X be a complex variety. A modification $f : M \longrightarrow X$ is called a resolution of singularities of X if M is non-singular and the restriction

$$M \setminus f^{-1}(\mathrm{Sing}(X)) \longrightarrow X \setminus \mathrm{Sing}(X)$$

is an isomorphism. We call M a resolution space. A resolution $f : M \longrightarrow X$ is called a good resolution if $f^{-1}(\mathrm{Sing}(X))$ is a subvariety of pure codimension

1 and has only simple normal crossings. If (X, x) is isolated, then we write the resolution as $f : (M, A) \longrightarrow (X, x)$, where $A = f^{-1}(\mathrm{Sing}(X))$: in this case we may regard $f : (M, A) \longrightarrow (X, x)$ as a morphism of germs.

Theorem 1.15 (Hironaka [**9**])**.** *Any singularity admits a good resolution.*

1.2. Blowing up

Definition 1.16. Let X be a complex analytic space and $\mathcal{I}$ a sheaf of ideals on X. Let $f : Y \longrightarrow X$ be a morphism of complex analytic spaces. We define the inverse image ideal sheaf $\mathcal{I}\mathcal{O}_Y \subset \mathcal{O}_Y$ to be the image of the natural homomorphism $f^*\mathcal{I} \longrightarrow \mathcal{O}_Y$.

Definition 1.17. Let X be a complex variety, C a closed subvariety and $\mathcal{I}$ its sheaf of ideals. Then there exists a unique proper morphism $f : Y \longrightarrow X$ of varieties which satisfies the following (see Fischer [**6**, 4.1]):

(1) the inverse image ideal sheaf $\mathcal{I}\mathcal{O}_Y$ is invertible;
(2) if $g : Z \longrightarrow X$ is a morphism of complex analytic spaces such that $\mathcal{I}\mathcal{O}_Z$ is invertible, then there exists a unique morphism $h : Z \longrightarrow Y$ such that $g = f \circ h$;
(3) the restriction $Y \setminus f^{-1}(C) \longrightarrow X \setminus C$ of f is an isomorphism;
(4) if X is a manifold and C is a submanifold, then Y is also a manifold.

We call the morphism f the blowing up of X with center C, or the blowing up of X with respect to the ideal sheaf $\mathcal{I}$. The morphism f is also called a blowing down when X is viewed as constructed from Y.

A resolution of a singularity is obtained by a finite succession of blowing ups with non-singular centers.

Example 1.18. We construct the blowing up of $\mathbb{C}^n$ with center the origin. Let $z_1, \ldots, z_n$ be the coordinates of $\mathbb{C}^n$, and $(Z_1 : \cdots : Z_n)$ the homogeneous coordinates of $\mathbb{P}^{n-1}$. Let M be a subvariety of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ defined by the equations

$$z_i Z_j - z_j Z_i = 0, \quad i, j = 1, \dots, n.$$

Then the blowing up $f : M \longrightarrow \mathbb{C}^n$ is induced by the projection $\mathbb{C}^n \times \mathbb{P}^{n-1} \longrightarrow \mathbb{C}^n$:

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{C}^n \times \mathbb{P}^{n-1} \arrow[d] \\ & \mathbb{C}^n \end{tikzcd}$$

Let $E = f^{-1}(o)$. We put

$$U_i = \{p \in \mathbb{P}^{n-1} | Z_i(p) \neq 0\},\ M_i = M \cap (\mathbb{C}^n \times U_i).$$

Then M_i is isomorphic to the affine space $\mathbb{C}^n$ and which has the coordinates

$$Z_1/Z_i, \dots, Z_{i-1}/Z_i,\ z_i,\ Z_{i+1}/Z_i, \dots, Z_n/Z_i.$$

Let $w_i = Z_i/Z_1,\ i = 2, \dots, n$. The restriction $f_1 : M_1 \longrightarrow \mathbb{C}^n$ of f is given by $z_1 = z_1, z_j = z_1 w_j,\ j = 2, \dots, n$, and $E \cap M_1$ is defined by the function z_1 in M_1. This shows that $E = \{o\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$. Let Y be a hypersurface in a neighborhood of the origin defined by a holomorphic function $g(z) = \sum_{i \geq k} g_i(z)$, where each $g_i(z) = g_i(z_1, \dots, z_n)$ denotes a homogeneous polynomial of degree i and $g_k(z) \neq 0$. Let

$$h(z_1, w) = g(z_1, z_1 w_2, \dots, z_1 w_n)/z_1^k.$$

Then the strict transform of Y is defined by $h(z_1, w)$ in M_1. Since $f^* g(z) = z_1^k h(z_1, w)$, we see that $f^* Y = kE + f_*^{-1} Y$.

Example 1.19. Let $X \subset \mathbb{C}^3$ be a hypersurface defined by $g(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 = 0$. Then $\mathrm{Sing}(X) = \{(0, 0, 0)\}$. Let $f : \tilde{X} \longrightarrow X$ be the blowing up of X at $o := (0, 0, 0)$. Following the situation of Example 1.18, the strict transform $f_*^{-1} X$ of X is defined by $h(z_1, w) := g(z_1, z_1 w_2, z_1 w_3)/z_1^2 = 1 + w_2^2 + w_3^2$

in M_1 and $f_*^{-1}X$ is non-singular. Thus the blowing up $f : \tilde{X} \longrightarrow X$ is a good resolution with exceptional set $E = f^{-1}(o) \cong \mathbb{P}^1$.

Let S be a non-singular surface, not necessarily compact. Let $D = \sum a_i D_i$ be a divisor on S, where D_i's are mutually distinct prime divisors. We put $D_{\mathrm{red}} = \sum_{a_i \neq 0} D_i$. The divisor D is said to be connected if the support of D is connected, and said to be positive if D is effective and $D \neq 0$. If the support of D is compact and each a_i is an integer, then we call D a $\mathbb{Z}$-cycle, or a cycle for short.

Let D be a positive divisor on S and $p \in \mathrm{Supp}(D)$. Let x, y be local coordinates at p, and $f = \sum_{i \geq 0} f_i(x, y) \in \mathcal{O}_{S,p}$ a function defining D near p, where $f_i(x, y)$ is a homogeneous polynomial of degree i. Then we define the multiplicity of D at p, denoted $\mathrm{mult}(D, p)$, to be the least integer m such that $f_m \neq 0$. If p is not a point of $\mathrm{Supp}(D)$, then put $\mathrm{mult}(D, p) = 0$. Note that p is a non-singular point of D if and only if $\mathrm{mult}(D, p) = 1$. If $h : S' \longrightarrow S$ is the blowing up of S with center p and E the exceptional divisor of h, then $h^*D = h_*^{-1}D + \mathrm{mult}(D, p)E$.

Theorem 1.20. *Let D be a reduced divisor on S. Then there exists a finite sequence of the blowing ups*

$$S_n \longrightarrow S_{n-1} \longrightarrow \cdots \longrightarrow S_0 = S$$

such that each $S_i \longrightarrow S_{i-1}$ is the blowing up with center a point, and that the support of the fiber of D on S_n has only simple normal crossings.

Proof. See Barth-Peters-Van de Ven [**3**, II, 7]. □

Proposition 1.21. *A curve singularity $(C, p) \subset (S, p)$ with $\mathrm{mult}(C, p) = 2$ is isomorphic to the germ of $\{x^r - y^2 = 0\} \subset \mathbb{C}^2$ at the origin for some $r \geq 2$: if $r = 2$ the singular point is called a node; if $r = 3$ it is called a cusp.*

Proof. See Barth-Peters-Van de Ven [**3**, II, 8]. □

Example 1.22. Let $C \subset S$ be a compact curve with a cusp $p \in C$. Let $S_1 \longrightarrow S$ be the blowing up with center p. Then the strict transform of C on S_1 is non-singular. However, we need three blowing ups so that the support of the fiber of C has only simple normal crossings. See Figure 1.1: C_i denotes the strict transform of C_{i-1}. Note that the fiber of C is the divisor $C_3 + 2E_2 + 3F_1 + 6G_0$ (see Example 1.18).

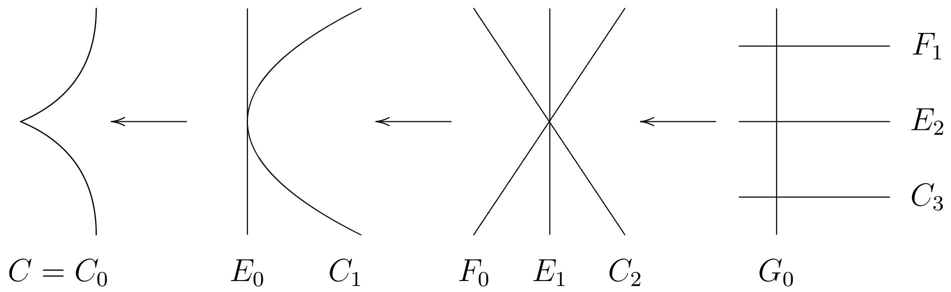


FIGURE 1.1. Resolution of a cusp

Let D and E be reduced divisors on S having no common irreducible component. Suppose that $p \in D \cap E$, and that D, E are defined by $f, g \in \mathcal{O}_{S,p}$, respectively. We define the intersection multiplicity $(D, E)_p$ of D and E at p by $(D, E)_p = \dim_{\mathbb{C}} \mathcal{O}_{S,p}/(f, g)$. If $(D, E)_p = 1$, then p is a node of $D \cup E$. For example, let $C = \{(z_1, z_2) \in \mathbb{C}^2 | z_1^2 - z_2^3 = 0\} \subset \mathbb{C}^2$ and $D_i = \{(z_1, z_2) \in \mathbb{C}^2 | z_i = 0\} \subset \mathbb{C}^2$ for $i = 1, 2$. Then $(C, D_1)_o = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,o}/(z_1^2 - z_2^3, z_1) = 3$ and $(C, D_2)_o = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,o}/(z_1^2 - z_2^3, z_2) = 2$.

Let C be a compact curve on S. Let $\sigma : C' \longrightarrow C$ be the normalization. For an invertible sheaf $\mathcal{L}$ on S, the intersection number $\mathcal{L} \cdot C$ is defined as $\deg \sigma^*(\mathcal{L} \otimes \mathcal{O}_C)$. Let $D = \sum_{i=1}^n m_i C_i$ be a cycle on S, where each C_i is a compact curve. Then the intersection number $\mathcal{L} \cdot D$ is defined by $\mathcal{L} \cdot D = \sum_{i=1}^n m_i \mathcal{L} \cdot C_i$. For any divisor E on S the intersection number $E \cdot D$ is defined by $E \cdot D = \mathcal{O}_S(E) \cdot D$. If D and E are cycles on S, then we have the following (see Barth-Peters-Van de Ven [**3**, II,10]):

(1) $D \cdot E = E \cdot D$;

(2) if $\alpha : Y \longrightarrow S$ is a proper morphism of non-singular surfaces, then

$$(\alpha^* D) \cdot (\alpha^* E) = \deg(\alpha) D \cdot E;$$

(3) if D and E are positive, and have no common component, then

$$D \cdot E = \sum_{p \in D \cap E} (D, E)_p.$$

For a divisor D and cycle E, we can naturally define the intersection number $D \cdot E$, and also obtain the properties (1) and (2) above. We denote by D^2 the self-intersection number $D \cdot D$.

Definition 1.23. A curve C on a surface S is called a $(-n)$-curve if $C \cong \mathbb{P}^1$ and $C^2 = -n$.

Theorem 1.24 (Castelnuovo)**.** *Let C be a curve on a surface S. Then C is a (-1)-curve if and only if there exists a blowing down $f : S \longrightarrow S'$ such that f induces an isomorphism $S \setminus C \cong S' \setminus f(C)$ and $f(C)$ is a non-singular point of S'.*

Theorem 1.25. *Let $f : S' \longrightarrow S$ be a modification of non-singular surfaces. Suppose that there exists a finite set F of points on S such that f induces an isomorphism $S' \setminus f^{-1}(F) \longrightarrow S \setminus F$. Then f is a finite sequence of blowing ups*

$$S' = S_n \longrightarrow S_{n-1} \longrightarrow \cdots \longrightarrow S_0 = S$$

such that each $S_i \longrightarrow S_{i-1}$ is the blowing up with center a point.

Proof. See Barth-Peters-Van de Ven [**3**, II, 7]. □

Proposition 1.26. *Let $\alpha : Y \longrightarrow S$ be the blowing up of S with center $p \in S$ and $E = \alpha^{-1}(p)$. Let D be a positive divisor on S, $D_1 = \alpha_*^{-1} D$ the strict transform of D and $n = \mathrm{mult}(D, p)$. Then we have the following:*

(1) $(\alpha^* D) \cdot E = 0$*;*

(2) $D_1 \cdot E = n$*;*

(3) *If C and D are positive cycles on S, then*

$$C_1 \cdot D_1 = C \cdot D - mn,$$

where $C_1 = \alpha_*^{-1} C$ and $m = \mathrm{mult}(C, p)$.

Proof. Since $\alpha^* \mathcal{O}_S(D)$ is trivial near E, we have (1). The assertion (2) follows from $0 = (\alpha^* D) \cdot E = (D_1 + nE) \cdot E$, since E is a (-1)-curve. The formula $(\alpha^* C) \cdot (\alpha^* D) = C \cdot D$ implies (3). □

Definition 1.27. Let $D = \sum_{i=1}^n C_i$ be a connected cycle on S, where C_i are mutually distinct curves. Then the matrix $(C_i \cdot C_j)$ is called the intersection matrix of D.

Theorem 1.28 (Artin [**2**, Proposition 2])**.** *Let D be as above.*

(1) *If the intersection matrix $(C_i \cdot C_j)$ is negative definite, then there exists a positive cycle $Z = \sum_{i=1}^n m_i C_i$ such that $Z \cdot C_i \leq 0$ for $i = 1, \ldots, n$.*

(2) *Conversely, if there exists a positive cycle $Z = \sum_{i=1}^n m_i C_i$ such that $Z \cdot C_i \leq 0$ for $i = 1, \ldots, n$, then $(C_i \cdot C_j)$ is negative semi-definite, and if in addition $Z^2 < 0$, then $(C_i \cdot C_j)$ is negative definite.*

Theorem 1.29 (Grauert [**8**, p. 367])**.** *Let D be as above. If the intersection matrix $(C_i \cdot C_j)$ is negative definite, then there uniquely exists a blowing down $f : S \longrightarrow X$ such that X is normal and f induces an isomorphism $S \backslash D \cong X \backslash \{x\}$, where $\{x\} = f(D)$. In this situation, we say that f contracts D, and that D is contractible to the singularity (X, x).*

1.3. Resolution of normal surface singularities

Let (X, x) be a surface singularity and $f : (\tilde{X}, E) \longrightarrow (X, x)$ a resolution. Then any cycle on $\tilde{X}$ is supported in E. Let $E = \bigcup_{i=1}^n E_i$ be the decomposition of E into irreducible components. We denote by $K_{\tilde{X}}$ the canonical divisor on $\tilde{X}$.

Theorem 1.30 (Mumford [**21**])**.** *The intersection matrix $(E_i \cdot E_j)$ is negative definite.*

Definition 1.31. A resolution $f : (\tilde{X}, E) \longrightarrow (X, x)$ is called a minimal resolution if for any resolution $f' : \tilde{X}' \longrightarrow X$ there exists a unique morphism $g : \tilde{X}' \longrightarrow \tilde{X}$ such that $f' = f \circ g$.

By the definition, a minimal resolution is unique if it exists.

Theorem 1.32. *Let $f : \tilde{X} \longrightarrow X$ be any resolution. Then the minimal resolution of the singularity (X, x) is obtained from $\tilde{X}$ by successively contracting all (-1)-curves.*

Proof. See Laufer [**15**, Theorem 5.9]. $\square$

Definition 1.33. A good resolution $f : (\tilde{X}, E) \longrightarrow (X, x)$ is called a minimal good resolution if for any good resolution $f' : \tilde{X}' \longrightarrow X$ there exists a unique morphism $g : \tilde{X}' \longrightarrow \tilde{X}$ such that $f' = f \circ g$.

Theorem 1.34. *For any surface singularity, there exists a unique minimal good resolution.*

Proof. See Laufer [**15**, Theorem 5.12]. $\square$

Remark 1.35. From the minimal resolution, we obtain the minimal good resolution by a finite succession of blowing ups (cf. Theorem 1.20 and Theorem 1.25).

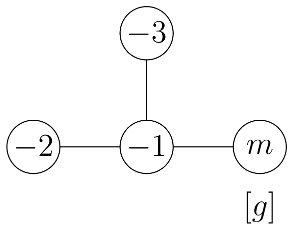
Definition 1.36. Let D be a reduced cycle on a non-singular surface. Suppose that D has only simple normal crossings. Then the weighted dual graph of D is the graph such that each vertex represents an irreducible component E_i of D weighted by E_i^2 and $g(E_i)$, while each edge connecting the vertices corresponding to E_i and E_j, $i \neq j$, corresponds to the point $E_i \bigcap E_j$. For example, if $E_i^2 = -b_i$ and $g(E_i) = g_i > 0$ (resp. $g_i = 0$), we write the vertex corresponding to E_i as follows:

$$\underset{[g_i]}{\boxed{-b_i}} \quad \left(\text{resp. } \boxed{-b_i} \right).$$

A graph obtained by removing the weights from a weighted dual graph is simply called a dual graph.

Let (X, x) be a surface singularity and $f : (\tilde{X}, E) \longrightarrow (X, x)$ the minimal good resolution. Then the weighted dual graph of (X, x) means the weighted dual graph of E. It is clear that giving the weighted dual graph of (X, x) is equivalent to giving the information on the genera of the E_i's and the intersection matrix $(E_i \cdot E_j)$.

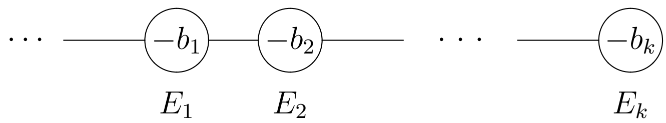
Example 1.37. Let C be a compact curve with a cusp on a non-singular surface. Suppose that $C^2 = -d < 0$. Then C is contractible to a surface singularity by Theorem 1.29. From Example 1.22 and Proposition 1.26, we see that the weighted dual graph of the singularity is as follows:



where $m = -d - 6$ and $g = p_a(C) - 1$.

Definition 1.38. Let D be a reduced connected cycle on $\tilde{X}$ having only simple normal crossings. Then D is called a tree of curves if the dual graph of D is a tree, and called a chain of curves if the dual graph is a chain.

Definition 1.39. A string S in E is a chain of non-singular rational curves $E_1, \ldots, E_k$ so that $E_i \cdot E_{i+1} = 1$ for $i = 1, \ldots, k-1$, and these account for all intersections in E among the E_i's, except that E_1 intersects exactly one other curve.



Definition 1.40. Suppose that $f : (\tilde{X}, E) \longrightarrow (X, x)$ is the minimal good resolution. The weighted dual graph of (X, x) is called a star-shaped graph, if E is not a chain of rational curves, and if $E = E_0 + \sum_{i=1}^{\beta} S_i$, where E_0 is a curve and S_i are the maximal strings. Then E_0 is called the central curve, and S_j are called branches. Let $S_i = \bigcup_{j=1}^{r_i} E_{ij}$ be the decomposition into irreducible components, where $E_0 \cdot E_{i1} = E_{ij} \cdot E_{i,j+1} = 1$. Let $g = g(E_0)$, $b = -E_0^2$ and $b_{ij} = -E_{ij}^2$. Then we obtain the weighted dual graph in Figure 1.2.

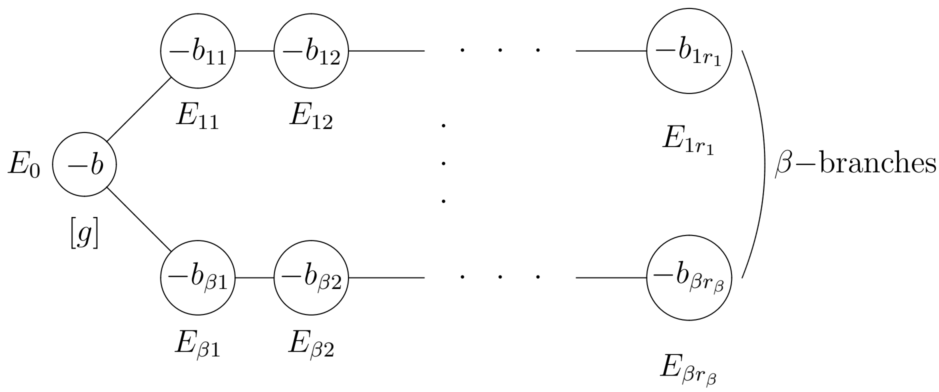


FIGURE 1.2. A star-shaped graph

For each branch S_i, the positive integers e_i and d_i are defined by

$$\frac{d_i}{e_i} = [[b_{i1}, \ldots, b_{ir_i}]] := b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{\ddots - \cfrac{1}{b_{ir_i}}}}$$

where $e_i < d_i$, and e_i and d_i are relatively prime. We call the set

$$\{g; b, (d_1, e_1), \ldots, (d_\beta, e_\beta)\}$$

the data of the star-shaped graph.

Remark 1.41. Let D be a reduced connected cycle on a non-singular surface. Suppose that the weighted dual graph of D is represented as in Figure 1.2. Then the intersection matrix of D is negative definite if and only if $b > \sum_{i=1}^{\beta}(e_i/d_i)$ (cf. Pinkham [**28**, p. 185])

Definition 1.42. Divisors D and C on $\tilde{X}$ are said to be f-numerically equivalent, written $D \equiv C$, if $(D - C) \cdot E_i = 0$ for all E_i. For a divisor D, $-D$ is said to be f-numerically effective, or f-nef for short, if $D \cdot E_i \leq 0$ for all E_i.

Lemma 1.43. *Let D be an f-nef cycle. Then $D = 0$, or $D < 0$ and* $\mathrm{Supp}(D) = E$.

Proof. Suppose that $D \neq 0$ and write D in the form $D = B - C$, where B and C are effective cycles without common components. Thus $B \cdot C \geq 0$. By assumption, we have $B^2 - B \cdot C = D \cdot B \geq 0$. Thus $B^2 \geq 0$. Since the intersection matrix is negative definite, $B = 0$. If $\mathrm{Supp}(C) \neq E$, then there exists a component E_i such that $C \cdot E_i > 0$ since E is connected. Hence $\mathrm{Supp}(D) = E$. □

Definition 1.44. A positive cycle Z on $\tilde{X}$ is called a fundamental cycle if $-Z$ is f-nef and for any positive cycle D with this property, $Z \leq D$.

Theorem 1.45. *There exists a unique fundamental cycle Z.*

Proof. By Theorem 1.28 there exists a positive cycle D such that $-D$ is f-nef. Let $D = \sum_{i=1}^{n} d_i E_i$ and $C = \sum_{i=1}^{n} e_i E_i$ be cycles having such the property. Let $a_i = \min\{d_i, e_i\}$ and $F = \sum_{i=1}^{n} a_i E_i$. It suffices to show that $-F$ is f-nef. If $a_j = d_j$, then

$$F \cdot E_j = d_j E_j^2 + \sum_{i \neq j} a_i E_i \cdot E_j \leq d_j E_j^2 + \sum_{i \neq j} d_i E_i \cdot E_j = D \cdot E_j \leq 0.$$

Hence $-F$ is f-nef. □

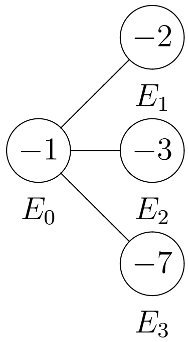
Proposition 1.46. *The fundamental cycle Z is computed via a computation sequence for Z:*

$$Z_1 = E_{i_1}, \ldots, Z_j = Z_{j-1} + E_{i_j}, \ldots, Z_t = Z_{t-1} + E_{i_t} = Z,$$

where E_{i_1} is arbitrary and $Z_{j-1} \cdot E_{i_j} > 0$ for $1 < j \leq t$.

Proof. Let $Z' = \sum a'_i E_i$ and $Z = \sum_{i=1}^{n} a_i E_i$. Suppose that $Z' \leq Z$ and $a'_j = a_j$. Then by the argument in the proof above, we obtain that $Z' \cdot E_j \leq Z \cdot E_j \leq 0$. This implies that $Z_j \leq Z$ for any Z_j occurring in a computation sequence. Hence any computation sequence reaches the fundamental cycle. $\square$

Example 1.47. Suppose that (X, x) is a surface singularity and $f : (\tilde{X}, E) \longrightarrow (X, x)$ a resolution of (X, x) such that the weighted dual graph of the exceptional set E is as follows:



Let $Z_1 = E_0,\ Z_2 = Z_1 + E_1,\ Z_3 = Z_2 + E_2,\ Z_4 = Z_3 + E_3,\ Z_5 = Z_4 + E_0,\ Z_6 = Z_5 + E_0,\ Z_7 = Z_6 + E_1,\ Z_8 = Z_7 + E_0,\ Z_9 = Z_8 + E_2,\ Z_{10} = Z_9 + E_0,\ Z_{11} = Z_{10} + E_1,\ Z_{12} = Z_{11} + E_0 = Z$. Then $\{Z_i\}$ is a computation sequence for the fundamental cycle Z on $\tilde{X}$. In fact, $Z_1 \cdot E_1 > 0,\ Z_2 \cdot E_2 > 0,\ Z_3 \cdot E_3 > 0,\ Z_4 \cdot E_0 > 0,\ Z_5 \cdot E_0 > 0,\ Z_6 \cdot E_1 > 0,\ Z_7 \cdot E_0 > 0,\ Z_8 \cdot E_2 > 0,\ Z_9 \cdot E_0 > 0,\ Z_{10} \cdot E_1 > 0,\ Z_{11} \cdot E_0 > 0$ and $Z \cdot E_0 = 0,\ Z \cdot E_1 = 0,\ Z \cdot E_2 = 0,\ Z \cdot E_3 = -1 < 0$. We obtain that $Z = 6E_0 + 3E_1 + 2E_2 + E_3$.

Proposition 1.48. *Let $g : \tilde{X}' \longrightarrow \tilde{X}$ be a modification, where $\tilde{X}'$ is a non-singular surface. Let Z and Z' be the fundamental cycles on $\tilde{X}$ and $\tilde{X}'$, respectively. Then $Z' = g^* Z$.*

Proof. By Theorem 1.25, we may assume that g is the blowing up with center $p \in E$. Let $E'_i = g_*^{-1} E_i$, the strict transform of E_i, and $E' = g^{-1}(p)$. Then

$$-g^* Z \cdot E'_i = -g^* Z \cdot g^* E_i = -Z \cdot E_i \geq 0.$$

Hence $-g^*Z$ is $f \circ g$-nef and $Z' \leq g^*Z$. Let $Z' = \sum_{i=1}^n a_i' E_i' + b'E'$ and $g^*Z = \sum_{i=1}^n a_i E_i' + bE'$. Suppose that $a_i' < a_i$ for some i. Then

$$g_*Z' = \sum_{i=1}^n a_i' E_i < \sum_{i=1}^n a_i E_i = Z.$$

Thus there exists a component E_j such that

$$0 < g_*Z' \cdot E_j = g^*(g_*Z') \cdot g^*E_j = g^*(g_*Z') \cdot E_j'.$$

Let $cE' = g^*(g_*Z') - Z'$, $c \in \mathbb{Z}$. Since $(g^*(g_*Z') - Z') \cdot E_j' > 0$, we have $c > 0$. But this implies that $0 = g^*(g_*Z') \cdot E' = (Z' + cE') \cdot E' < 0$. Hence we obtain that $a_i' = a_i$ for all i. Since $0 \geq Z' \cdot E' = (Z' - g^*Z) \cdot E' = -b' + b$, we have $b' = b$. $\square$

Let (X, x) be a normal surface singularity and $f : (\tilde{X}, E) \longrightarrow (X, x)$ a resolution with exceptional set E. For any non-zero function $h \in \mathcal{O}_{X,x}$, the zero divisor of $h \circ f$ is written as

$$\mathrm{div}_{\tilde{X}}(h) := \mathrm{div}_{\tilde{X}}(h \circ f) = (h)_E + H$$

where $(h)_E$ is supported in E and H does not contain any irreducible component of E.

Definition 1.49 ([**37**])**.** Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Then the smallest positive cycle among the cycles $(h)_E$, $h \in \mathfrak{m} \setminus \{x\}$, is called the maximal ideal cycle.

Remark 1.50. The fundamental cycle Z is a topological invariant of the resolution, in fact, it is determined by the weighted dual graph of exceptional set E. The maximal ideal cycle $Z_{\mathfrak{m}}$ is an analytic invariant of the resolution and cannot be determined by the weighted dual graph of E in general. We have $Z \leq Z_{\mathfrak{m}}$ by the definitions of these cycles.

1.4. Cyclic quotient singularities

In this section, we introduce the cyclic quotient singularities and their fundamental facts.

Definition 1.51. Let n and μ be positive integers with $\mu < n$ and $\gcd(n, \mu) = 1$. Let ϵ_n denote the primitive n-th root of unity $\exp(2\pi\sqrt{-1}/n)$. Then the singularity of the quotient

$$\mathbb{C}^2 \Big/ \left\langle \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{\mu} \end{pmatrix} \right\rangle$$

is called the cyclic quotient singularity of type $C_{n,\mu}$.

A non-singular point is regarded as of type $C_{1,0}$. For integers $c_i \geq 2$, $i = 1, \ldots, r$, we put

$$[[c_1, \ldots, c_r]] := c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_r}}}$$

Lemma 1.52. *If $n/\mu = [[c_1, \ldots, c_r]]$, then the weighted dual graph of the minimal resolution of the cyclic quotient singularity of type $C_{n,\mu}$ is as in Figure 1.3,*

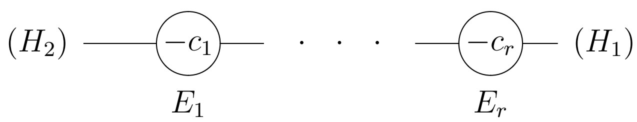


Figure 1.3.

where all prime exceptional divisors E_i are rational and H_i denotes the strict transform of the image of the coordinate axis $\{x_i = 0\} \subset \mathbb{C}^2$ by the quotient map, and (H_i) the vertex corresponding to H_i.

Proof. See Brieskorn [**4**]. $\square$

It is known that the complex structure of quotient surface singularity is determined by its resolution graph (cf. [**4**], [**16**]).

In the situation above, for any positive integer λ_0, let

$$\mathcal{L}(\lambda_0) := \left\{ \lambda_0 E_0 + \sum_{i=1}^{r} m_i E_i \,\middle|\, m_1, \ldots, m_r \in \mathbb{Z} \right\},$$

where $E_0 = H_2$. Then we define a set $\mathcal{D}(\lambda_0)$ as follows:

$$\mathcal{D}(\lambda_0) := \{ D \in \mathcal{L}(\lambda_0) \mid DE_i \leq 0,\, i = 1, \ldots, r \}.$$

We see that $\mathcal{D}(\lambda_0)$ is not empty and has the smallest element.

Lemma 1.53 ([**18**, Lemma 2.2])**.** *Let $D \in \mathcal{D}(\lambda_0)$. Assume that $DE_i = 0$ for $i < r$ and $DE_r \geq -1$. Then D is the smallest element of $\mathcal{D}(\lambda_0)$.*

Proof. Suppose that $D_0 \in \mathcal{D}(\lambda_0)$ is the smallest element. Let $\triangle = D - D_0$. Then

$$\triangle E_r = (D - D_0)E_r \geq -1. \tag{1.1}$$

Assume $\triangle = \sum_{i=k}^{r} m_i E_i$ and $m_k \neq 0$. Then

$$\triangle E_i = m_{i-1} - c_i m_i + m_{i+1} \quad (m_{k-1} = m_{r+1} = 0).$$

For $1 \leq i < r$, since $\triangle E_i = (D - D_0)E_i = -D_0 E_i \geq 0$ and $c_i \geq 2$,

$$m_{i+1} \geq c_i m_i - m_{i-1} \geq m_i + (m_i - m_{i-1}).$$

Therefore, $m_{i+1} > m_i$ for $k - 1 \leq i < r$, and

$$\triangle E_r = m_{r-1} - c_r m_r < m_r(1 - c_r) \leq -1.$$

It contradicts (1.1). □

For any $x \in \mathbb{R}$, we write $\lceil x \rceil = \min\{t \in \mathbb{Z} \mid x \leq t\}$. Let $e_i := [[c_i, \ldots, c_r]]$ for $1 \leq i \leq r$, then $c_i = e_i + 1/e_{i+1}$ for $1 \leq i < r$ and $c_r = e_r$.

Lemma 1.54 ([**13**, Lemma 1.1]). *Take a positive integer* λ_0 *and define the sequence* $\{\lambda_i\}_{i=0}^r$ *by the recurrence formula* $\lambda_i = \lceil \lambda_{i-1}/e_i \rceil$ *for* $1 \le i \le r$. *Then the cycle* $\sum_{i=0}^r \lambda_i E_i$ *is the smallest element of* $\mathcal{D}(\lambda_0)$.

Corollary 1.55. *Let* Y_0 *and* Y_0' *be the smallest element of* $\mathcal{D}(\lambda_0)$ *and* $\mathcal{D}(\lambda_0')$, *respectively. Then* $Y_0 \ge Y_0'$ *if and only if* $\lambda_0 \ge \lambda_0'$.

Proof. If $Y_0 \ge Y_0'$, it is clear that $\lambda_0 \ge \lambda_0'$.

Conversely, assume that $\lambda_0 \ge \lambda_0'$. Then $\lambda_1 = \lceil \lambda_0/e_1 \rceil \ge \lceil \lambda_0'/e_1 \rceil = \lambda_1'$. Suppose that $\lambda_k \ge \lambda_k'$ for some integer k with $1 \le k < r$. Then

$$\lambda_{k+1} = \lceil \lambda_k/e_{k+1} \rceil \ge \lceil \lambda_k'/e_{k+1} \rceil = \lambda_{k+1}'.$$

By induction, we have $\lambda_i \ge \lambda_i'$ for any i with $1 \le i \le r$. Therefore, $Y_0 \ge Y_0'$. $\square$

Lemma 1.56 ([**13**, Lemma 1.2]). *Let the sequence* $\{\lambda_i\}_{i=0}^r$ *be as in Lemma 1.54, and for* $1 \le i \le r$, *take relatively prime positive integers* n_i *and* μ_i *satisfying* $n_i/\mu_i = e_i$. *Put* $\lambda_{r+1} := \lambda_r c_r - \lambda_{r-1}$.

(1) *If* $\lambda_{i-1} = \lambda_i c_i - \lambda_{i+1}$ *holds for* $1 \le i \le r$, *then* $\lambda_1 = (\mu\lambda_0 + \lambda_{r+1})/n$.

(2) *If* $\lambda_0 \equiv 0 \pmod{n}$, *then* $\lambda_i = \mu_i \lambda_{i-1}/n_i$ *for* $1 \le i \le r$. *If* $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$, *then* $\lambda_i = (\mu_i\lambda_{i-1} + 1)/n_i$ *for* $1 \le i \le r$.

(3) *If either* $\lambda_0 \equiv 0 \pmod{n}$ *or* $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$, *then* $\lambda_{i-1} = \lambda_i c_i - \lambda_{i+1}$ *holds for* $1 \le i \le r$. *Furthermore,* $\lambda_{r+1} = 0$ *when* $\lambda_0 \equiv 0 \pmod{n}$, *and* $\lambda_{r+1} = 1$ *when* $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$.

(4) *If* $\lambda_0 \equiv 0 \pmod{n}$, *then* $\lambda_r = \lambda_0/n$. *If* $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$, *then* $\lambda_r = \lceil \lambda_0/n \rceil$.

Proof. (1) Note that we have $n_1 = n$, $\mu_1 = \mu$ and $c_r = n_r$, $\mu_r = 1$. Suppose $\lambda_{i-1} = \lambda_i c_i - \lambda_{i+1}$ for $1 \le i \le r$. Put $n_{r+1} = 1, \mu_{r+1} = 0$. For $1 \le i \le r$, since $\gcd(n_{i+1}, \mu_{i+1}) = 1$ and

$$\frac{n_i}{\mu_i} = c_i - \frac{1}{\frac{n_{i+1}}{\mu_{i+1}}} = \frac{c_i n_{i+1} - \mu_{i+1}}{n_{i+1}},$$

we have $\mu_i = n_{i+1}$ and $n_i = c_i n_{i+1} - \mu_{i+1} = c_i n_{i+1} - n_{i+2}$. Thus,

$$\begin{aligned}
(\mu\lambda_0 + \lambda_{r+1})/n &= (\mu\lambda_0 + \lambda_r c_r - \lambda_{r-1})/n \\
&= (\mu_1\lambda_0 + (\lambda_{r-1}c_{r-1} - \lambda_{r-2})n_r - \lambda_{r-1})/n \\
&= (\mu_1\lambda_0 + \lambda_{r-1}(c_{r-1}n_r - 1) - \lambda_{r-2}n_r)/n \\
&= (\mu_1\lambda_0 + \lambda_{r-1}n_{r-1} - \lambda_{r-2}n_r)/n \\
&= \cdots \\
&= (\mu_1\lambda_0 + \lambda_1 n_1 - \lambda_0 n_2)/n \\
&= \lambda_1.
\end{aligned}$$

(2) Suppose $\lambda_0 \equiv 0 \pmod{n}$, then $\lambda_1 = \lceil \lambda_0/e_1 \rceil = \lceil \mu_1\lambda_0/n_1 \rceil = \mu_1\lambda_0/n_1$.

Assume that $\lambda_k = \mu_k\lambda_{k-1}/n_k$ for some integer k with $1 \leq k < r$. Then

$$\begin{aligned}
\lambda_{k+1} &= \lceil \lambda_k/e_{k+1} \rceil = \lceil \mu_{k+1}\lambda_k/n_{k+1} \rceil \\
&= \lceil \mu_{k+1}\mu_k\lambda_{k-1}/(n_k n_{k+1}) \rceil \\
&= \lceil \mu_{k+1}\lambda_{k-1}/n_k \rceil.
\end{aligned}$$

Since $\gcd(n_k, \mu_k) = 1$, we have $\lambda_{k+1} = \mu_{k+1}\lambda_{k-1}/n_k = \mu_{k+1}\lambda_k/\mu_k = \mu_{k+1}\lambda_k/n_{k+1}$. By induction, we have $\lambda_i = \mu_i\lambda_{i-1}/n_i$ for $1 \leq i \leq r$.

Next, we suppose that $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$. Then

$$\lambda_1 = \lceil \lambda_0/e_1 \rceil = \lceil \mu_1\lambda_0/n_1 \rceil = \lceil (\mu_1\lambda_0 + 1)/n_1 - 1/n_1 \rceil = (\mu_1\lambda_0 + 1)/n_1.$$

Assume that $\lambda_j = (\mu_j\lambda_{j-1} + 1)/n_j$ for some integer j with $1 \leq j < r$. We have

$$\begin{aligned}
\mu_{j+1}\lambda_j + 1 &= \frac{n_{j+1}}{e_{j+1}}\lambda_j + 1 \\
&= n_{j+1}(c_j - e_j)\lambda_j + 1 \\
&= n_{j+1}\left(c_j - \frac{n_j}{\mu_j}\right)\lambda_j + 1 \\
&= n_{j+1}c_j\lambda_j - n_{j+1}\lambda_j \cdot \frac{n_j}{\mu_j} + 1
\end{aligned}$$

$$
\begin{aligned}
&= n_{j+1}c_j\lambda_j - n_{j+1}\cdot\frac{n_j}{\mu_j}\cdot\frac{\mu_j\lambda_{j-1}+1}{n_j}+1\\
&= n_{j+1}c_j\lambda_j - \mu_j\lambda_{j-1} - 1 + 1\\
&= n_{j+1}(c_j\lambda_j - \lambda_{j-1})\\
&= n_{j+1}\lambda_{j+1}.
\end{aligned}
$$

By induction, we have $\lambda_i = (\mu_i\lambda_{i-1}+1)/n_i$ for $1 \le i \le r$.

(3) Suppose that $\lambda_0 \equiv 0 \pmod{n}$, then we have $\lambda_i = \mu_i\lambda_{i-1}/n_i$ for $1 \le i \le r$ from (2). Thus

$$
\lambda_i c_i - \lambda_{i+1} = \lambda_i c_i - \lambda_i/e_{i+1} = \lambda_i c_i - \lambda_i(c_i - e_i) = \lambda_i e_i = (\lambda_{i-1}/e_i)\cdot e_i = \lambda_{i-1}.
$$

Assume that $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$, then $\lambda_j = (\mu_j\lambda_{j-1}+1)/n_j$ for $1 \le j \le r$ from (2). Thus

$$
\begin{aligned}
\lambda_j c_j - \lambda_{j+1} &= \lambda_j c_j - (\mu_{j+1}\lambda_j + 1)/n_{j+1}\\
&= \lambda_j c_j - \lambda_j(c_j - e_j) - \frac{1}{n_{j+1}}\\
&= \frac{\mu_j\lambda_{j-1}+1}{n_j}\cdot\frac{n_j}{\mu_j} - \frac{1}{n_{j+1}}\\
&= \lambda_{j-1} + \frac{1}{\mu_j} - \frac{1}{n_{j+1}} = \lambda_{j-1}.
\end{aligned}
$$

When $\lambda_0 \equiv 0 \pmod{n}$, we have

$$
\lambda_{r+1} = \lambda_r c_r - \lambda_{r-1} = \mu_r\lambda_{r-1}c_r/n_r - \lambda_{r-1} = \lambda_{r-1} - \lambda_{r-1} = 0.
$$

When $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$, we have

$$
\lambda_{r+1} = \lambda_r c_r - \lambda_{r-1} = (\mu_r\lambda_{r-1}+1)c_r/n_r - \lambda_{r-1} = \lambda_{r-1} + 1 - \lambda_{r-1} = 1.
$$

(4) Let $\mu^{'}$ be the positive integer determined by $\mu\mu^{'} \equiv 1 \pmod{n}$ with $1 \le \mu^{'} < n$. Then $n/\mu^{'} = [[c_r, \ldots, c_1]]$. Thus, by (1), we have $\lambda_r = (\mu^{'}\lambda_{r+1} + \lambda_0)/n$.

When $\lambda_0 \equiv 0 \pmod{n}$, we have $\lambda_{r+1} = 0$ from (3), and then $\lambda_r = \lambda_0/n$.

When $\mu\lambda_0 + 1 \equiv 0 \pmod{n}$, we have $\lambda_r = (\mu^{'} + \lambda_0)/n = \lceil \lambda_0/n \rceil$ following (3) and the definition of μ'. $\square$

1.5. Results of Konno and Nagashima

In 2012, Konno and Nagashima consider the Brieskorn hypersurface singularities $(V_{a_0,a_1,a_2}, o) := (\{x_0^{a_0} + x_1^{a_1} = x_2^{a_2}\}, o)$, where a_i's are integers and $2 \leq a_0 \leq a_1 \leq a_2$, and give the concrete descriptions of the fundamental cycle and the maximal ideal cycle over (V_{a_0,a_1,a_2}, o). We consider the two-dimensional Brieskorn complete intersection singularity which is a generalization of Brieskorn hypersurface singularity. In order to compare with the results of Konno and Nagashima, we mainly review the main results of Konno and Nagashima in this section.

Let $f = x_i^{a_i} + x_j^{a_j}$ and let $C \subset \mathbb{C}^2$ be the plane curve defined by $f = 0$. We define the positive integers d, n_1 and n_2 as follows:

$$d := \operatorname{lcm}(a_i, a_j),\ \ n_1 := a_i/\gcd(a_i, a_j),\ \ n_2 := a_j/\gcd(a_i, a_j).$$

In addition, we define the non-negative integers μ_1, μ_2 by the following conditions:

$$n_2\mu_1 + 1 \equiv 0 \pmod{n_1}, \quad 0 \leq \mu_1 < n_1,$$

$$n_1\mu_2 + 1 \equiv 0 \pmod{n_2}, \quad 0 \leq \mu_2 < n_2.$$

Let $\phi : Y \longrightarrow \mathbb{C}^2$ be the minimal embedded good resolution of the curve singularity (C, o) with exceptional set F and $\bar{C}$ the strict transform of C. Using a result in [**34**, Theorem 2.3], Konno and Nagashima give the following results:

- F is a chain of rational curves with unique (-1)-curve F_0.
- The multiplicity of the zero divisor $\operatorname{div}_Y(f \circ \phi)$ along F_0 is d.
- The strict transform $\bar{C}$ of C has $\gcd(a_i, a_j)$ irreducible components.

The weighted dual graph of the minimal embedded good resolution of C is given as in Figure 1.4.

In the Figure 1.4, F_{m,ν_m} is the exceptional curve arising from C_{n_m,μ_m} with self-intersection number $-c_{m,\nu_m}$, where $n_m/\mu_m = [[c_{m,1}, \ldots, c_{m,s_m}]]$, and ρ_{m,ν_m} is the multiplicity of the zero divisor $\operatorname{div}_Y(f \circ \phi)$ along F_{m,ν_m}, where $m = 1, 2$ and $1 \leq \nu_m \leq s_m$ (see [**13**, §2]).

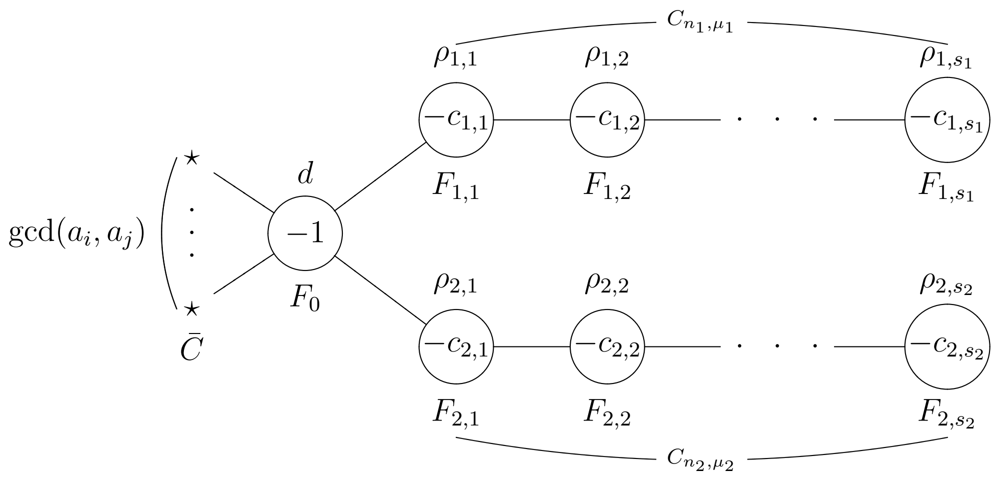


FIGURE 1.4.

For $i \in \{0,1,2\}$, we define the integers $l,\ l_i,\ \alpha_i$ as follows:

$$l := \mathrm{gcd}(a_0, a_1, a_2),\ l_i := \frac{\mathrm{gcd}(a_j, a_k)}{l},\ \alpha_i := \frac{a_i}{l_j l_k l} \quad (\{i,j,k\} = \{0,1,2\}).$$

Furthermore, we define p_0, p_1, p_2 be the integers determined by

$$p_i \alpha_j \alpha_k l_i + 1 \equiv 0 \pmod{\alpha_i}, \quad 0 \le p_i < \alpha_i,\ \{i,j,k\} = \{0,1,2\}.$$

When $\alpha_w > 1$, we put $\alpha_w / p_w = [[d_{w,1}, d_{w,2}, \ldots, d_{w,r_w}]]$. For $w \in \{0,1,2\}$, let

$$e_{w,\nu} := [[d_{w,\nu}, d_{w,\nu+1}, \ldots, d_{w,r_w}]],$$

where $1 \le \nu \le r_w$.

By [**27**], there exists a resolution $\pi : (\tilde{X}, E_\pi) \longrightarrow (V_{a_0,a_1,a_2}, o)$ where $E_\pi := \pi^{-1}(o)$ is the exceptional set such that the weighted dual graph of E_π is as in Figure 1.5.

Theorem 1.57 ([**13**, Proposition 1.3, Theorem 2.1])**.** *The genus g and the self-intersection number $-d_0$ of E_0 are given respectively as follows:*

$$2g - 2 = l(l_0 l_1 l_2 l - l_0 - l_1 - l_2),\ d_0 = l\left(\sum_{w=0}^{2} \frac{p_w l_w}{\alpha_w} + \frac{1}{\alpha_0 \alpha_1 \alpha_2}\right).$$

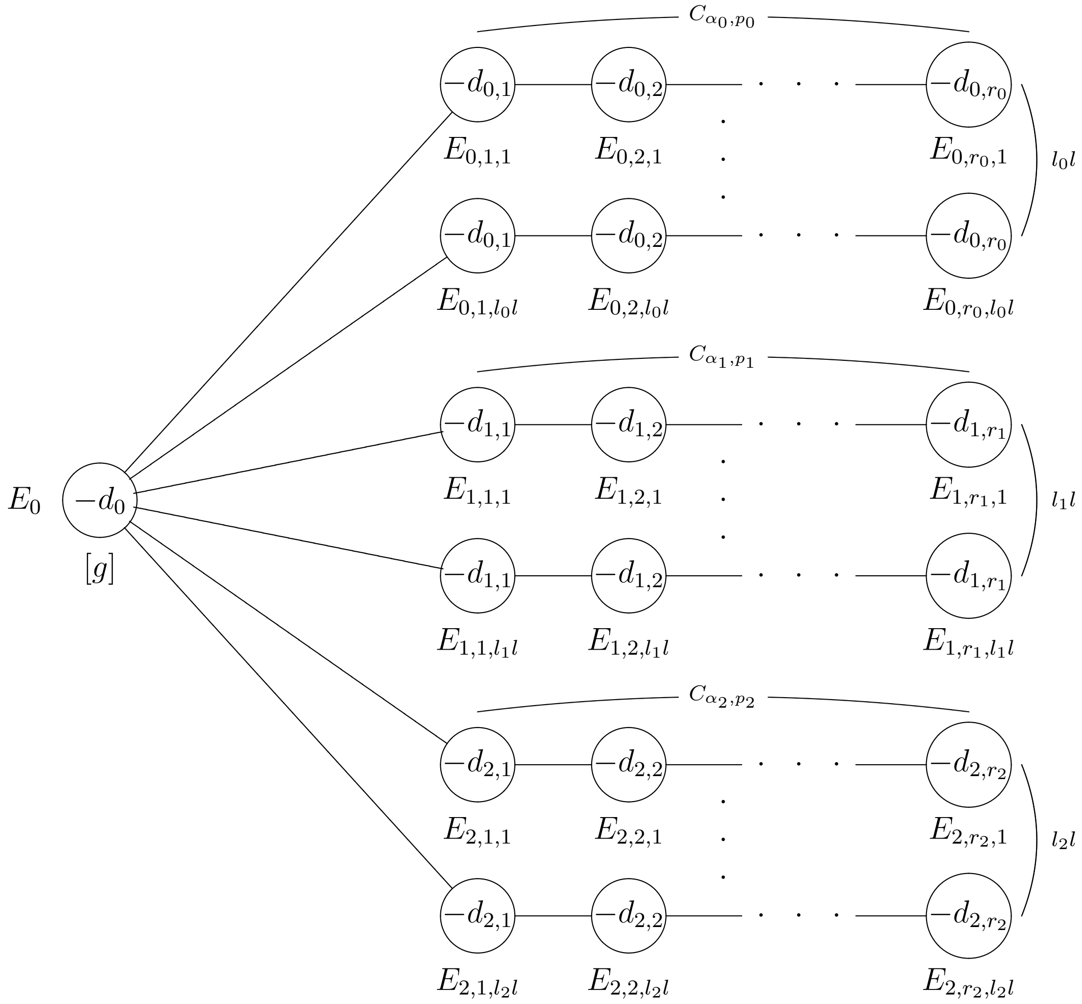


FIGURE 1.5.

Furthermore, let $Z^{(k)} := (x_k)_{E_\pi},\ k = 0,1,2$. *Then*

$$Z^{(k)} = \lambda_0^{(k)} E_0 + \sum_{w=0}^{2} \sum_{\nu=1}^{r_w} \sum_{\xi=1}^{l_w l} \lambda_{w,\nu,\xi}^{(k)} E_{w,\nu,\xi} \quad (0 \le k \le 2),$$

where $\lambda_0^{(k)}$ *and the sequence* $\{\lambda_{w,\nu,\xi}^{(k)}\}$ *are determined by the following:*

$$\lambda_{w,0,\xi}^{(k)} := \lambda_0^{(k)} := \alpha_i \alpha_j l_k \quad (\{i,j,k\} = \{0,1,2\}),$$

$$\lambda_{w,r_w+1,\xi}^{(k)} := \begin{cases} 1 & \text{if } w = k, \\ 0 & \text{if } w \ne k, \end{cases}$$

$$\lambda_{w,\nu-1,\xi}^{(k)} = \lambda_{w,\nu,\xi}^{(k)} d_{w,\nu} - \lambda_{w,\nu+1,\xi}^{(k)}.$$

Lemma 1.58 ([**13**, Lemma 3.8]). *We have* $-(Z^{(k)})^2 = l_k l \lceil \alpha_i \alpha_j l_k / \alpha_k \rceil$, *where* $\{i, j, k\} = \{0, 1, 2\}$.

Theorem 1.59 ([**13**, Theorem 1.4]). *Let*

$$Z = \theta_0 E_0 + \sum_{w=0}^{2} \sum_{\nu=1}^{r_w} \sum_{\xi=1}^{l_w l} \theta_{w,\nu,\xi} E_{w,\nu,\xi}$$

be the fundamental cycle for resolution π. *Then* θ_0 *and the sequence* $\{\theta_{w,\nu,\xi}\}$ *are defined by the following:*

$$\theta_{w,0,\xi} := \theta_0 := \begin{cases} \alpha_0\alpha_1\alpha_2 & \text{if } \alpha_2 \le l_2, \\ \alpha_0\alpha_1 l_2 & \text{if } \alpha_2 \ge l_2, \end{cases}$$

$$\theta_{w,\nu,\xi} = \lceil \theta_{w,\nu-1,\xi}/e_{w,\nu} \rceil, \quad 1 \le \nu \le r_w.$$

Proposition 1.60 ([**13**, Proposition 1.6]). *The self-intersection number of the fundamental cycle is given by*

$$-Z^2 = \begin{cases} l\alpha_0\alpha_1\alpha_2 & if\ \alpha_2 \le l_2, \\ l_2 l \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil & if\ \alpha_2 \ge l_2. \end{cases}$$

Lemma 1.61 ([**13**, Theorem 3.2]). *We have* $Z = Z^{(2)}$ *if and only if* $\alpha_2 \ge l_2$.

The arithmetic genus of the fundamental cycle Z, namely,

$$1 - \chi(Z) = (1/2)Z(K_{\tilde{X}} + Z) + 1$$

is called the fundamental genus of (V_{a_0,a_1,a_2}, o). This invariant is independent of the resolution and denoted by p_f.

Theorem 1.62 ([**13**, Theorem 1.7]). *The fundamental genus* p_f *of* (V_{a_0,a_1,a_2}, o), $2 \le a_0 \le a_1 \le a_2$ *is given as follows.*

(i) If $\alpha_2 \le l_2$, *then*

$$p_f = \frac{1}{2} l \left\{ \mathrm{lcm}(a_0, a_1, a_2) - \alpha_1\alpha_2 l_0 - \alpha_0\alpha_2 l_1 - \alpha_0\alpha_1 l_2 - \alpha_0\alpha_1\alpha_2 + 1 \right\} + 1.$$

(ii) If $\alpha_2 \geq l_2$, then

$$p_f = \frac{1}{2}\left\{(a_0-1)(a_1-1) - \left(2\left\lceil\frac{\alpha_0\alpha_1 l_2}{\alpha_2}\right\rceil - 1\right)\gcd(a_0,a_1)+1\right\}.$$

Theorem 1.63 ([**13**, Theorem 3.1])**.** *We have $Z^{(2)} \leq Z^{(1)} \leq Z^{(0)}$. In particular, $Z^{(2)}$ is the maximal ideal cycle for resolution π.*

Theorem 1.64 ([**13**, Theorem 3.2])**.** *The maximal ideal cycle coincides with the fundamental cycle for resolution π if and only if $\alpha_2 \geq l_2$.*

Example 1.65 ($\alpha_2 \geq l_2$)**.** If $(a_0, a_1, a_2) = (6, 20, 45)$, then $l = 1,\ l_0 = 5,\ l_1 = 3,\ l_2 = 2,\ \alpha_0 = 1,\ \alpha_1 = 2,\ \alpha_2 = 3,\ p_0 = 0,\ p_1 = 1,\ p_2 = 2$. By Theorem 1.57, we obtain that $d_0 = 3$ and $g = 11$. Hence the weighted dual graph of the maximal ideal cycle $Z^{(2)}$ is as in Figure 1.6.

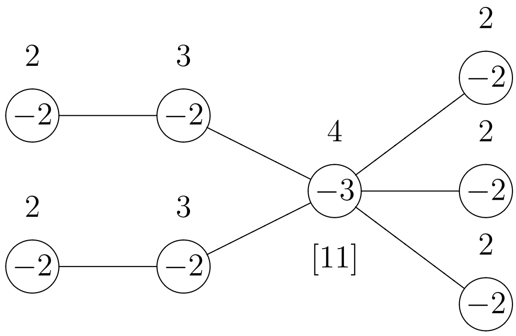


FIGURE 1.6.

Note that we have $\alpha_2 > l_2$, and by Theorem 1.59, Theorem 1.63 and Theorem 1.57, we can compute that the fundamental cycle coincides with the maximal ideal cycle. Furthermore, following Theorem 1.62, we have $p_f = 45$.

Example 1.66 ($\alpha_2 < l_2$)**.** If $(a_0, a_1, a_2) = (15, 18, 20)$, then $l = 1,\ l_0 = 2,\ l_1 = 5,\ l_2 = 3,\ \alpha_0 = 1,\ \alpha_1 = 3,\ \alpha_2 = 2,\ p_0 = 0,\ p_1 = 2,\ p_2 = 1$. By Theorem 1.57, we obtain that $d_0 = 5$ and $g = 11$. Hence the weighted dual graph of the maximal ideal cycle $Z^{(2)}$ is as in Figure 1.7.

Following Theorem 1.59, we obtain that the weighted dual graph of the fundamental cycle is as in Figure 1.8.

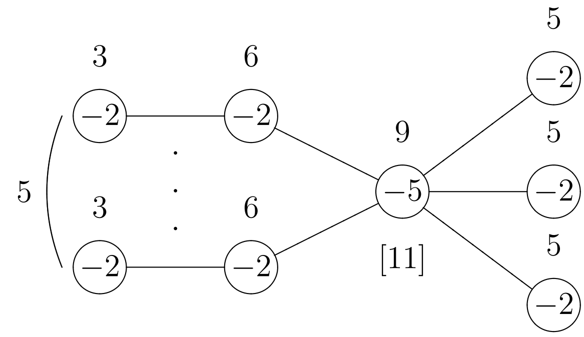


FIGURE 1.7.

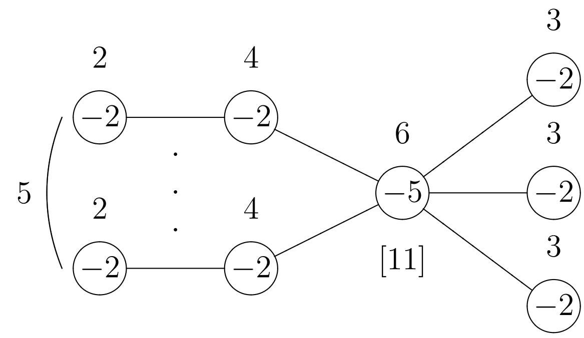


FIGURE 1.8.

From Figure 1.7 and Figure 1.8, we have that the maximal ideal cycle $Z^{(2)}$ does not coincide with the fundamental cycle Z. Moreover, by Theorem 1.62, we obtain that $p_f = 72$.

Chapter 2

The main results

In this chapter, we consider a germ $(X, o) \subset (\mathbb{C}^m, o)$ of a complete intersection singularity of Brieskorn type defined by

$$X = \{(x_i) \in \mathbb{C}^m | q_{j1}x_1^{a_1} + \cdots + q_{jm}x_m^{a_m} = 0, \quad j = 3, \ldots, m\},$$

where $a_i \geq 2$ are integers. We assume that (X, o) is an isolated singularity. Then (X, o) is a normal surface singularity by Serre's criterion for normality. Neumann [**24**] proved that the universal abelian cover of a weighted homogeneous normal surface singularity with rational homology sphere link is a complete intersection surface singularity of this type. The aim of this chapter is to identify the maximal ideal cycle on the minimal good resolution of (X, o). We give concrete descriptions of the maximal ideal cycle and the fundamental cycle, and a condition for the coincidence of these cycles.

This chapter is organized as follows. In Section 2.1, we give the construction of a partial resolution of (X, o) with cyclic quotient singularities. In Section 2.2, we compute the zero divisors of the pull-back of the coordinate functions $x_1, x_2, \ldots, x_m$. In Section 2.3, we compute the fundamental cycle, the canonical cycle and the fundamental genus. In Section 2.4, we identify the maximal ideal cycle and give a condition for the coincidence of the fundamental cycle and the maximal ideal cycle. In Section 2.5, we give a condition for (X, o) to be a Kodaira singularity following Konno and Nagashima.

2.1. The construction of a partial resolution with cyclic quotient singularities

Definition 2.1. A Brieskorn polynomial is a polynomial of the form

$$c_1 x_1^{a_1} + \cdots + c_m x_m^{a_m}, \quad c_i \in \mathbb{C}$$

where $a_i \geq 2$ are integers for $i = 1, \ldots, m$.

Let $(X, o) \subset (\mathbb{C}^m, o)$ be a germ of a complete intersection singularity of Brieskorn type defined by

$$X = \{(x_i) \in \mathbb{C}^m | q_{j1} x_1^{a_1} + \cdots + q_{jm} x_m^{a_m} = 0, \ \ j = 3, \ldots, m\},$$

where $a_i \geq 2$ are integers. We assume that (X, o) is an isolated singularity; this condition is equivalent to that every maximal minor of the matrix (q_{ji}) does not vanish (see [**10**, §7]). Therefore, by row operations and a diagonal linear change of coordinates, we may assume that

$$(q_{ij}) = \begin{pmatrix} p_3 & q_3 & -1 & 0 & \cdots & 0 \\ p_4 & q_4 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_m & q_m & 0 & 0 & \cdots & -1 \end{pmatrix},$$

where $p_i, q_i \neq 0$ and $p_i q_j \neq p_j q_i$ for $i \neq j$.

Suppose that $f : \tilde{X} \longrightarrow X$ is the minimal good resolution and E the exceptional set. Assume that E is not a chain of rational curves. Then the dual graph of E is star-shaped. Let E_0 denote the central curve of E and $f' : \tilde{X} \to X'$ the morphism which contracts the divisor $E - E_0 \subset \tilde{X}$. Then X' has cyclic quotient singularities along the exceptional set $E' := f'(E_0)$ and f' is the minimal resolution of those singularities. Thus we can read the weighted dual graph of E from the information of $E' \subset X'$ and those cyclic quotient singularities.

In [**13**, §2], Konno and Nagashima constructed a good resolution of the hypersurface singularity $\{x_1^{a_1} + x_2^{a_2} = x_3^{a_3}\}$ via cyclic covering as an application of

Tomaru's results [**36**] and [**34**]. We adopt their method to obtain a good resolution of (X, o) and the information of the divisors on it.

The singularity (X, o) can be obtained by a sequence of branched cyclic coverings over $\mathbb{C}^2$ as follows. Let $f_j = p_j x_1^{a_1} + q_j x_2^{a_2}$ for $j = 3, \ldots, m$. Put $X_2 = \mathbb{C}^2$ and $X_k = \{f_k = x_k^{a_k}\} \subset X_{k-1} \times \mathbb{C}$ for $k \geq 3$, where x_k is the coordinate function of the second component $\mathbb{C}$. Then we have the sequence of coverings

$$X = X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_2 = \mathbb{C}^2.$$

We shall construct the sequence of branched coverings

$$\tilde{X}_m \xrightarrow{\pi_m} \tilde{X}_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_3} \tilde{X}_2,$$

where $\tilde{X}_2$ is a partial embedded resolution of the branch locus of $X_3 \longrightarrow X_2 = \mathbb{C}^2$, and for each $k \geq 3$, $\tilde{X}_k$ is a partial resolution of the singularity of X_k with irreducible exceptional set and cyclic quotient singularities. Then we obtain that $X' = \tilde{X}_m$.

For $2 \leq k \leq m$ and $1 \leq i \leq k$, we define positive integers d_{ik}, n_{ik} and e_{ik} as follows:

$$\begin{aligned}
d_{ik} &:= \operatorname{lcm}(a_1, \ldots, \hat{a}_i, \ldots, a_k), \\
n_{ik} &:= \frac{a_i}{\gcd(a_i, d_{ik})}, \\
e_{ik} &:= \frac{d_{ik}}{\gcd(a_i, d_{ik})}.
\end{aligned}$$

(The symbol ˆ in the definition of d_{ik} indicates an omitted term.) In addition, we define integers μ_{ik} by the following condition:

$$e_{ik}\mu_{ik} + 1 \equiv 0 \pmod{n_{ik}}, \quad 0 \leq \mu_{ik} < n_{ik}. \tag{2.1}$$

We also write

$$d_{k-1} := d_{kk}, \ d_m := \operatorname{lcm}(a_1, \ldots, a_m),$$

$$n_k := n_{kk}, \ e_k := e_{kk}, \ \mu_k := \mu_{kk}.$$

We can easily see that

$$d_k = d_{ik} n_{ik} = a_i e_{ik}, \tag{2.2}$$

$$\gcd(n_{ik}, n_{jk}) = 1 \quad (1 \le i < j \le k \le m). \tag{2.3}$$

Let $f_i = x_i$ for $i \in \{1, 2\}$, and $f_i = p_i x_1^{a_1} + q_i x_2^{a_2}$ for $i \in \{3, \dots, m\}$, where $p_i, q_i \ne 0$ and $p_i q_j \ne p_j q_i$ for $i \ne j$. For $i \in \{1, \dots, m\}$, let $C_i \subset \mathbb{C}^2$ and $C \subset \mathbb{C}^2$ be the plane curves defined by $f_i = 0$ and $\prod_{i=1}^m f_i = 0$, respectively. Then $C = \sum_{i=1}^m C_i$ is a reduced divisor.

Lemma 2.2. *Let $\phi : Y \longrightarrow \mathbb{C}^2$ be the minimal embedded good resolution of the curve singularity (C, o) with exceptional set F. Let $\bar{C}_i \subset Y$ be the strict transform of C_i. Then we have the following.*

(1) *F is a chain of rational curves with unique (-1)-curve. Let $F_0 \subset F$ denote the (-1)-curve.*
(2) *$\bigcup_{i=3}^m \bar{C}_i$ does not intersect any component of $F - F_0$.*
(3) *$\bar{C}_1$ and $\bar{C}_2$ intersect distinct ends of F if F is not irreducible.*
(4) *For $i \ge 3$, each $\bar{C}_i$ has $\gcd(a_1, a_2)$ components.*
(5) *The multiplicity of the zero divisor $\mathrm{div}_Y(f_i \circ \phi)$ along F_0 is e_{i2} for $i \in \{1, 2\}$, and d_2 for $i \ge 3$.*
(6) *For $i \in \{1, 2\}$, the weighted dual graph of the minimal connected chain of curves with ends F_0 and $\bar{C}_i$ is as follows:*

$$(F_0) \;—\; -c_{i1} \;—\; \cdot \;\; \cdot \;\; \cdot \;—\; -c_{is_i} \;—\; (\bar{C}_i)$$

where $n_{i2}/\mu_{i2} = [[c_{i1}, \dots, c_{is_i}]]$.

Proof. From the above notation, we have

$$d_2 = \mathrm{lcm}(a_1, a_2),$$

$$n_{12} = e_{22} = a_1 / \gcd(a_1, a_2),$$

$$n_{22} = e_{12} = a_2 / \gcd(a_1, a_2).$$

Let $f'_i = \bar{x}_i^{e_{i2}}$ for $i \in \{1,2\}$ and $f'_i = p_i\bar{x}_1^{d_2} + q_i\bar{x}_2^{d_2}$ for $i \in \{3,\dots,m\}$. For $i \in \{1,\dots,m\}$, let $C'_i \subset \mathbb{C}^2_{(\bar{x}_1,\bar{x}_2)}$ and $C' \subset \mathbb{C}^2_{(\bar{x}_1,\bar{x}_2)}$ be the plane curve defined by $f'_i = 0$ and $\prod_{i=1}^m f'_i = 0$, respectively. Let $\Psi : \mathbb{C}^2_{(\bar{x}_1,\bar{x}_2)} \longrightarrow \mathbb{C}^2_{(x_1,x_2)}$ be the holomorphic map defined by $x_1 = \bar{x}_1^{n_{22}}$, $x_2 = \bar{x}_2^{n_{12}}$. Since $d_2 = a_1a_2/\gcd(a_1,a_2) = a_1n_{22} = a_2n_{12}$, we have $\Psi(C') = C$. The map Ψ can be regarded as the quotient map by the natural action to $\mathbb{C}^2_{(\bar{x}_1,\bar{x}_2)}$ of the group

$$G = \left\langle \begin{pmatrix} \epsilon_{n_{22}} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_{n_{12}} \end{pmatrix} \right\rangle,$$

where $\epsilon_{n_{i2}}$ is the primitive n_{i2}-th root of unity $\exp(2\pi\sqrt{-1}/n_{i2})$ for $i \in \{1,2\}$.

Let $\Phi' : \bar{N} \longrightarrow \mathbb{C}^2_{(\bar{x}_1,\bar{x}_2)}$ be the blowing up at the origin $\bar{o}$ of $\mathbb{C}^2_{(\bar{x}_1,\bar{x}_2)}$ and $\bar{E} := \Phi'^{-1}(\bar{o})$ the exceptional set. Then $\bar{N}$ is covered by two open sets U_0 and U_1, each of which is isomorphic to $\mathbb{C}^2$. The action of G is lifted onto $\bar{N}$ through Φ'. From (2.1), we have

$$e_{12}\mu_{12} + 1 \equiv 0 \pmod{n_{12}}, \quad 0 \le \mu_{12} < n_{12},$$

$$e_{22}\mu_{22} + 1 \equiv 0 \pmod{n_{22}}, \quad 0 \le \mu_{22} < n_{22}.$$

Then, from [**34**, Theorem 2.3], we can easily see that the quotient space $\bar{N}/G$ is covered by two cyclic quotient singularity spaces U_0/G and U_1/G whose respective types are $C_{n_{12},\mu_{12}}$ and $C_{n_{22},\mu_{22}}$; also the cyclic quotient singularity of type $C_{n_{12},\mu_{12}}$ (resp. $C_{n_{22},\mu_{22}}$) is located on $\psi(\bar{E}) \cap \psi({\Phi'}_*^{-1}C'_1)$ (resp. $\psi(\bar{E}) \cap \psi({\Phi'}_*^{-1}C'_2)$) and $\psi(\bar{E}) \simeq \mathbb{P}^1$, where $\psi : \bar{N} \longrightarrow \bar{N}/G$ is the quotient map. Furthermore, for $i \in \{3,\dots,m\}$, we have that $\psi({\Phi'}_*^{-1}C'_i)$ does not intersect $\psi({\Phi'}_*^{-1}C'_1)$ and $\psi({\Phi'}_*^{-1}C'_2)$.

Let $\eta : Y \longrightarrow \bar{N}/G$ be the minimal resolution of those two cyclic quotient singularities of type $C_{n_{12},\mu_{12}}$ and $C_{n_{22},\mu_{22}}$, and $\Phi : \bar{N}/G \longrightarrow \mathbb{C}^2_{(x_1,x_2)}$ the natural map to $\mathbb{C}^2_{(x_1,x_2)}$. Then $\phi = \Phi \circ \eta : Y \longrightarrow \mathbb{C}^2_{(x_1,x_2)}$ gives us the minimal embedded good resolution of the curve singularity (C,o) with exceptional set F. Thus we have the following diagram:

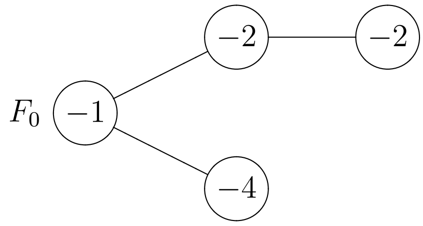
$$
\begin{tikzcd}
\bar{N} \arrow[r, "\Phi'"] \arrow[d, "\psi"'] & \mathbb{C}^2_{(\bar{x}_1,\bar{x}_2)} \arrow[d, "\Psi"] \\
\bar{N}/G \arrow[r, "\Phi"] & \mathbb{C}^2_{(x_1,x_2)} \\
Y \arrow[u, "\eta"] \arrow[ur, "\phi"'] &
\end{tikzcd}
$$

We see that the strict transform of $\psi(\bar{E})$ by η is necessarily the unique (-1)-curve. Thus we have (1), (6) and (2). Following (6), we have (3).

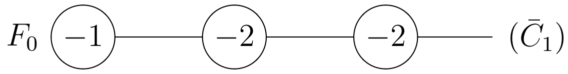
For $i \in \{3, \ldots, m\}$, the strict transform ${\Phi'}_*^{-1} C'_i$ of C'_i by Φ' consists of disjoint d_2 branches, each of which intersects $\bar{E}$ transversely at a point. Then $\psi({\Phi'}_*^{-1} C'_i)$ consists of $d_2/(n_{22}n_{12}) = \gcd(a_1, a_2)$ irreducible components, each of which intersects $\psi(\bar{E})$ transversely at a point, and then the strict transform $\bar{C}_i$ of C_i intersect F_0 transversely at $\gcd(a_1, a_2)$ distinct points by ϕ for $i \in \{3, \ldots, m\}$. Hence (4) holds.

The multiplicity of $f'_i \circ \Phi'$ along $\bar{E}$ is e_{i2} for $i \in \{1, 2\}$ and d_2 for $i \in \{3, \ldots, m\}$, then the multiplicity of $f_i \circ \Phi$ along $\psi(\bar{E})$ is also e_{i2} for $i \in \{1, 2\}$ and d_2 for $i \in \{3, \ldots, m\}$, and then the multiplicity of $f_i \circ \phi$ along F_0 is e_{i2} for $i \in \{1, 2\}$ and d_2 for $i \in \{3, \ldots, m\}$. Thus we have (5). $\square$

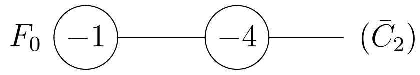
Example 2.3. Let $f_1 = x, f_2 = y$ and $f_3 = x^3 + y^4$. Then $d_2 = \operatorname{lcm}(3, 4) = 12$, $n_{12} = e_{22} = 3/\gcd(3, 4) = 3$, $n_{22} = e_{12} = 4/\gcd(3, 4) = 4$, $\mu_{12} = 2$ and $\mu_{22} = 1$. For $i \in \{1, 2, 3\}$, let $C_i \subset \mathbb{C}^2$ and $C \subset \mathbb{C}^2$ be the plane curve defined by $f_i = 0$ and $\prod_{i=1}^{3} f_i = 0$, respectively. Let $\phi : Y \longrightarrow \mathbb{C}^2$ be the minimal embedded good resolution of the curve singularity (C, o) with exceptional set F. Then the weighted dual graph of the exceptional set F is a chain of rational curves with unique (-1)-curve F_0 which is as follows:



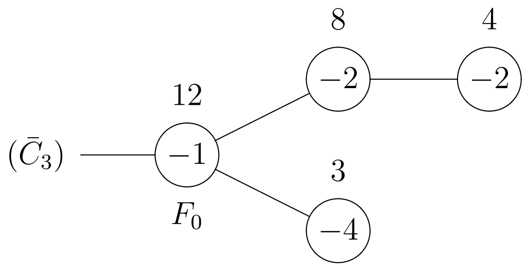
For $i \in \{1,2,3\}$, let $\bar{C}_i \subset Y$ be the strict transform of C_i. Then the weighted dual graph of the minimal connected chain of curves with ends F_0 and $\bar{C}_1$ is as follows:



and the multiplicity of the zero divisor $\mathrm{div}_Y(x \circ \phi)$ along F_0 is $e_{12} = 4$. The weighted dual graph of the minimal connected chain of curves with ends F_0 and $\bar{C}_2$ is as follows:



and the multiplicity of the zero divisor $\mathrm{div}_Y(y \circ \phi)$ along F_0 is $e_{22} = 3$. The strict transform $\bar{C}_3$ of C_3 has $\gcd(3,4) = 1$ component which intersects F_0 transversely at a point and the weighted dual graph of $\phi^* C_3$ is as follows:



Following the situation of Lemma 2.2, let $\eta : Y \longrightarrow \tilde{X}_2$ be the morphism which contracts the divisor $F - F_0 \subset Y$. Let $D_{i,2} = \eta_*(\bar{C}_i)$ and $F_2 = \eta_*(F_0)$. By Lemma 2.2, $\tilde{X}_2$ has only two singular points of types $C_{n_{12},\mu_{12}}$ and $C_{n_{22},\mu_{22}}$.

Let $\Phi : \tilde{X}_2 \longrightarrow \mathbb{C}^2$ be the natural projection and let $f_{j,2} = f_j \circ \Phi$. Suppose that $\tilde{X}_k$ and $\{f_{j,k}\}$ are obtained for $2 \leq k < m$. Then we define $\tilde{X}_{k+1}$ to be the normalization of a surface $\{f_{k+1,k} = x_{k+1}^{a_{k+1}}\} \subset \tilde{X}_k \times \mathbb{C}$, where we regard x_{k+1} as the coordinate function of the second component $\mathbb{C}$. Let $\pi_{k+1} : \tilde{X}_{k+1} \longrightarrow \tilde{X}_k$ be the natural morphism and $f_{j,k+1} = f_{j,k} \circ \pi_{k+1}$. Let F_k (resp. $D_{j,k}$) denote the fiber of F_2 (resp. $D_{j,2}$) on $\tilde{X}_k$. We have the following commutative diagram:

$$\begin{tikzcd}
 & (\tilde{X}_m, F_m) \arrow[r, "\pi_m"] \arrow[d] & \cdots \arrow[r, "\pi_4"] & (\tilde{X}_3, F_3) \arrow[r, "\pi_3"] \arrow[d] & (\tilde{X}_2, F_2) \arrow[d] & \\
(X, o) \arrow[r, equal] & (X_m, o) \arrow[r] & \cdots \arrow[r] & (X_3, o) \arrow[r] & (X_2, o) \arrow[r, equal] & \mathbb{C}^2
\end{tikzcd}$$

Theorem 2.4 (Tomaru [**36**, §3], cf. [**13**, Theorem 2.2])**.** *Let (U, o) be the cyclic quotient singularity of type $C_{n,\mu}$, $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{U,o}$, and $h \in \mathfrak{m}$. Assume that the zero divisor of the pull-back of h on the minimal resolution of (U, o) has the weighted dual graph as in Figure 2.1,*

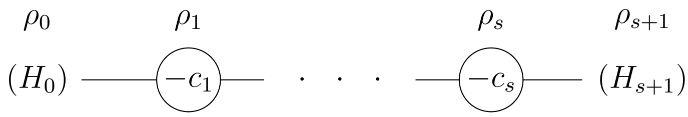


FIGURE 2.1.

where $n/\mu = [[c_1, \ldots, c_s]]$, $H_0 \cup H_{s+1}$ is the strict transform of $\{h = 0\}$ with irreducible components H_0 and H_{s+1}, and the ρ_i's denote multiplicities (if $n = 1$, then $(U, o) = (\mathbb{C}^2, o)$ and $s = 0$). Let a be a positive integer. We define integers α and p as follows. Let

$$\bar{a} = \frac{a}{\gcd(a, \operatorname{lcm}(\rho_0, \rho_{s+1}))}, \quad \bar{n} = \frac{n \gcd(a, \rho_0, \rho_1, \ldots, \rho_{s+1})}{\gcd(a, \rho_0, \rho_{s+1})},$$

and $\alpha = \bar{a}\bar{n}$. Then p is defined by the following condition:

$$p \equiv \frac{a}{\gcd(a, \rho_{s+1})}\mu\beta + \frac{\rho_{s+1}}{\gcd(a, \rho_{s+1})}\gamma \pmod{\alpha}, \quad 0 \le p < \alpha,$$

where β and γ are integers determined by

$$\frac{a}{\gcd(a, \rho_0)}\beta \equiv 1 \pmod{\rho_0/\gcd(a, \rho_0)}, \quad 0 \le \beta < \frac{\rho_0}{\gcd(a, \rho_0)},$$
$$\frac{\rho_0}{\gcd(a, \rho_0)}\gamma = \frac{a}{\gcd(a, \rho_0)}\beta - 1.$$

Then the normalization W of the a-fold covering of U defined by $z^a = h$ has exactly $\gcd(a, \rho_0, \ldots, \rho_{s+1})$ connected components. Each component W_i of W has *a cyclic quotient singularity of type $C_{\alpha,p}$ and the divisor of the function z on W_i has the multiplicity $\frac{\rho_j}{\gcd(a,\rho_j)}$ along the fiber of H_j for $j = 0, s+1$.*

Example 2.5 ([**36**, Example 3.5])**.** Let (U, o) be a cyclic quotient singularity of type $C_{30,7}$ and h an element of the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{U,o}$, such that the zero divisor of the pull-back of h on the minimal resolution of (U, o) has the weighted dual graph as in Figure 2.2.

$$\begin{array}{ccccccccccc} 30 & & 9 & & 15 & & 21 & & 27 & & 60 \\ (H_0) & \text{———} & -5 & \text{——} & -2 & \text{——} & -2 & \text{——} & -3 & \text{———} & (H_5) \end{array}$$

FIGURE 2.2.

Let W be the normalization of the 45-fold cyclic cover of U defined by $z^{45} = h$.

Since $a = 45$, $\rho_0 = 30$, and $\rho_5 = 60$, we have

$$\gcd(a, \rho_0, \rho_5) = \gcd(45, 30, 60) = 15,$$

$$\gcd(a, \rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5) = \gcd(45, 30, 9, 15, 21, 27, 60) = 3,$$

$$\bar{n} = \frac{30 \times 3}{15} = 6, \ \bar{a} = \frac{45}{\gcd(45, \operatorname{lcm}(30, 60))} = 3, \ \alpha = \bar{n}\bar{a} = 18.$$

Then $\beta = \gamma = 1$ and $p = 7$. Hence W has exactly 3 connected components. Each component W_i of W has a cyclic quotient singularity of type $C_{18,7}$, and the divisor of the function z on W_i has the multiplicities $\frac{30}{\gcd(30,45)} = 2$ and $\frac{60}{\gcd(45,60)} = 4$ along the fiber of H_0 and H_5, respectively.

Lemma 2.6. *For $2 \le k \le m$ and $j > k$, we have the following.*

(1) *$\tilde{X}_k$ and F_k are non-singular outside $F_k \cap \left(\bigcup_{i \le k} D_{i,k}\right)$.*

(2) *Each of the divisors $D_{j,k}$ and $D_{k+1,k+1}$ has $\prod_{i=2}^{k} \gcd(a_i, d_{i-1})$ components.*

(3) *Every point $x \in F_k \cap D_{k,k}$ is of type C_{n_k,μ_k} and the dual graph of the minimal embedded good resolution of the germ of the curve singularity $(F_k \cup D_{k,k}, x) \subset (\tilde{X}_k, x)$ is as follows:*

$$\begin{array}{ccccccc} (\bar{F}_k) & \text{———} & -c_1 & \text{——} \ \cdot \ \cdot \ \cdot \ \text{——} & -c_s & \text{———} & (\bar{D}_{k,k}) \end{array}$$

where $\bar{F}_k$ and $\bar{D}_{k,k}$ denote the strict transforms, and $n_k/\mu_k = [[c_1, \ldots, c_s]]$.

(4) $\mathrm{div}_{\tilde{X}_k}(x_k) = e_k F_k + D_{k,k}$.

(5) $\mathrm{div}_{\tilde{X}_k}(f_{j,k}) = d_k F_k + D_{j,k}$.

(6) *F_k is irreducible.*

Proof. In case $k = 2$, the assertion follows from Lemma 2.2. Assume that it holds for some $k \geq 2$. Let x be a point of $F_k \cap D_{j,k}$ and $U \subset \tilde{X}_k$ a sufficiently small neighborhood of x. Then the point $x \in \tilde{X}_k$ is of type $C_{1,0}$ and the multiplicity of $f_{k+1,k}$ along F_k is d_k by the assumption. We apply Theorem 2.4 to the germ (U, x) and the covering $x_{k+1}^{a_{k+1}} = f_{k+1,k}$; then α in the theorem is $n_{k+1} = n_{k+1k+1}$. Let W be the normalization of the covering. If $j > k+1$, putting $(s, \rho_0, \rho_{s+1}) = (0, d_k, 0)$, we obtain that W has exactly $\gcd(a_{k+1}, d_k)$ components, which are non-singular. Thus (1) holds by induction. Suppose $j = k+1$ and put $\rho_0 = d_k$ and $\rho_1 = 1$. Then W is an irreducible surface with a cyclic quotient singularity of type C_{n_j,μ_j}, and $\mathrm{div}_W(x_j) = e_j F_j + D_{j,j}$. These imply (2), (3) and (4). It also follows that F_k is locally irreducible. Since $\tilde{X}_k$ is a partial resolution of a normal surface singularity, the exceptional set $F_k \subset \tilde{X}_k$ is connected. Hence (6) holds. We have

$$a_j \mathrm{div}_W(x_j) = \mathrm{div}_W(f_{j,j}) = (\pi_j|_W)^*(d_k F_k + D_{j,k}).$$

By (2.2), we have $\pi_{k+1}^*(d_k F_k) = d_{k+1} F_{k+1}$. This proves (5). $\square$

For $1 \leq i \leq m$, we define integers $\hat{g}$ and $\hat{g}_i$ as follows:

$$\hat{g} := \frac{a_1 \cdots a_m}{\mathrm{lcm}(a_1, \ldots, a_m)},$$
$$\hat{g}_i := \frac{a_1 \cdots \hat{a}_i \cdots a_m}{\mathrm{lcm}(a_1, \ldots, \hat{a}_i, \ldots, a_m)}.$$

For $m = 3$, we have $\gcd(a_2, d_1) = \gcd(a_1, a_2) = \hat{g}_3 = \dfrac{a_1 a_2}{\mathrm{lcm}(a_1, a_2)}$. If we assume that $\prod_{i=2}^{k-1} \gcd(a_i, d_{i-1}) = \hat{g}_k$ for some integer k with $3 \leq k < m$. Then

$$\prod_{i=2}^{k} \gcd(a_i, d_{i-1}) = \hat{g}_k \cdot \gcd(a_k, d_{k-1}) = \frac{a_1 \cdots a_k}{\mathrm{lcm}(a_1, \ldots, a_k)} = \hat{g}_{k+1}.$$

Thus, by induction, we have $\prod_{i=2}^{m-1} \gcd(a_i, d_{i-1}) = \hat{g}_m$, and Lemma 2.6 immediately implies the following.

Lemma 2.7. *Let $X' = \tilde{X}_m$, $E' = F_m$, and $D_i = D_{i,m}$. Then we have the following.*

(1) *Every D_i is reduced and $D := \bigcup_{i=1}^{m} D_i$ is a disjoint union of irreducible components.*
(2) *The set of the singular points of X' is a subset of $E' \cap D$.*
(3) *D_m has $\hat{g}_m$ components.*
(4) *Suppose $x \in E' \cap D_m$. Then $x \in X'$ is a cyclic quotient singularity of type C_{n_m,μ_m} and the weighted dual graph of the minimal embedded good resolution of the germ $(E' \cup D_m, x) \subset (X', x)$ is as follows:*

$$(\bar{E}') \;\text{———}\; -c_1 \;\text{——}\; \cdot\;\cdot\;\cdot \;\text{——}\; -c_s \;\text{———}\; (\bar{D}_m)$$

where $\bar{E}'$ and $\bar{D}_m$ denote the strict transforms, and $n_m/\mu_m = [[c_1, \ldots, c_s]]$.
(5) $\dim_{X'}(x_m) = e_m E' + D_m$.

2.2. Zero divisors of the pull-back of the coordinate functions

We use the same notation as in Section 2.1. Let $f' : \tilde{X} \to X'$ be the minimal resolution of $X' = \tilde{X}_m$ and E the fiber of $E' = F_m$. Let $E_0 \subset E$ denote the strict transform of E' and $f : \tilde{X} \longrightarrow X$ be the composite of the resolution $f' : \tilde{X} \longrightarrow X'$ and the partial resolution $X' = \tilde{X}_m \longrightarrow X_m = X$. Clearly f is a good resolution of (X, o) with exceptional set E. We have the following diagram:

$$\begin{tikzcd}
E_0 \subset E \subset \tilde{X} \arrow[r, "f'"] \arrow[dr, "f"'] & \tilde{X}_m = X' \supset E' = F_m \arrow[d] \\
& X_m = X
\end{tikzcd}$$

For any non-zero function $h \in \mathcal{O}_{X,o}$, we write

$$\mathrm{div}_{\tilde{X}}(h) := \mathrm{div}_{\tilde{X}}(h \circ f) = (h)_E + H,$$

where $(h)_E$ is supported in E and H does not contain any irreducible component of E. Let

$$Z^{(i)} = (x_i)_E, \ \alpha_i := n_{im}, \ \beta_i := \mu_{im}.$$

Lemma 2.8 (Hurwitz formula). *Let R and S be non-singular compact algebraic curves and $\varphi : R \longrightarrow S$ be a surjective holomorphic map. Let $P_1, P_2, \ldots, P_l$ be the ramification points with ramification indices $e_1, e_2, \ldots, e_l$, respectively. Then*

$$2g(R) - 2 = \deg(\varphi)(2g(S) - 2) + \sum_{j=1}^{l}(e_j - 1),$$

where $g(R), g(S)$ are the genus of R, S, respectively.

Theorem 2.9. *Let g and $-c_0$ denote the genus and the self-intersection number of E_0, respectively. Then the weighted dual graph of the exceptional set E is as Figure 2.3, where the invariants are as follows:*

$$2g - 2 = (m-2)\hat{g} - \sum_{i=1}^{m} \hat{g}_i,$$

$$c_0 = \sum_{w=1}^{m} \frac{\hat{g}_w \beta_w}{\alpha_w} + \frac{a_1 \cdots a_m}{d_m^2},$$

$$\beta_w/\alpha_w = \begin{cases} [[c_{w,1}, \ldots, c_{w,s_w}]]^{-1} & if \ \alpha_w \geq 2, \\ 0 & if \ \alpha_w = 1. \end{cases}$$

Furthermore,

$$Z^{(i)} = \lambda_0^{(i)} E_0 + \sum_{w=1}^{m} \sum_{\nu=1}^{s_w} \sum_{\xi=1}^{\hat{g}_w} \lambda_{w,\nu,\xi}^{(i)} E_{w,\nu,\xi} \ (1 \leq i \leq m),$$

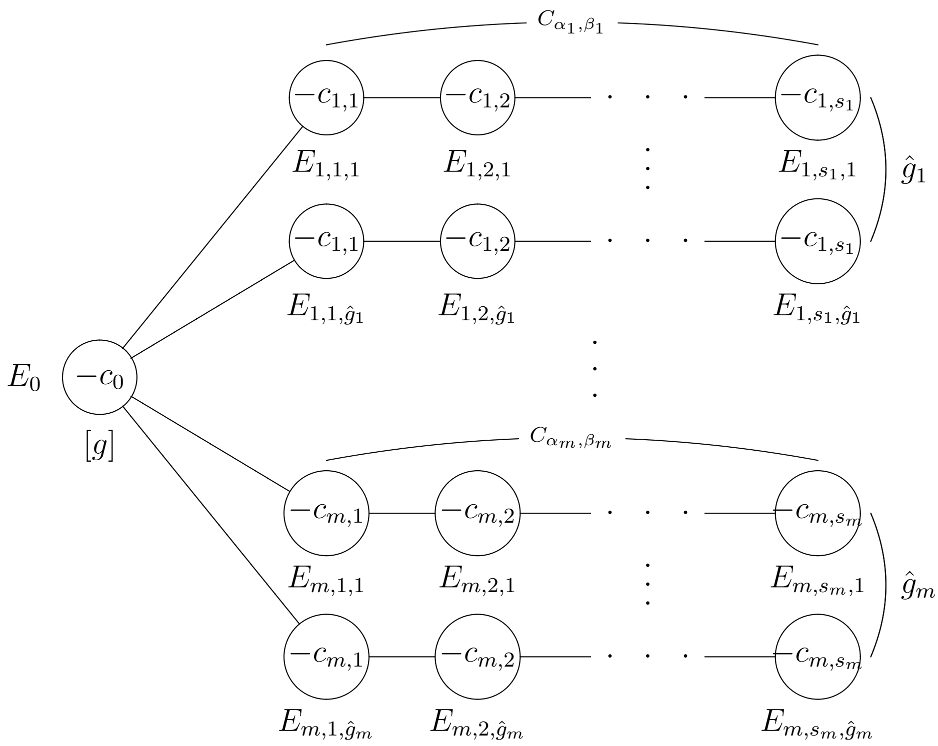


FIGURE 2.3.

where $\lambda_0^{(i)}$ *and the sequence* $\{\lambda_{w,\nu,\xi}^{(i)}\}$ *are determined by the following:*

$$\lambda_{w,0,\xi}^{(i)} := \lambda_0^{(i)} := e_{im},$$

$$\lambda_{w,s_w+1,\xi}^{(i)} := \begin{cases} 1 & if\ w = i \\ 0 & if\ w \neq i, \end{cases}$$

$$\lambda_{w,\nu-1,\xi}^{(i)} = \lambda_{w,\nu,\xi}^{(i)} c_{w,\nu} - \lambda_{w,\nu+1,\xi}^{(i)}.$$

The cycle $Z^{(i)}$ *is the smallest one among the cycles* $Z > 0$ *such that* $-Z$ *is nef and the coefficients of* E_0 *in* Z *is* e_{im}.

Proof. From (1) and (2) of Lemma 2.7, we see that the claims (3)-(5) of Lemma 2.7 also hold for every $i \in \{1, \dots, m\}$ instead of m, by taking permutations of variables. These data immediately show the dual graph except for c_0 and g. Since $\mathrm{div}_{X'}(x_i) = e_{im}E' + D_i$ by Lemma 2.7 (5), $\lambda_0^{(i)}$ should be e_{im} and the coefficient of the cycle $Z^{(i)}$ can be determined by the following:

$$0 = (Z^{(i)} + \bar{D}_i)E_{w,k,\xi} = \lambda^{(i)}_{w,k-1,\xi} - \lambda^{(i)}_{w,k,\xi}c_{w,k} + \lambda^{(i)}_{w,k+1,\xi}.$$

The last assertion follows from Lemma 1.53.

Recall that F_0 is the (-1)-curve on Y (see Lemma 2.2). Let $p : E_0 \longrightarrow F_0 \cong \mathbb{P}^1$ be the natural map and $\pi := \pi_3 \circ \cdots \circ \pi_m$. From the proof of Lemma 2.6 and Lemma 2.7 (3), we obtain the following.

- $\pi^*F_2 = (d_m/d_2)F_m$, and thus $\deg p = d := d_2a_3 \cdots a_m/d_m$.
- The ramification index of a point $x \in E_0$ which corresponds to a point of $E' \cap D_i$ is $d\gcd(a_1, a_2)/\hat{g}_i$ for $i \geq 3$, and $d/\hat{g}_i$ for $i = 1, 2$.

By Lemma 2.8,

$$\begin{aligned}
2g - 2 &= d(-2) + \sum_{i=3}^{m} \hat{g}_i \left(\frac{d\gcd(a_1, a_2)}{\hat{g}_i} - 1 \right) + \sum_{i=1}^{2} \hat{g}_i \left(\frac{d}{\hat{g}_i} - 1 \right) \\
&= (m-2)d\gcd(a_1, a_2) - \sum_{i=1}^{m} \hat{g}_i \\
&= \frac{(m-2)d_2a_3 \cdots a_m}{\operatorname{lcm}(a_1, \ldots, a_m)} \gcd(a_1, a_2) - \sum_{i=1}^{m} \hat{g}_i \\
&= (m-2)\hat{g} - \sum_{i=1}^{m} \hat{g}_i.
\end{aligned}$$

By Lemma 1.56 (1), we obtain (even in the case $s_w = 0$) that

$$\lambda^{(m)}_{w,1,\xi} = (\beta_w e_m + \lambda^{(m)}_{w,s_w+1,\xi})/\alpha_w.$$

Since the intersection number of E_0 and $\operatorname{div}_{\tilde{X}}(x_m)$ is zero,

$$\begin{aligned}
c_0e_m = c_0\lambda^{(m)}_0 &= \sum_{w=1}^{m} \hat{g}_w \lambda^{(m)}_{w,1,\xi} \\
&= \sum_{w=1}^{m-1} \frac{\hat{g}_w\beta_w e_m}{\alpha_w} + \frac{\hat{g}_m(\beta_m e_m + 1)}{\alpha_m}.
\end{aligned}$$

Hence

$$c_0 = \sum_{w=1}^{m} \frac{\hat{g}_w\beta_w}{\alpha_w} + \frac{\hat{g}_m}{\alpha_m e_m}.$$

Since $\hat{g}_m/\alpha_m e_m = \hat{g}_m a_m d_{m-1}/d_m^2 = a_1 \cdots a_m/d_m^2$, we obtain the assertion. $\square$

Example 2.10. Let $a_1 = \cdots = a_{m-3} = 2,\ a_{m-2} = 3,\ a_{m-1} = 4,\ a_m = 5,$ $m \geq 4$. Then

$$d_{1m} = \cdots = d_{m-3,m} = 60,\ d_{m-2,m} = 20,\ d_{m-1,m} = 30,\ d_{mm} = 12,\ d_m = 60;$$

$$e_{1m} = \cdots = e_{m-3,m} = 30,\ e_{m-2,m} = 20,\ e_{m-1,m} = 15,\ e_{mm} = 12;$$

$$n_{1m} = \cdots = n_{m-3,m} = 1,\ n_{m-2,m} = 3,\ n_{m-1,m} = 2,\ n_{mm} = 5;$$

$$\mu_{1m} = \cdots = \mu_{m-3,m} = 0,\ \mu_{m-2,m} = 1,\ \mu_{m-1,m} = 1,\ \mu_{mm} = 2;$$

$$\hat{g}_1 = \cdots = \hat{g}_{m-3} = 2^{m-4},\ \hat{g}_{m-2} = 2^{m-3},\ \hat{g}_{m-1} = 2^{m-4},\ \hat{g}_m = 2^{m-3},\ \hat{g} = 2^{m-3}.$$

By Theorem 2.9, we have $c_0 = 2^{m-3}$ and $g = (m-6)\cdot 2^{m-5}+1$. Then the weighted dual graph of E is as Figure 2.4. Furthermore, by Theorem 2.9, the zero divisors

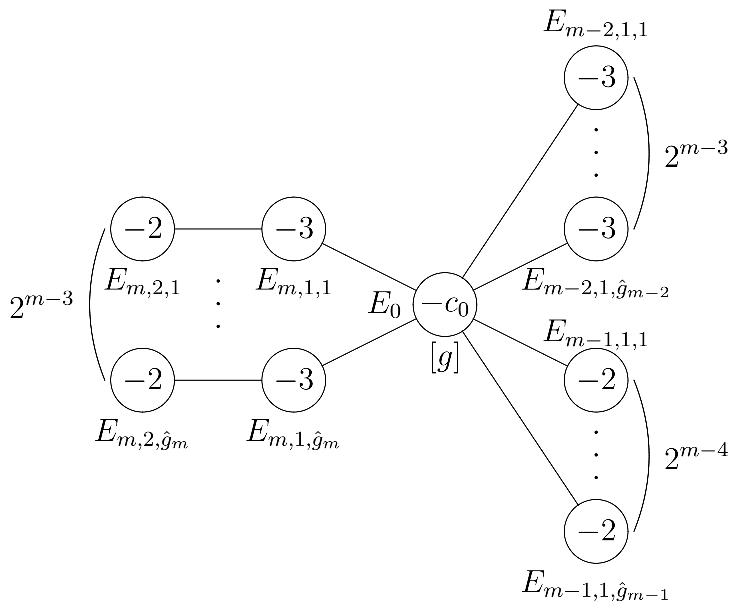


FIGURE 2.4.

of the pull-back of the coordinate functions $x_1, \ldots, x_m$ are as follows:

$$Z^{(1)} = \cdots = Z^{(m-3)}$$
$$= 30E_0 + 10 \cdot \sum_{\xi=1}^{\hat{g}_{m-2}} E_{m-2,1,\xi} + 15 \cdot \sum_{\xi=1}^{\hat{g}_{m-1}} E_{m-1,1,\xi} + \sum_{\xi=1}^{\hat{g}_m} (12E_{m,1,\xi} + 6E_{m,2,\xi});$$

$$Z^{(m-2)} = 20E_0 + 7 \cdot \sum_{\xi=1}^{\hat{g}_{m-2}} E_{m-2,1,\xi} + 10 \cdot \sum_{\xi=1}^{\hat{g}_{m-1}} E_{m-1,1,\xi} + \sum_{\xi=1}^{\hat{g}_m}(8E_{m,1,\xi} + 4E_{m,2,\xi});$$

$$Z^{(m-1)} = 15E_0 + 5 \cdot \sum_{\xi=1}^{\hat{g}_{m-2}} E_{m-2,1,\xi} + 8 \cdot \sum_{\xi=1}^{\hat{g}_{m-1}} E_{m-1,1,\xi} + \sum_{\xi=1}^{\hat{g}_m}(6E_{m,1,\xi} + 3E_{m,2,\xi});$$

$$Z^{(m)} = 12E_0 + 4 \cdot \sum_{\xi=1}^{\hat{g}_{m-2}} E_{m-2,1,\xi} + 6 \cdot \sum_{\xi=1}^{\hat{g}_{m-1}} E_{m-1,1,\xi} + \sum_{\xi=1}^{\hat{g}_m}(5E_{m,1,\xi} + 3E_{m,2,\xi}).$$

Note that we have $Z^{(m)} < Z^{(m-1)} < Z^{(m-2)} < Z^{(m-3)} = \cdots = Z^{(1)}$, and by computation, we have

$$\begin{aligned}
&-(Z^{(1)})^2 = \cdots = -(Z^{(m-3)})^2 = 15 \cdot 2^{m-3}, \\
&-(Z^{(m-2)})^2 = 7 \cdot 2^{m-3}, \\
&-(Z^{(m-1)})^2 = 4 \cdot 2^{m-3}, \\
&-(Z^{(m)})^2 = 3 \cdot 2^{m-3}.
\end{aligned}$$

Lemma 2.11. *For* $1 \le w \le m$, $-(Z^{(w)})^2 = \hat{g}_w \lceil e_{wm}/\alpha_w \rceil$.

Proof. Let $E_{i,0,\xi} = E_0$. From Theorem 2.9,

$$-Z^{(w)} E_{i,\nu,\xi} = \begin{cases} 1 & if\ s_w > 0\ and\ (i,\nu) = (w, s_w), \\ \hat{g}_w & if\ s_w = 0\ and\ \nu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lambda^{(w)}_{w,s_w,\xi} = \lceil e_{wm}/\alpha_w \rceil$ by Lemma 1.56 (4), we have

$$-Z^{(w)} Z^{(w)} = \hat{g}_w \lceil e_{wm}/\alpha_w \rceil.$$

□

In the situation of Example 2.10, by Lemma 2.11, we also have

$-(Z^{(1)})^2 = \cdots = -(Z^{(m-3)})^2 = \hat{g}_w \lceil e_{wm}/\alpha_w \rceil = 15 \cdot 2^{m-3},\ w \in \{1, \cdots, m-3\},$

$-(Z^{(m-2)})^2 = \hat{g}_{m-2} \lceil e_{m-2,m}/\alpha_{m-2} \rceil = 7 \cdot 2^{m-3},$

$-(Z^{(m-1)})^2 = \hat{g}_{m-1} \lceil e_{m-1,m}/\alpha_{m-1} \rceil = 4 \cdot 2^{m-3},$

$-(Z^{(m)})^2 = \hat{g}_m \lceil e_{mm}/\alpha_m \rceil = 3 \cdot 2^{m-3}.$

Lemma 2.12. [**13**, Lemma 4.3]. *The resolution $f : \tilde{X} \longrightarrow X$ is not the minimal good resolution, i.e., E_0 is a (-1)-curve and intersects at most two curves, if and only if $m = 3$ and $(a_1, a_2, a_3) = (2, 2, 2l + 1)$ for an integer $l > 0$.*

Proof. If f is not the minimal good resolution, X is a cyclic quotient singularity. It is well-known that a Gorenstein quotient surface singularity is a rational double point, hence a hypersurface. Thus the assertion follows from the result of hypersurface singularities [**13**, Lemma 4.3]. □

2.3. The fundamental cycle and the canonical cycle

Let Z_E denote the fundamental cycle on E, i.e., the smallest anti-nef cycle supported on E. Since (X, o) is a Gorenstein singularity, there exists a cycle Z_K such that $-Z_K$ is a canonical divisor of $\tilde{X}$. We call Z_K the canonical cycle on $\tilde{X}$. Let $\alpha = \prod_{w=1}^{m} \alpha_w$.

Assume that $a_1 \leq \cdots \leq a_m$. It follows from (2.2) that $e_{1m} \geq \cdots \geq e_{mm} = e_m$.

Theorem 2.13. *Let $\epsilon_{w,\nu} = [[c_{w,\nu}, \ldots, c_{w,s_w}]]$ if $s_w > 0$ (i.e., $\alpha_w > 1$), and let*

$$Z_E = \theta_0 E_0 + \sum_{w=1}^{m} \sum_{\nu=1}^{s_w} \sum_{\xi=1}^{\hat{g}_w} \theta_{w,\nu,\xi} E_{w,\nu,\xi}.$$

Then θ_0 and the sequence $\{\theta_{w,\nu,\xi}\}$ are determined by the following:

$$\theta_{w,0,\xi} := \theta_0 := \min(e_m, \alpha),$$

$$\theta_{w,\nu,\xi} = \lceil \theta_{w,\nu-1,\xi}/\epsilon_{w,\nu} \rceil \ (1 \leq \nu \leq s_w).$$

Proof. We follow the proof of [**13**, Theorem 1.4]. By Lemma 1.54, we only need to identify θ_0. Let u_w $(1 \leq w \leq m)$ be the integers determined by

$$\beta_w\theta_0 + u_w \equiv 0 \pmod{\alpha_w}, \quad 0 \leq u_w < \alpha_w.$$

Then

$$\theta_{w,1,\xi} = \lceil \theta_0\beta_w/\alpha_w \rceil = \frac{\theta_0\beta_w + u_w}{\alpha_w}.$$

Note that $\beta_w = s_w = 0$ if $\alpha_w = 1$. The condition $Z_E E_0 \leq 0$ is equivalent to that

$$\theta_0 c_0 \geq \sum_{w=1}^{m} \frac{\hat{g}_w(\theta_0\beta_w + u_w)}{\alpha_w}.$$

Using Theorem 2.9, this inequality is equivalent to the following:

$$\frac{\theta_0 a_1 \cdots a_m}{d_m^2} \geq \sum_{w=1}^{m} \frac{\hat{g}_w u_w}{\alpha_w}.$$

By (2.2), we have

$$\frac{\hat{g}_w d_m^2}{\alpha_w a_1 \cdots a_m} = \frac{d_m}{a_w} = e_{wm}.$$

Thus the inequality is equivalent to the following:

$$\theta_0 \geq \sum_{w=1}^{m} u_w e_{wm}.$$

Let Λ be the set of positive integers λ satisfying the following condition: there exist integers $0 \leq v_w < \alpha_w$ for $1 \leq w \leq m$ such that

$$\lambda \geq \sum_{w=1}^{m} v_w e_{wm}, \quad \beta_w\lambda + v_w \equiv 0 \pmod{\alpha_w}. \tag{2.4}$$

By the definition of the fundamental cycle, $\theta_0 = \min\Lambda$. Let

$$\Lambda_0 = \{\lambda \in \Lambda \mid (2.4) \text{ with } v_1 = \cdots = v_m = 0\} \text{ and}$$

$$\Lambda_i = \{\lambda \in \Lambda \mid (2.4) \text{ with } v_i = 1,\ v_j = 0,\ j \neq i\}.$$

We see that $\gcd(\alpha_w, \beta_w) = \gcd(\alpha_w, \alpha_{w'}) = 1$ and $\min\Lambda_i = e_{im}$ for $1 \leq i \leq m$ by the definition of these integers and (2.3). Thus we have

$$\min\Lambda_0 = \alpha \quad \text{and} \quad \min(\Lambda \setminus \Lambda_0) = \min\Lambda_m = e_m.$$

Therefore, we obtain that $\min\Lambda = \min(e_m, \alpha)$. $\square$

Example 2.14. Let $a_1 = a_2 = 2,\ a_3 = \cdots = a_{m-1} = 3,\ a_m = 4,\ m \geq 4$. Then

$$d_{1m} = d_{2m} = 12,\ d_{3m} = \cdots = d_{m-1,m} = 12,\ d_{mm} = 6,\ d_m = 12;$$

$$e_{1m} = e_{2m} = 6,\ e_{3m} = \cdots = e_{m-1,m} = 4,\ e_{mm} = 3;$$

$$\alpha_{1m} = \cdots = \alpha_{m-1,m} = 1,\ \alpha_{mm} = 2;$$

$$\mu_{1m} = \cdots = \mu_{m-1,m} = 0,\ \mu_{mm} = 1;$$

$$\hat{g}_1 = \hat{g}_2 = 2 \cdot 3^{m-4},\ \hat{g}_3 = \cdots = \hat{g}_{m-1} = 4 \cdot 3^{m-5},\ \hat{g}_m = 2 \cdot 3^{m-4},\ \hat{g} = 4 \cdot 3^{m-4}.$$

By Theorem 2.9, we have $c_0 = 4 \cdot 3^{m-5}$ and $g = (4m - 15) \cdot 3^{m-5} + 1$. Then the weighted dual graph of E is as in Figure 2.5. Furthermore, by Theorem 2.9,

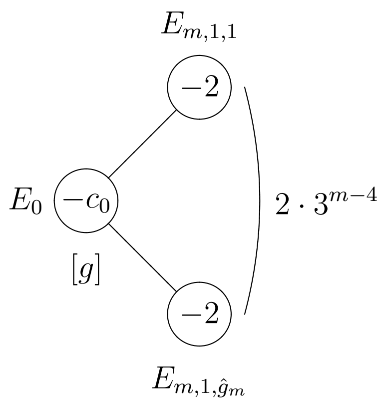


FIGURE 2.5.

the zero divisors of the pull-back of the coordinate functions $x_1, \ldots, x_m$ are as follows:

$$Z^{(1)} = Z^{(2)} = 6E_0 + 3 \cdot \sum_{\xi=1}^{\hat{g}_m} E_{m,1,\xi},$$

$$Z^{(3)} = \cdots = Z^{(m-1)} = 4E_0 + 2 \cdot \sum_{\xi=1}^{\hat{g}_m} E_{m,1,\xi},$$

$$Z^{(m)} = 3E_0 + 2 \cdot \sum_{\xi=1}^{\hat{g}_m} E_{m,1,\xi}.$$

By Theorem 2.13, we obtain that $\theta_0 = \min(e_m, \alpha) = 2$ and the fundamental cycle Z_E on E is as follows:

$$Z_E = 2E_0 + \sum_{\xi=1}^{\hat{g}_m} E_{m,1,\xi}.$$

Note that $\alpha = 2$ and $\alpha < e_{mm} = 3$, and $Z_E < Z^{(m)}$.

In the situation of Example 2.10, we have $a_1 = \cdots = a_{m-3} \le a_{m-2} \le a_{m-1} \le a_m,\ e_{mm} = e_m = 12,\ \alpha = 30$ and $\theta_0 = \min(e_m, \alpha) = 12$. Then, by Theorem 2.13, the fundamental cycle Z_E on E is as follows:

$$Z_E = 12E_0 + 4 \cdot \sum_{\xi=1}^{\hat{g}_{m-2}} E_{m-2,1,\xi} + 6 \cdot \sum_{\xi=1}^{\hat{g}_{m-1}} E_{m-1,1,\xi} + \sum_{\xi=1}^{\hat{g}_m} (5E_{m,1,\xi} + 3E_{m,2,\xi}).$$

Note that we have $\alpha > e_m$ and $Z_E = Z^{(m)}$.

Lemma 2.15. *$Z_E = Z^{(m)}$ if and only if $e_m \le \alpha$.*

Proof. It follows from Theorem 2.9, Theorem 2.13 and Lemma 1.56 (3). □

Theorem 2.16. *Let Z_0 be the cycle which is obtained as Z_E with the condition that $\theta_0 = \alpha$ in Theorem 2.13. Then*

$$Z_K = E + \frac{(m-2)d_m}{\alpha} Z_0 - \sum_{w=1}^{m} Z^{(w)}.$$

Proof. Let $N_0 = \{w \in \{1, \ldots, m\} \mid \alpha_w = 1\}$ and $N_1 = \{1, \ldots, m\} \setminus N_0$. Note that for $w \in N_0$, $\beta_w = 0$ and $Z^{(w)} E_0 = -\hat{g}_w$ (cf. the proof of Lemma 2.11). Let B be any irreducible component of $E - E_0$. By the adjunction formula and

Theorem 2.9,

$$
\begin{aligned}
(-Z_K + E)E_0 &= 2g - 2 + \sum_{w \in N_1} \hat{g}_w \\
&= 2g - 2 + \sum_{w=1}^{m} \hat{g}_w + \sum_{w \in N_0} Z^{(w)} E_0 \\
&= (m-2)\hat{g} + \sum_{w \in N_0} Z^{(w)} E_0, \\
(-Z_K + E)B &= -2 + (E - B)B \\
&= \begin{cases} -1 & \text{if } B = E_{w,s_w,\xi} \text{ for some } w \text{ and } \xi, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

It follows from Lemma 1.56 (3) and (1) that $Z_0(E - E_0) = 0$ and $\theta_{w,1,\xi} = \alpha\beta_w/\alpha_w$. By Theorem 2.9,

$$
\begin{aligned}
-Z_0 E_0 &= c_0 \alpha - \sum_{w=1}^{m} \frac{\hat{g}_w \alpha \beta_w}{\alpha_w} \\
&= \alpha \left(c_0 - \sum_{w=1}^{m} \frac{\hat{g}_w \beta_w}{\alpha_w} \right) \\
&= \frac{\alpha a_1 \cdots a_m}{d_m^2} = \frac{\alpha \hat{g}}{d_m}.
\end{aligned}
$$

We see that for $w \in N_0$,

$$
\frac{1}{Z_0 E_0} Z_0 = \frac{1}{Z^{(w)} E_0} Z^{(w)}, \tag{2.5}
$$

since they are numerically equivalent. Form the data of the intersection numbers of $-Z_K + E$ and (2.5), we obtain that

$$
\begin{aligned}
-Z_K + E &= \frac{(-Z_K + E)E_0}{Z_0 E_0} Z_0 + \sum_{w \in N_1} Z^{(w)} \\
&= \frac{(m-2)\hat{g}}{Z_0 E_0} Z_0 + \sum_{w \in N_0} Z^{(w)} + \sum_{w \in N_1} Z^{(w)} \\
&= -\frac{(m-2)d_m}{\alpha} Z_0 + \sum_{w=1}^{m} Z^{(w)}.
\end{aligned}
$$

Thus the formula follows. □

In the situation of Example 2.10, we have

$$Z_0 = 30E_0 + 10\cdot\sum_{\xi=1}^{\hat{g}_{m-2}} E_{m-2,1,\xi} + 15\cdot\sum_{\xi=1}^{\hat{g}_{m-1}} E_{m-1,1,\xi} + \sum_{\xi=1}^{\hat{g}_m}(12E_{m,1,\xi} + 6E_{m,2,\xi})$$

and

$$Z_K = E + (30m-77)E_0 + (10m-26)\cdot\sum_{\xi=1}^{\hat{g}_{m-2}} E_{m-2,1,\xi} + (15m-39)\cdot\sum_{\xi=1}^{\hat{g}_{m-1}} E_{m-1,1,\xi}$$

$$+\sum_{\xi=1}^{\hat{g}_m}((12m-31)E_{m,1,\xi} + (6m-16)E_{m,2,\xi}).$$

In the situation of Example 2.14, we have $Z_0 = 2E_0 + \sum_{\xi=1}^{\hat{g}_m} E_{m,1,\xi}$ and

$$Z_K = E + (8m-27)E_0 + (4m-14)\cdot\sum_{\xi=1}^{\hat{g}_m} E_{m,1,\xi}.$$

The arithmetic genus of the fundamental cycle, namely,

$$1-\chi(Z_E) = (1/2)Z_E(K_{\tilde{X}} + Z_E) + 1,$$

is called the fundamental genus. This invariant is independent of the resolution and denoted by $p_f(X,o)$. A formula of p_f for weighted homogeneous surface singularities was established by Tomari (cf. [**31**, Theorem 3.1]). Applying the formula, Tomaru [**32**] obtained the following result in hypersurface case (cf. [**13**, Theorem 1.7]).

Theorem 2.17. *If $e_m \geq \alpha$, then $-Z_E^2 = \alpha^2\hat{g}/d_m$ and*

$$p_f(X,o) = \frac{1}{2}\alpha\left\{(m-2)\hat{g} - \frac{(\alpha-1)\hat{g}}{d_m} - \sum_{w=1}^{m}\frac{\hat{g}_w}{\alpha_w}\right\} + 1.$$

If $e_m \leq \alpha$, then $-Z_E^2 = \hat{g}_m\lceil e_m/\alpha_m\rceil$ and

$$p_f(X,o) = \frac{1}{2}e_m\left\{(m-2)\hat{g} - \frac{(2\lceil e_m/\alpha_m\rceil - 1)\hat{g}_m}{e_m} - \sum_{w=1}^{m-1}\frac{\hat{g}_w}{\alpha_w}\right\} + 1.$$

Proof. Assume that $e_m \geq \alpha$, then $Z_E = Z_0$ and $\theta_0 = \alpha$. By the proof of Theorem 2.16, we have $-Z_0E_0 = \alpha\hat{g}/d_m$. Therefore

$$-Z_E^2 = Z_0(\alpha E_0) = \alpha^2\hat{g}/d_m.$$

We see that $\hat{g}e_{wm} = d_m\hat{g}_w/\alpha_w$. Thus

$$\begin{aligned}
2p_f(X,o) - 2 &= (Z_K - Z_0)(-Z_0) \\
&= \left(E + \frac{(m-2)d_m - \alpha}{\alpha} Z_0 - \sum_{w=1}^{m} Z^{(w)}\right)(-Z_0) \\
&= \left(1 + \frac{(m-2)d_m - \alpha}{\alpha} \cdot \alpha - \sum_{w=1}^{m} e_{wm}\right)\alpha\hat{g}/d_m \\
&= \left((m-2)\hat{g} + (1-\alpha)\frac{\hat{g}}{d_m} - \sum_{w=1}^{m}\frac{\hat{g}_w}{\alpha_w}\right)\alpha.
\end{aligned}$$

Next, assume that $e_m \le \alpha$. Then $Z_E = Z^{(m)}$ by Lemma 2.20. It follows from Lemma 1.56 (4) that $\lambda^{(w)}_{m,s_m,\xi} = e_m/\alpha_w$ for $w \ne m$. By Lemma 2.11 and its proof, we have $-Z_E^2 = -(Z^{(m)})^2 = \hat{g}_m\lceil e_m/\alpha_m\rceil$ and

$$\begin{aligned}
2p_f(X,o) - 2 &= (Z_K - Z^{(m)})(-Z^{(m)}) \\
&= \left(E + \frac{(m-2)d_m}{\alpha} Z_0 - \sum_{w=1}^{m-1} Z^{(w)} - 2Z^{(m)}\right)(-Z^{(m)}) \\
&= \hat{g}_m + \frac{(m-2)d_m}{\alpha} \cdot \frac{\alpha\hat{g}}{d_m} \cdot e_m - \sum_{w=1}^{m-1}\frac{\hat{g}_w e_m}{\alpha_w} - 2\hat{g}_m\lceil e_m/\alpha_m\rceil \\
&= \left((m-2)\hat{g} + \frac{\hat{g}_m}{e_m}(1 - 2\lceil e_m/\alpha_m\rceil) - \sum_{w=1}^{m-1}\frac{\hat{g}_w}{\alpha_w}\right) e_m.
\end{aligned}$$

□

In the situation of Example 2.10, we have

$\alpha = 30 > e_{mm} = 12$, $-Z_E^2 = 3\cdot 2^{m-3} = \hat{g}_m\lceil e_{mm}/\alpha_m\rceil$ and $p_f(X,o) = (3m-9)\cdot 2^{m-3}+1$.

In the situation of Example 2.14, we have

$\alpha = 2 < e_{mm} = 3$, $-Z_E^2 = 4\cdot 3^{m-5} = \alpha^2\hat{g}/d_m$ and $p_f(X,o) = (8m-28)\cdot 3^{m-5}+1$.

2.4. The maximal ideal cycle

In this section, we identify the maximal ideal cycle. We keep the assumption that $a_1 \le \cdots \le a_m$. Let $\mathfrak{m}$ denote the maximal ideal of the local ring $\mathcal{O}_{X,o}$ and

$Z_{\mathfrak{m}}$ the maximal ideal cycle on $\tilde{X}$. By the definition, we have

$$Z_{\mathfrak{m}} = \min\{(L)_E | L = c_1x_1 + \cdots + c_mx_m \in \mathfrak{m},\ c_i \in \mathbb{C},\ L \neq 0\}.$$

Theorem 2.18. *We have $Z^{(m)} \leq \cdots \leq Z^{(1)}$. Hence $Z_{\mathfrak{m}} = Z^{(m)}$. Furthermore, the maximal ideal cycle coincides with the fundamental cycle on the minimal good resolution space and on $\tilde{X}$ if and only if $e_m \leq \alpha$.*

Proof. Since $e_{1m} \geq \cdots \geq e_{mm} = e_m$, the first assertion follows from Theorem 2.9 and Corollary 1.55. The last assertion follows from Lemma 2.20 and Lemma 2.12, and the fact that these two cycles coincide on any resolution of every rational surface singularity ([**2**]). □

Example 2.19 ($\alpha < e_m$)**.** Let $a_1 = a_2 = 2,\ a_3 = \cdots = a_m = 3,\ m \geq 3$. Then

$$e_{1m} = e_{2m} = 3, e_{3m} = \cdots = e_{mm} = 2,$$
$$\alpha_1 = \cdots = \alpha_m = 1.$$

Also, we have $\beta_1 = \cdots = \beta_m = 0$. Therefore E is irreducible, $Z_E = E$, and $Z_{\mathfrak{m}} = 2E$. We also have $g = (2m-7) \cdot 3^{m-4} + 1,\ c_0 = 3^{m-4}$ by Theorem 2.9.

Example 2.20 ($\alpha \geq e_m$)**.** Let $a_1 = \cdots = a_{m-2} = 2,\ a_{m-1} = 3,\ a_m = 7,\ m \geq 4$. Then

$$e_{1m} = \cdots = e_{m-2,m} = 21, e_{m-1,m} = 14, e_m = 6,$$
$$\alpha_1 = \cdots = \alpha_{m-2} = 1, \alpha_{m-1} = 3, \alpha_m = 7.$$

Also, we have

$$\beta_1 = \cdots = \beta_{m-2} = 0, \beta_{m-1} = \beta_m = 1,$$
$$\hat{g}_{m-1} = \hat{g}_m = 2^{m-3}.$$

Then the weighted dual graph of E is as in Figure 2.6.

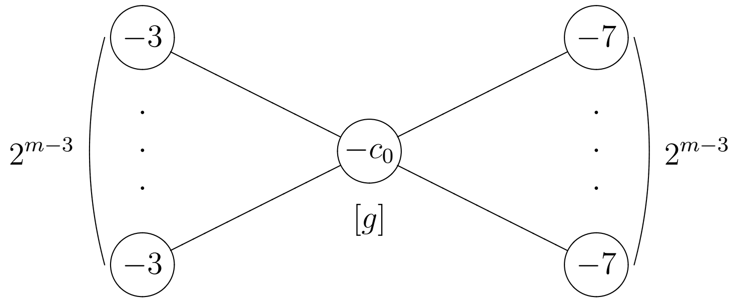


FIGURE 2.6.

The components $E_{m,s_m,\xi}$ correspond to the vertices with weight -7. By Theorem 2.9, we have $c_0 = 2^{m-4}$ and $g = (m-6)\cdot 2^{m-5}+1$. Since $\alpha = 21 > 6 = e_m$, we have $Z_{\mathfrak{m}} = Z_E$ by Theorem 2.18.

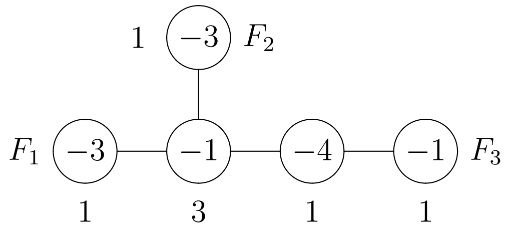
2.5. Kodaira singularities

After Kulikov's results ([**14**]) for Arnold's classification ([**1**]) of unimodal and bimodal singularities, U. Karras ([**11**]) introduced the notation of Kodaira singularities for normal surface singularities in terms of pencils of curves. He proved several fundamental properties for them and applied his results to deformation theory of surface singularities. In this section, we give a condition for (X, o) to be a Kodaira singularity following Konno and Nagashima.

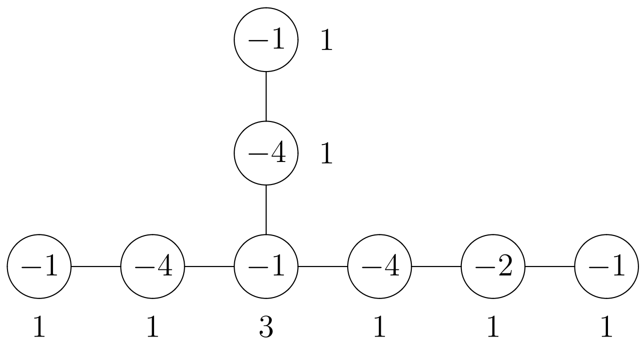
Let S be a non-singular complex surface and $\Delta \subset \mathbb{C}$ a small open disc around the origin. A surjective holomorphic map $\Phi : S \longrightarrow \Delta$ is said to be a pencil of curves of genus g, if it is proper and connected, and fibers $S_t := \Phi^{-1}(t)$ $(t \neq 0)$ are smooth curves of genus g. In this situation, we call $S_o := \Phi^{-1}(o)$ the singular fiber.

Definition 2.21 (Karras [**11**]). A normal surface singularity (W, p) is called a Kodaira singularity if there exists a pencil of curves $\Phi : S \to \Delta$ such that, after a finite number of blowing ups at finitely many non-singular points $P_1, \dots, P_r$ in non-multiple components of the singular fiber S_o, $\Psi : S' \to S$, there is a holomorphic map $\phi : M \to W$ from an open neighborhood M of the strict transform F of $\mathrm{Supp}(S_0)$ in S' which is a resolution of (W, p) with exceptional divisor F.

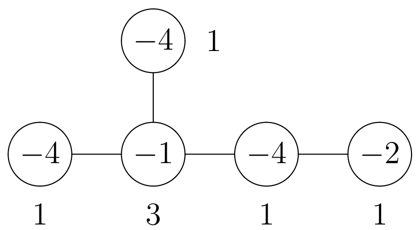
Example 2.22 ([**35**, Example 2.4])**.** There exists a pencil of curves $\Phi : S \longrightarrow \Delta$ of genus $g = 1$, , such that the singular fiber $S_o = \Phi^{-1}(o)$ is as follows:



Let $\Psi : S' \longrightarrow S$ be blowing ups at non-singular points $P_1 \in F_1, P_2 \in F_2, P_3 \in F_3$. Then we have the following figure



Let M be an open neighborhood of the strict transform F of $\mathrm{Supp}(S_o)$, where F is as follows:

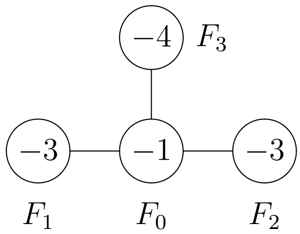


Contracting F in M, we obtain a Kodaira surface singularity (W, o).

Karras ([**12**]) proved a fine criteria for normal surface singularities to be a Kodaira singularities in terms of the maximal ideal cycle on the minimal good resolution.

Proposition 2.23 ([**11**, §2], [**12**]). *Let $\phi : (M, F) \to (W, p)$ be the minimal good resolution of a normal surface singularity and Z_F the fundamental cycle on F. Then (W, p) is a Kodaira singularity if and only if the coefficient of F_j in Z_F is 1 for every component F_j satisfying $Z_F F_j < 0$ and there exists an element $h \in \mathcal{O}_{W,p}$ such that the divisor $\mathrm{div}_M(h \circ \phi)$ is normal crossing with exceptional part $(h)_F = Z_F$.*

Example 2.24. Let $W = \{x^2 + y^3 + z^8 = 0\}$ and $\phi : (M, F) \longrightarrow (W, o)$ be the minimal good resolution of (W, o) with exceptional set F. From Theorem 2.9, the weighted dual graph of F is as follows:



Following Theorem 2.13, the fundamental cycle $Z_F = 3F_0 + F_1 + F_2 + F_3$. By computation, we have $Z_F F_0 = 0$, $Z_F F_1 = 0$, $Z_F F_2 = 0$ and $Z_F F_3 = -1 < 0$. The coefficient of F_3 in Z_F is 1. Also, there exists an element $z \in \mathcal{O}_{W,o}$ such that the divisor $\mathrm{div}_M(z \circ \phi)$ is normal crossing with exceptional part $(z)_F = 3F_0 + F_1 + F_2 + F_3$ following Theorem 2.9. Note that $(z)_F = Z_F$. By Proposition 2.23, we have that (W, o) is a Kodaira singularity.

Theorem 2.25. *(X, o) is a Kodaira singularity if and only if $d_{m-1} \leq a_m$.*

Proof. By Lemma 2.12, if the resolution $f : \tilde{X} \longrightarrow X$ is not the minimal good resolution, then the condition $d_{m-1} \leq a_m$ is satisfied. On the other hand, a rational singularity with reduced fundamental cycle is a Kodaira singularity ([**11**, Theorem 2.9]).

We assume that $f : \tilde{X} \longrightarrow X$ is the minimal good resolution. We have seen that $\mathrm{div}_{\tilde{X}}(x_m)$ is normal crossing (cf. Lemma 2.7). Therefore, it follows from Proposition 2.23, Theorem 2.18 and the proof of Lemma 2.11 that (X, o) is a

Kodaira singularity if and only if $e_m \leq \alpha$ and $\lceil e_m/\alpha_m \rceil = 1$; this condition is equivalent to that $d_{m-1} \leq a_m$. $\square$

In the situation of Example 2.19, we have $d_{m-1} = 6 > a_m = 3$. Hence (X, o) is not a Kodaira singularity by Theorem 2.25.

In the situation of Example 2.20, we have $d_{m-1} = 6 < a_m = 7$, and then (X, o) is a Kodaira singularity by Theorem 2.25.

Bibliography

[1] V. I. Arnold, *Normal forms of functions*, Invent. Math. **35** (1976), 87–109.

[2] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.

[3] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Math. (3), vol. 4, Springer-Verlag, New York, Heidelberg, Berlin, 1984.

[4] E. Brieskorn, *Rationale Singularitäten komplexer Flächen*, Invent. Math. **4** (1968), 336–358.

[5] D. J. Dixon, *The fundamental divisor of normal double points of surfaces*, Pacific J. Math. **80** (1979), no. 1, 105–115.

[6] G. Fischer, *Complex Analytic Geometry*, Lecture Notes in Math., vol. 538, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[7] Akira Fujiki, *On resolutions of cyclic quotient singularities*, Publ. Res. Inst. Math. Sci. **10** (1974/75), no. 1, 293–328.

[8] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[9] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1964), 109–326.

[10] M. Jankins and W. D. Neumann, *Lectures on Seifert manifolds*, Brandeis Lecture Notes, vol. 2, Brandeis University, Waltham, MA, 1983.

[11] U. Karras, *On pencils of curves and deformations of minimally elliptic singularities*, Math. Ann. **247** (1980), 43–65.

[12] U. Karras, *Methoden zur Berechnung von Algebraischen Invarianten und zur Konstruktion von Deformationen Normaler Flächensingularitäten*, Habilitationschrift, Dortmund, 1981.

[13] K. Konno and D. Nagashima, *Maximal ideal cycles over normal surface singularities of Brieskorn type*, Osaka J. Math. **49** (2012), no. 1, 225–245.

[14] V. S. Kulikov., *Degenerate elliptic curves and resolution of uni-and bimodal singularities*, Funct. Anal. Appl. **9** (1975), 69–70.

[15] H. Laufer, *Normal two-dimensional singularities*, Ann. of Math. Studies no. **71**, Princeton University Press 1971.

[16] H. Laufer, *Taut two-dimensional singularities*, Math. Ann. **205** (1973), 131–164.

[17] H. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99** (1977), 1257–1295.

[18] H. Laufer, *Tangent cones for deformations of two-dimensional quasi-homogeneous singularities*, Singularities (Iowa City, IA, 1986), Contemp. Math., vol. 90, Amer. Math. Soc., Providence, RI, 1989, pp. 183–197.

[19] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Math., vol. 8, Cambridge University Press, 1986.

[20] F. N. Meng and T. Okuma, *The maximal ideal cycles over complete intersection surface singularities of Brieskorn type*, to appear in Kyushu J. Math.

[21] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. **9**(1961), 5–22.

[22] A. Némethi, *"Weakly" elliptic Gorenstein singularities of surfaces*, Invent. Math. **137** (1999), no. 1, 145–167.

[23] A. Némethi, *Invariants of normal surface singularities*, Real and complex singularities, Contemp. Math., vol. 354, Amer. Math. Soc., Providence, RI, 2004, pp. 161–208.

[24] W. D. Neumann, *Abelian covers of quasihomogeneous surface singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 233–243.

[25] T. Okuma, *Plurigenera of surface singularities*, Nova Science Publishers, New York, 2000.

[26] T. Okuma, *Numerical Gorenstein elliptic singularities*, Math. Z. **249** (2005), 31–62.

[27] P. Orlik and P. Wagreich, *Isolated singularities of algebraic surface with $\mathbb{C}^*$ action*, Ann. of Math. **93** (1971), 205–228.

[28] H. Pinkham, *Normal surface singularities with $\mathbb{C}^*$-action*, Math. Ann. **227** (1977), 183–193.

[29] J. Stevens, *Elliptic surface singularities and smoothings of curves*, Math. Ann. **267** (1984), no. 2, 239–249.

[30] M. Tomari, *A p_g-formula and elliptic singularities*, Publ. Res. Inst. Math. Sci. **21** (1985), no. 2, 297-354.

[31] T. Tomaru, *On Gorenstein surface singularities with fundamental genus $p_f \geq 2$ which satisfy some minimally conditions*, Pacific J. Math. **170** (1995), no. 1, 271–295.

[32] T. Tomaru, *A formula of the fundamental genus for hypersurface singularities of Brieskorn type*, Ann. Rep. Coll. Med. Care Technol. Gunma Univ. **17** (1996), 145–150.

[33] T. Tomaru, *On Kodaira singularities defined by $z^n = f(x, y)$*, Math. Z. **236** (2001), no. 1, 133–149.

[34] T. Tomaru, *Pinkham-Demazure construction for two dimensional cyclic quotient singularities*, Tsukuba J. Math. **25** (2001), no. 1, 75–83.

[35] T. Tomaru, *Complex surface singularities and degenerations of compact complex curves*, Demonstratio Math., **XLIII** (2010), no. 2, 339–359.

[36] T. Tomaru, *$\mathbb{C}^*$-equivariant degenerations of curves and normal surface singularities with $\mathbb{C}^*$-action*, J. Math. Soc. Japan, **65** (2013), no. 3, 829–885.

[37] S. S.-T. Yau, *On maximal elliptic singularities*, Trans. Amer. Math. Soc., **257** (1980), 269–329.