学 位 論 文

Bounded linear operators on Morrey spaces

March, 2014

Graduate school of Science and Engineering
Yamagata University

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DOCTORAL THESIS

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Contents

Introduction

The classical Morrey spaces were introduced by Morrey in 1938 for investigating the local behavior of solutions to second order elliptic partial differential equations and calculus of variations ([**42**]). In 1961, John-Nirenberg [**32**] introduced BMO spaces for studying PDE, and Campanato [**7**] introduced the function spaces in 1963, which are called Morrey-Campanato spaces. At this time, Stampacchia [**50**] and Peetre [**45**] considered the Morrey-Campanato spaces. These spaces were studied in close connection with the theory of partial differential equations and harmonic analysis, and helped to obtain many interesting results. On the other hand, Giga-Miyakawa [**19**] introduced a Morrey type space with respect to a Radon measure for three dimensional Navier-Stokes equations. Kato [**33**] and Kozono-Yamazaki [**36**] also applied Morrey spaces to Navier-Stokes equations. Moreover, we have another applications of Morrey spaces to Schrödinger equations, elliptic problems with discontinuous coefficients and potential theory ([**4**], [**5**], [**9**], [**12**], [**15**], [**39**]).

From these facts, Morrey spaces are important function spaces. The definition of Morrey spaces on $\mathbb{R}^n$ are as follows:

DEFINITION ([**42**]). Let p and λ be in $1 \leq p < \infty$, $0 \leq \lambda \leq 1$. Morrey spaces are the space of all measurable function $f : \mathbb{R}^n \to \mathbb{C}$

such that

$$||f||_{L^{p,\lambda}} = \sup_{Q \subset \mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{|Q|^\lambda} \int_Q |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

Especially, Morrey spaces are L^p spaces when $\lambda = 0$, and $L^{p,1}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. Therefore, we can consider Morrey spaces from the point of view of a generalization of L^p spaces which are function spaces such that pth powers are integrable.

The overall aim of this dissertation is to study some properties of Morrey spaces and bounded linear operators on Morrey spaces. The thesis consists of three chapters.

In Chapter 1 is divided into two parts.

Firstly, we review some results about Morrey-Campanato spaces on the unit circle $\mathbb{T}$. Although Morrey-Campanato spaces were introduced by Morrey and Campanato, we define this space based on Torchinsky [**53**] and Kufner [**37**] here.

DEFINITION. Let p and λ be in $1 < p < \infty$, $0 \leq \lambda < \infty$. Then, Morrey-Campanato spaces are the space of all measurable function $f : \mathbb{T} \to \mathbb{C}$ such that

$$||f||_{\mathcal{L}^{p,\lambda}} = \sup_{\substack{I \subset \mathbb{T} = [-\pi, \pi) \\ I \neq \phi:\text{interval}}} \left(\frac{1}{|I|^\lambda} \int_I |f(y) - f_I|^p dy \right)^{\frac{1}{p}} < \infty,$$

where f_I denotes the average of f over I, that is $\frac{1}{|I|} \int_I f(y) dy$.

If $\lambda = 0$, Morrey-Campanato spaces and L^p spaces are same spaces. And, when $\lambda = 1$, it is BMO spaces, $1 < \lambda < 1 + p$, it is Lipschitz function and $0 < \lambda < 1$, it is Morrey spaces. Moreover, if $\lambda = 1 + p$, f

is absolutely continuous function. In the case $\lambda > 1+p$, f is a constant function ([**53**], [**37**]). These are all well-known results, but important for properties of function spaces. Therefore, we mention these proofs in this part.

Secondary, we give new results in Morrey spaces on the unit circle $\mathbb{T}$. As a preparation, we define bounded linear functionals in functional analysis.

DEFINITION. Suppose that X is a norm space, and $T : X \to \mathbb{C}$. Then, T is called a bounded linear functional if T satisfies following conditions:

(1) For all α, $\beta \in \mathbb{C}$, and $f, g \in X$, we have

$$T(\alpha f + \beta g) = \alpha T f + \beta T g;$$

(2) For all $f \in X$, there exists $C > 0$ such that

$$|Tf| \leq C||f||_X.$$

Next, we define dual and predual space of X.

DEFINITION.

(1) The dual space of X is defined by the space of all bounded linear functionals on a norm space X. It is denoted by X^*.

(2) The norm space Y is called predual of X if Y^* equals X.

Let $L_0^{p,\lambda}(\mathbb{T})$ be the closure of $C(\mathbb{T})$ in $L^{p,\lambda}(\mathbb{T})$, where $C(\mathbb{T})$ is the set of all continuous functions on $\mathbb{T}$. Firstly, we show a property of $L_0^{p,\lambda}(\mathbb{T})$.

Theorem ([**31**]). *Let* $1 \leq p < \infty$, *and* $0 < \lambda < 1$. *Also let* ϕ *be an infinitely differentiable function such that supp* $\phi \subset [-1, 1]$, $\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(x)dx = 1$ *and* $\phi \geq 0$, *and let* $\phi_j(x) = j\phi(jx)$ $(j = 1, 2, \cdots)$. *Then, the following properties are equivalent:*

(1) $f \in L_0^{p,\lambda}(\mathbb{T})$

(2) $f \in L^{p,\lambda}(\mathbb{T})$ *and* $||\tau_y f - f||_{p,\lambda} \to 0$ $(y \to 0)$,

where $\tau_y f(x) = f(x - y)$

(3) $f \in L^{p,\lambda}(\mathbb{T})$ *and* $||f - f * \phi_j||_{p,\lambda} \to 0$ $(j \to \infty)$

(4) $\lim_{\delta\to 0}\sup_{|I|\leq\delta, I\subset\mathbb{T}:\text{interval}} \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx = 0$

Like Adams-Xiao [**3**], $L^{p,\lambda}(\mathbb{T})$ and $L_0^{p,\lambda}(\mathbb{T})$ are similar to BMO and VMO ([**11**], [**47**]). Moreover, it is known that the dual of VMO is Hardy space H^1.

On the other hand, Zorko [**55**] gave the predual space $Z^{q,\lambda}(\mathbb{T})$ $(1/p + 1/q = 1)$ of $L^{p,\lambda}(\mathbb{T})$ in 1986. $Z^{q,\lambda}(\mathbb{T})$ is defined by the set of all functions f such that

$$\begin{aligned}&||f||_{Z^{q,\lambda}}\\&= \inf\left\{\sum_{k=1}^{\infty}|c_k| \;\middle|\; f(x) = \sum_{k=1}^{\infty}c_k a_k(x),\ c_k \in \mathbb{C},\ a_k(x) : (q,\lambda)\text{-block}\right\}\\&< \infty,\end{aligned}$$

where $a_k(x)$ is called (q, λ)-block, if

(1) supp $a_k \subset I$

(2) $||a_k||_q \leq \frac{1}{|I|^{\lambda/p}}$, where $1/p + 1/q = 1$,

for some interval I.

Adams-Xiao [**3**] pointed out that $L_0^{p,\lambda}(\mathbb{T})$ is the predual of Zorko space $Z^{q,\lambda}(\mathbb{T})$ in 2012. But, they did not give the reason why they insisted that the proof is akin to that of BMO-H^1-VMO in Stein [**51**]. We prove in the detail in this part.

Theorem ([**31**]). *Let $1 < p < \infty$, and $0 < \lambda < 1$. Then $L_0^{p,\lambda}(\mathbb{T})$ is the predual of $Z^{q,\lambda}(\mathbb{T})$, where $1/p + 1/q = 1$.*

In Chapter 2, we study Fourier multipliers on $\mathbb{T}$. Let $M(X,Y)$ be the set of all translation invariant bounded linear operators from X to Y, where X and Y are translation invariant function spaces which is contained in $L^1(\mathbb{T})$. We note $M(X,Y)$ is a Banach space with the norm of $||\cdot||_{M(X,Y)}$. An element of $M(X,Y)$ is called a Fourier multiplier (operator). In 1970, Figa-Talamanca and Gaudry [**16**] showed $M(L^p, L^p) \neq M(L^q, L^q)$ $(1 \le p < q \le 2)$. In this chapter, we generalize Figa-Talamanca and Gaudry's result.

Theorem ([**30**]). *Let $1 \le p, q < \infty$ and $0 < \lambda, \nu < 1$. Suppose $\frac{\lambda}{p} \neq \frac{\nu}{q}$. Then we have*

$$M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu}).$$

Theorem ([**30**]). *Let $0 < \lambda, \nu < 1$. Also let p, q be positive numbers with $1 + \lambda < p < q$ and $\frac{1}{p} + \frac{1}{q} < 1$. Suppose $\frac{\lambda}{p} = \frac{\nu}{q}$. Then we have*

$$M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu}).$$

Moreover, we show a relation between $M(L^p, L^{p,\lambda})$ and the measure whose distribution function satisfies a Lipschitz condition (cf. [**21**]).

DEFINITION. Let μ be in $M(\mathbb{T})$ and $0 < \alpha < 1$. We say that $\mu \in Lip_\alpha(M(\mathbb{T}))$ for $\mu \in M(\mathbb{T})$ with $\mu \geq 0$ if for any interval $I = [x, x+h]$,

$$\mu(I) \leq C|I|^\alpha = C|h|^\alpha$$

for some constant $C > 0$ independent of I.

THEOREM ([**30**]). *Let $f \in L^1(\mathbb{T})$ be a non-negative function. Then we have that μ_f is in $Lip_\alpha(M(\mathbb{T}))$ for some $0 < \alpha < 1$, if and only if $T_f \in M(L^p, L^{p,\lambda})$ for some $1 < p < \infty$ and $0 < \lambda < 1$, where $T_f g = f * g$.*

In Chapter 3, we deal with function spaces with weighted norm. The theory of weights apply to boundary value problems for Laplace's equation on Lipschitz domains, extrapolation of operators, vector-valued inequalities, and certain classes of nonlinear partial differential and integral equations.

Here, we research the fractional integral operators on weighted Morrey spaces on $\mathbb{R}^n$. First, we define the fractional integral operator and weighted Morrey spaces on $\mathbb{R}^n$.

DEFINITION. Let $0 < \alpha < n$. Then, the fractional integral operator I_α is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

DEFINITION. Let $1 < p < \infty$, $0 \leq \lambda < 1$, and u, v are weight. Then, weighted Morrey spaces $L^{p,\lambda}(u,v)(\mathbb{R}^n)$ are the space of all measurable function $f \in L^1_{loc}(u)$ such that

$$||f||_{L^{p,\lambda}(u,v)} = \sup_{Q \subset \mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{v(Q)^\lambda} \int_Q |f(y)|^p u(y) dy \right)^{\frac{1}{p}} < \infty.$$

At an early age, Hardy-Littlewood [**23**], [**24**] and Sobolev [**49**] proved the boundedness of the fractional integral operators.

THEOREM ([**23**], [**24**], [**49**]). *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$. *Then, the fractional integral operator* I_α *is bounded from* L^p *to* L^{q_1}, *where* $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$.

After these results, Muckenhoupt and Wheeden [**43**] proved the boundedness of the fractional integral operators on weighted L^p spaces in 1974.

THEOREM ([**43**]). *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* w *is weight. Then,* $w \in A_{p,q_1}(\mathbb{R}^n)$ *if and only if the fractional integral operator* I_α *is bounded from* $L^p(w^p)$ *to* $L^{q_1}(w^{q_1})$, *where* $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$, *and a weight* w *belongs to* $A_{p,q_1}(\mathbb{R}^n)$ *if*

$$\sup_{Q \subset \mathbb{R}^n, Q: ball} \left(\frac{1}{|Q|} \int_Q w^{q_1}(y) dy \right)^{\frac{1}{q_1}} \left(\frac{1}{|Q|} \int_Q w^{-p'}(y) dy \right)^{\frac{1}{p'}} < \infty.$$

In 1975, Adams [**2**] showed the boundedness of the fractional integral operators on Morrey spaces.

THEOREM ([**2**]). *Let* $0 < \alpha < n$, $0 \leq \lambda < 1 - \frac{\alpha}{n}$ *and* $1 < p < \frac{n(1-\lambda)}{\alpha}$. *Then, the fractional integral operator* I_α *is bounded from* $L^{p,\lambda}$ *to* $L^{q_2,\lambda}$, *where* $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$.

In 1987, Chiarenza and Frasca [**8**] gave an alternative proof of this result. Komori and Shirai [**35**] generalized the boundedness of the fractional integral operators on weighted Morrey spaces in 2009.

THEOREM ([**35**]). *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 \leq \lambda < \frac{p}{q_1}$, *and* $w \in A_{p,q_1}(\mathbb{R}^n)$. *Then, the fractional integral operator* I_α *is bounded from* $L^{p,\lambda}(w^p, w^{q_1})$ *to* $L^{q_1, \frac{\lambda q_1}{p}}(w^{q_1}, w^{q_1})$, *where* $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$.

In this chapter, we obtain the including results of Muckenhoupt-Wheeden [**43**], Adams [**2**] and Komori-Shirai [**35**].

THEOREM ([**29**]). *Let* $0 < \alpha < n$, $1 < p < \frac{n(1-\lambda)}{\alpha}$, $0 \leq \lambda < \frac{p}{q_1}$, *and* $w \in A_{p,q_1}(\mathbb{R}^n)$. *Then, the fractional integral operator* I_α *is bounded from* $L^{p,\lambda}(w^p, w^{q_1})$ *to* $L^{q_2,\lambda}(w^{q_1}, w^{q_1})$, *where* $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$.

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CHAPTER 1

Some properties of Morrey spaces on the unit circle

1. Preliminaries

1.1. L^p spaces.

In this section, we recall the definition and basic properties of L^p spaces.

Definition 1.1. (1) Let $C(\mathbb{T})$ denote

$$C(\mathbb{T}) := \{f \mid f(x) \text{ is a continuous function of period } 2\pi \text{ on } \mathbb{R}\},$$

where f is called a function of period 2π on $\mathbb{R}$ if f satisfies $f(x) = f(x+2\pi) \quad (x \in \mathbb{R})$.

(2) Let p and q be $1 \leq p \leq \infty$, and f be a continuous function of period 2π on $\mathbb{R}$. Then, $L^p(\mathbb{T})$ are defined by

$$L^p(\mathbb{T}) := \left\{ f \;\middle|\; ||f||_{L^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\} \quad (1 \leq p < \infty)$$

$$L^\infty(\mathbb{T}) := \{f \mid \inf\{M \mid |f(x)| < M \quad (a.e.)\} < \infty\} \quad (p = \infty).$$

Lemma 1.2 (the Hölder inequality). *Let p and q be $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. And suppose $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$. Then,*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)g(x)| dx \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right)^{\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} |g(x)|^q dx \right)^{\frac{1}{q}}.$$

Remark 1.3. If $1 \leq q < p \leq \infty$, then $L^p(\mathbb{T}) \subsetneqq L^q(\mathbb{T})$. In fact, by the Hölder inequality, we have

$$\begin{aligned}
||f||_{L^q}^q &= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p \cdot 1dx \\
&\leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^{q\cdot\frac{p}{q}}dx\right)^{\frac{q}{p}} \left(\frac{1}{2\pi}\int_0^{2\pi} 1^{\frac{p}{p-q}}dx\right)^{\frac{p-q}{p}} \\
&\leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}\cdot q} \\
&= ||f||_{L^p}^q
\end{aligned}$$

if $1 < p < \infty$. Therefore, we get $L^p(\mathbb{T}) \subset L^q(\mathbb{T})$. Moreover, when we define

$$f(x) = x^{-\frac{1}{p}},$$

it is easy to show $f \in L^q(\mathbb{T})$ and $f \notin L^p(\mathbb{T})$. By $1 - \frac{q}{p} > 0$, we have

$$\begin{aligned}
||f||_{L^q}^q &= \int_0^{2\pi} x^{-\frac{1}{p}\cdot q}dx \\
&= \frac{1}{1-\frac{q}{p}}(2\pi)^{1-\frac{q}{p}} \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}
||f||_{L^p}^p &= \int_0^{2\pi} x^{-\frac{1}{p}\cdot p}dx \\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^{2\pi} \frac{dx}{x} \\
&= \lim_{\varepsilon\to 0}(\log 2\pi - \log\varepsilon) \\
&= \infty.
\end{aligned}$$

We obtain $L^p(\mathbb{T}) \subsetneqq L^q(\mathbb{T}) \ \ (1 \leq q < p \leq \infty)$.

1.2. BMO spaces.

DEFINITION 1.4. Suppose $f \in L^1(\mathbb{T})$ and I is an interval. And f_I denotes the average of f over I, that is, $f_I = \frac{1}{|I|}\int_I f$. Then, sharp maximal function $M^\sharp f$ is defined by

$$M^\sharp f(x) := \sup_{x \in I} \frac{1}{|I|}\int_I |f(t) - f_I| dt.$$

Moreover, if we put

$$||f||_* = ||M^\sharp f||_{L^\infty},$$

BMO spaces on $\mathbb{T}$ are defined by

$$BMO(\mathbb{T}) := \left\{ f \in L^1(\mathbb{T}) \mid ||f||_* < \infty \right\}.$$

REMARK 1.5 ([**53**]). $L^\infty(\mathbb{T}) \subsetneqq BMO(\mathbb{T})$. In fact,

$$\begin{aligned}
\int_I |f(t) - f_I| dt &\leq \int_I |f(t)| dt + \int_I |f_I| dt \\
&\leq \int_I |f(t)| dt + \int_I \left(\frac{1}{|I|} \int_I |f(y)| dy \right) dt \\
&= \int_I |f(t)| dt + \int_I |f(t)| dt \\
&= 2\int_I |f(t)| dt.
\end{aligned}$$

We obtain

$$\frac{1}{|I|}\int_I |f(t) - f_I| dx \leq 2\frac{1}{|I|}\int_I |f(t)| dt \leq 2||f||_{L^\infty}.$$

And if we take

$$f(t) = \log |t| \quad (|t| < \pi),$$

we get $f \in BMO(\mathbb{T})$ and $f \notin L^\infty(\mathbb{T})$. In this check, put $I = (a, b) \subset \mathbb{T}$, and devided into three cases of $0 < a < b$, $-b < a < 0 < b$ and $b < 0$.

2. Morrey-Campanato spaces

2.1. Definition.

Morrey-Campanato spaces are generalization of L^p spaces and BMO spaces. The definition of this spaces is based on Torchinsky [**53**] and Kufner [**37**].

Definition 1.6 ([**37**], [**53**]). Let p, λ be $1 < p < \infty$ and $0 \leq \lambda < \infty$. Then, Morrey-Campanato spaces $\mathcal{L}^{p,\lambda}$ and this norm are defined by

$$\mathcal{L}^{p,\lambda}(\mathbb{T}) := \left\{ f \in L^1(\mathbb{T}) \;\middle|\; \int_I |f(t) - f_I|^p dt < C|I|^\lambda \quad (\forall I \subsetneqq \mathbb{T}) \right\}$$

and

$$||f||_{\mathcal{L}^{p,\lambda}} := ||f||_{L^p} + [f]_{p,\lambda},$$

where the letter C stands for a constant independent of interval I and $[f]_{p,\lambda}$ is defined by

$$[f]_{p,\lambda} := \sup_{\substack{I \subset \mathbb{T} = [-\pi,\pi) \\ I \neq \emptyset : \text{interval}}} \left(\frac{1}{|I|^\lambda} \int_I |f(t) - f_I|^p dt \right)^{\frac{1}{p}}.$$

Remark 1.7. We have the following:

(1) $\mathcal{L}^{p,\lambda}(\mathbb{T}) \subsetneqq L^p(\mathbb{T})$.

(2) $\mathcal{L}^{p,\lambda}(\mathbb{T}) \subsetneqq \mathcal{L}^{p_1,\lambda_1}(\mathbb{T}) \quad (1 < p_1 \leq p < \infty,\ \frac{\lambda_1 - 1}{p_1} \leq \frac{\lambda - 1}{p})$.

We research the behavior of λ in this spaces. Throughout the rest of this section, q the conjugate exponent of p, that is $\frac{1}{p} + \frac{1}{q} = 1$.

2.2. In the case of $\lambda = 0$.

REMARK 1.8. When $\lambda = 0$, we have $\mathcal{L}^{p,0}(\mathbb{T}) \cong L^p(\mathbb{T})$. In fact, by Remark 1.7, we get $\mathcal{L}^{p,0}(\mathbb{T}) \subset L^p(\mathbb{T})$. On the other hand, suppose $f \in L^p(\mathbb{T})$. For all $I \subsetneqq \mathbb{T}$, we note

$$\begin{aligned}
|f_I| &\le \frac{1}{|I|}\int_I |f(y)|dy \\
&\le \left(\frac{1}{|I|}\int_I |f(y)|^p dy\right)^{\frac{1}{p}} \left(\frac{1}{|I|}\int_I 1^q dy\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{|I|}\int_I |f(y)|^p dy\right)^{\frac{1}{p}}
\end{aligned}$$

by the Hölder inequality. Then, we have

$$\begin{aligned}
\left(\int_I |f(t) - f_I|^p dt\right)^{\frac{1}{p}} &\le \left(\int_I |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_I |f_I|^p dt\right)^{\frac{1}{p}} \\
&\le \left(\int_{-\pi}^{\pi} |f(t)|^p dt\right)^{\frac{1}{p}} + \left\{\int_I \left(\frac{1}{|I|}\int_I |f(y)|^p dy\right) dt\right\}^{\frac{1}{p}} \\
&= (2\pi)^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_I |f(t)|^p dt\right)^{\frac{1}{p}} \\
&\le 2 \cdot (2\pi)^{\frac{1}{p}} ||f||_{L^p(\mathbb{T})} \\
&\le C
\end{aligned}$$

by the Minkowski inequality. Therefore, $\mathcal{L}^{p,0}(\mathbb{T}) \cong L^p(\mathbb{T})$.

2.3. In the case of $\lambda = 1$.

REMARK 1.9 ([**53**]). When $\lambda = 1$, we have $\mathcal{L}^{p,1}(\mathbb{T}) \cong BMO(\mathbb{T})$. In fact, for all $I \subsetneqq \mathbb{T}$, we have

$$\begin{aligned}
\int_I |f(x) - f_I| dx &\leq \left(\int_I |f(x) - f_I|^p dx \right)^{\frac{1}{p}} \left(\int_I 1^q dx \right)^{\frac{1}{q}} \\
&= |I|^{\frac{1}{q}} \left(\int_I |f(x) - f_I|^p dx \right)^{\frac{1}{p}} \\
&\leq |I|^{\frac{1}{q}} (C|I|)^{\frac{1}{p}} \\
&= C|I| \\
&\leq C.
\end{aligned}$$

To prove the reverse inclusion relation, we use the following result:

LEMMA 1.10 (John-Nirenberg inequality). *For all $f \in BMO(\mathbb{T})$ and $I \subsetneqq \mathbb{T}$, there exist $C_1 = C_1(f, I)$ and $C_2 = C_2(f, I) > 0$ such that for all $t > 0$,*

$$|\{x \in I : |f(x) - f_I| > t\}| \leq C_1 e^{-\frac{C_2 t}{||f||_*}} |I|.$$

In this fact, suppose $f \in BMO(\mathbb{T})$, for all $0 < \forall C < C_2$, we have

$$\begin{aligned}
\int_I e^{\frac{C|f(x) - f_I|}{||f||_*}} dx &\leq C \int_{[0,\infty)} |\{x \in I : |f(x) - f_I| \cdot ||f||_*^{-1} > t\}| e^{Ct} dt \\
&\leq C \int_{[0,\infty)} C_1 e^{-C_2 t} |I| e^{Ct} dt \\
&= CC_1 |I| \int_{[0,\infty)} e^{-(C_2 - C)t} dt.
\end{aligned}$$

We note

$$\begin{aligned}
\int_0^\infty e^{-(C_2 - C)t} dt &= \lim_{M \to \infty} \int_0^M e^{-(C_2 - C)t} dt \\
&= \lim_{M \to \infty} \left(\frac{1}{C_2 - C} - \frac{1}{C_2 - C} e^{-(C_2 - C)} \right) \\
&= \frac{1}{C_2 - C}.
\end{aligned}$$

We get

$$\int_I e^{\frac{C|f(x)-f_I|}{||f||_*}} dx \leq \frac{CC_1}{C_2 - C}|I|.$$

Now, if $p \in \mathbb{N}$, we obtain

$$\begin{aligned}
\int_I \frac{C^p}{p!||f||_*^p}|f(x) - f_I|^p dx &\leq \int_I \sum_{n=0}^{\infty} \frac{(\frac{C|f(x)-f_I|}{||f||_*})^n}{n!} dx \\
&= \int_I e^{\frac{C|f(x)-f_I|}{||f||_*}} dx \\
&\leq \frac{CC_1}{C_2 - C}|I|
\end{aligned}$$

because of

$$e^{cx} = \sum_{n=0}^{\infty} \frac{(Cx)^n}{n!}.$$

Therefore, $\int_I |f(x) - f_I|^p dx \leq C|I|$. Moreover, if $p \notin \mathbb{N}$, for N such that

$$\begin{cases} p > N \text{ if } \frac{1}{|I|}\int_I |f(x) - f_I|^p dx \geq 1 \\ p < N \text{ if } \frac{1}{|I|}\int_I |f(x) - f_I|^p dx < 1, \end{cases}$$

we have

$$\left(\frac{1}{|I|}\int_I |f(x) - f_I|^p dx\right)^{\frac{1}{p}} \leq \left(\frac{1}{|I|}\int_I |f(x) - f_I|^p dx\right)^{\frac{1}{N}} \leq C.$$

Therefore, $\mathcal{L}^{p,1}(\mathbb{T}) \cong BMO(\mathbb{T})$.

2.4. In the case of $1 < \lambda < 1 + p$.

Definition 1.11. For $0 < \alpha < 1$, we exist $C > 0$ such that for all $x, y \in I$

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}.$$

Then, f is called Lipshitz function of order α in I, and denote by $f \in \mathrm{Lip}_{\alpha}(I)$ this. Moreover, Lip_{α} norm of f denoted by

$$||f||_{\Lambda_\alpha(I)} := \sup_{x,y \in I, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$

THEOREM 1.12 ([**53**]). *Suppose* $f \in L^1(I)$ *and* $0 < \alpha < 1$. *Then the following statements are equivalent:*

(i) $|f(x) - f(y)| \leq C_1|x-y|^\alpha$ *for all* $x, y \in I$,

(ii) $\dfrac{1}{|J|^{1+\alpha}} \displaystyle\int_J |f(x) - f_J| dx \leq C_2$ *for all* $J \subsetneqq I$,

(iii) $|f(x) - f_J| \leq C_3|J|^\alpha$ *for all* $x \in J$ *and* $J \subsetneqq I$,

(iv) $\left(\dfrac{1}{|J|^{1+\alpha p}} \displaystyle\int_J |f(x) - f_J|^p dx\right)^{\frac{1}{p}} \leq C_4$ *for all* $J \subsetneqq I$ *and* $1 < p < \infty$.

REMARK 1.13. In Theorem 1.12 of (iv), if we take $I = \mathbb{T}$ and $\alpha = \frac{\lambda - 1}{p}$, we have

$$\left(\frac{1}{|J|^\lambda} \int_J |f(x) - f_J|^p dx\right)^{\frac{1}{p}} \leq C$$

for all $J \subsetneqq \mathbb{T}$ and $1 < p < \infty$. Therefore, we have $\mathcal{L}^{p,\lambda}(\mathbb{T}) \cong \mathrm{Lip}_{\frac{\lambda-1}{p}}(\mathbb{T})$ if $1 < \lambda < 1 + p$.

PROOF OF THEOREM 1.12. We show this eauivalence as follows:

$$\text{(i)} \ \Rightarrow \ \text{(iii)} \ \Rightarrow \ \text{(iv)} \ \Rightarrow \ \text{(ii)} \ \Rightarrow \ \text{(i)}$$

We show (ii) implies (i). Assume $x < y$, $x, y \in I$, and $J = [x, y]$. Then, we define A and B as

$$|f(x) - f(y)| \leq |f(x) - f_J| + |f_J - f(y)| =: A + B.$$

We only consider A. Let a sequence of subinterval $\{J_k\}$ of J such that

$$J_1 = J, \quad |J_{n+1}| = \frac{1}{2}|J_n| \ \text{ and } \ x \in J_n \text{ for all } n \in \mathbb{N}.$$

For $k \geq 2$, we take A_1 and A_2 for

$$\begin{aligned} A &= |f(x) - f_{J_k} + f_{J_k} - f_{J_1}| \\ &\leq |f(x) - f_{J_k}| + \sum_{n=1}^{k-1} |f_{J_{n+1}} - f_{J_n}| =: A_1 + A_2. \end{aligned}$$

By the Lebesgue differentiation theorem, we get

$$\lim_{|J_k| \to 0} |f(x) - f_{J_k}| = 0 \quad a.e.\ x \in I.$$

As for A_2, because of

$$\begin{aligned} |f_{J_{n+1}} - f_{J_n}| &= \left| \frac{1}{|J_{n+1}|} \int_{J_{n+1}} f(x)dx - f_{J_n} \cdot \frac{1}{|J_{n+1}|} \int_{J_{n+1}} dx \right| \\ &= \left| \frac{1}{|J_{n+1}|} \int_{J_{n+1}} (f(x) - f_{J_n})dx \right| \\ &\leq \frac{1}{|J_{n+1}|} \int_{J_{n+1}} |f(x) - f_{J_n}|dx, \end{aligned}$$

we have

$$\begin{aligned} A_2 &\leq \sum_{n=1}^{k-1} \frac{1}{|J_{n+1}|} \int_{J_{n+1}} |f(x) - f_{J_n}|dx \\ &\leq \sum_{n=1}^{k-1} \frac{2}{|J_n|} \int_{J_n} |f(x) - f_{J_n}|dx \\ &\leq \sum_{n=1}^{k-1} 2C_2 |J_n|^\alpha \\ &= 2C_2 \sum_{n=1}^{k-1} \left(\frac{1}{2^{n-1}} |J| \right)^\alpha \\ &\leq C_2 C_\alpha |J|^\alpha. \end{aligned}$$

Hence, if $k \geq 2$, we obtain $A \leq CC_2|J|^\alpha = CC_2|x-y|^\alpha \quad a.e.\ x \in I$. Therefore, $|f(x) - f(y)| \leq CC_2|x-y|^\alpha \quad a.e.\ x, y \in I$. $\square$

2.5. In the case of $0 < \lambda < 1$.

Definition 1.14 ([**37**], [**53**]). Let p, λ be $1 < p < \infty$ and $0 \leq \lambda \leq 1$. Then, Morrey spaces $L^{p,\lambda}$ and this norm are defined by

$$L^{p,\lambda}(\mathbb{T}) := \left\{ f \in L^1(\mathbb{T}) \;\middle|\; \int_I |f(t)|^p dt < C|I|^\lambda \quad (\forall I \subsetneqq \mathbb{T}) \right\}$$

and

$$||f||_{L^{p,\lambda}} := \sup_{\substack{I \subset \mathbb{T} = [-\pi,\pi) \\ I \neq \emptyset : \text{interval}}} \left(\frac{1}{|I|^\lambda} \int_I |f(t)|^p dt \right)^{\frac{1}{p}},$$

where C stands for a constant independent of interval I.

Remark 1.15. When $\lambda = 0$ and 1, $L^{p,0}(\mathbb{T}) = L^p(\mathbb{T})$, $L^{p,1}(\mathbb{T}) = L^\infty(\mathbb{T})$, respectively. Therefore, we consider $0 < \lambda < 1$.

Theorem 1.16 (cf. [**37**]). *Let p, λ be $1 < p < \infty$ and $0 < \lambda < 1$. Then, we have*

$$L^{p,\lambda}(\mathbb{T}) \cong \mathcal{L}^{p,\lambda}(\mathbb{T}).$$

To prove this theorem, we give some lemmas.

Lemma 1.17 (cf. [**37**]). *Let p, λ be $1 < p < \infty$ and $0 < \lambda < 1$. Then, we have*

$$f \in \mathcal{L}^{p,\lambda}(\mathbb{T}) \quad \Longleftrightarrow \quad f \in L^p(\mathbb{T}) \quad \text{and} \quad |||f|||_{p,\lambda} < \infty,$$

where

$$|||f|||_{p,\lambda} := \sup_{\substack{I \subset \mathbb{T} = [-\pi,\pi) \\ I \neq \emptyset : \text{interval}}} \left\{ \frac{1}{|I|^\lambda} \left(\inf_{c \in \mathbb{C}} \int_I |f(t) - c|^p dt \right) \right\}^{\frac{1}{p}}.$$

Lemma 1.18 (cf. [**37**]). *Let $1 < p < \infty$, $0 < \lambda < 1$ and $0 < \alpha < \beta < \pi$. Then, for all $f \in \mathcal{L}^{p,\lambda}(\mathbb{T})$, $x \in \mathbb{T}$, we exist $C > 0$ such that*

$$|f_{x,\beta} - f_{x,\alpha}| \leq C \left(\frac{\beta^\lambda + \alpha^\lambda}{\alpha} \right)^{\frac{1}{p}} [f]_{p,\lambda},$$

where

$$f_{x,\alpha} = \frac{1}{2\alpha}\int_{x-\alpha}^{x+\alpha} f(y)dy.$$

LEMMA 1.19 (cf. [**37**]). *Let $1 < p < \infty$, $0 < \lambda < 1$ and $0 < \gamma \leq \pi$. Then, for all $f \in \mathcal{L}^{p,\lambda}(\mathbb{T})$ and $n \in \mathbb{N}$, we exist $C > 0$ such that*

$$|f_{x,\gamma} - f_{x,\frac{\gamma}{2^n}}| \leq C[f]_{p,\lambda}\gamma^{\frac{\lambda-1}{p}}\sum_{m=0}^{n-1} 2^{\frac{m(1-\lambda)}{p}}.$$

LEMMA 1.20 (cf. [**37**]). *Let p, λ be $1 < p < \infty$ and $0 < \lambda < 1$. Then, for all $f \in \mathcal{L}^{p,\lambda}(\mathbb{T})$, we exist $C > 0$ such that*

$$|f_I| \leq |f_{\mathbb{T}}| + C[f]_{p,\lambda}|I|^{\frac{\lambda-1}{p}}.$$

PROOF OF THEOREM 1.16. Let $f \in L^{p,\lambda}(\mathbb{T})$. Then, we have

$$\begin{aligned}
||f||_{\mathcal{L}^{p,\lambda}}^p &\leq 3^p(||f||_{L^p}^p + |||f|||_{p,\lambda}^p) \\
&= 3^p\left\{||f||_{L^p}^p + \sup_{\substack{I\subset\mathbb{T}=[-\pi,\pi) \\ I\neq\emptyset:\text{interval}}} \frac{1}{|I|^\lambda}\left(\inf_{c\in\mathbb{C}}\int_I |f(t)-c|^p dt\right)\right\} \\
&\leq 3^p(||f||_{L^p}^p + ||f||_{L^{p,\lambda}}^p) \\
&= 3^p\left(\frac{(2\pi)^\lambda}{2\pi}\frac{1}{(2\pi)^\lambda}\int_{-\pi}^{\pi}|f(t)|^p dt + ||f||_{L^{p,\lambda}}^p\right) \\
&\leq 3^p \cdot 2||f||_{L^{p,\lambda}}^p \\
&\leq C
\end{aligned}$$

by Lemma 1.17. Therefore, $f \in \mathcal{L}^{p,\lambda}(\mathbb{T})$. On the other hand, suppose $f \in \mathcal{L}^{p,\lambda}(\mathbb{T})$. We have

$$\begin{aligned}
\int_I |f(t) - f_I|^p dt &= |I|^\lambda \frac{1}{|I|^\lambda}\int_I |f(t)-f_I|^p dt \\
&\leq |I|^\lambda [f]_{p,\lambda}^p
\end{aligned}$$

and by Lemma 1.20, we obtain

$$\begin{aligned}\int_I |f_I|^p dt &\leq C\int_I |f_{\mathbb{T}}|^p dt + C\int_I [f]^p_{p,\lambda}|I|^{\lambda-1}dt\\ &\leq C|I|^\lambda [f]^p_{p,\lambda} + C|I|\ |f_{\mathbb{T}}|^p.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_I |f(t)|^p dt &\leq 2^{p-1}\left(\int_I |f(t)-f_I|^p dt + \int_I |f_I|^p dt\right)\\ &\leq 2^{p-1}(|I|^\lambda [f]^p_{p,\lambda} + C|I|^\lambda [f]^p_{p,\lambda} + C|I|\ |f_{\mathbb{T}}|^p)\\ &\leq C(|I|^\lambda [f]^p_{p,\lambda} + |I|\ ||f||^p_{L^1})\\ &\leq C|I|^\lambda([f]^p_{p,\lambda} + ||f||^p_{L^p}).\end{aligned}$$

Hence,

$$\frac{1}{|I|^\lambda}\int_I |f(t)|^p dt \leq C([f]^p_{p,\lambda} + ||f||^p_{L^p}) \leq C||f||_{\mathcal{L}^p_{p,\lambda}}.$$

Therefore, $L^{p,\lambda}(\mathbb{T}) \cong \mathcal{L}^{p,\lambda}(\mathbb{T})$ if $0<\lambda<1$. $\square$

The following is a summary of the above:

$$\mathcal{L}^{p,\lambda}(\mathbb{T}) \cong \begin{cases} L^p(\mathbb{T}) & \text{if } \lambda = 0\\ BMO(\mathbb{T}) & \text{if } \lambda = 1\\ \mathrm{Lip}_{\frac{\lambda-1}{p}}(\mathbb{T}) & \text{if } 1<\lambda<1+p\\ L^{p,\lambda}(\mathbb{T}) & \text{if } 0<\lambda<1.\end{cases}$$

Remark 1.21. When $\lambda = 1+p$, f is absolutely continuous. And in the case $\lambda > 1+p$, we get

$$\frac{|f(x+h)-f(x)|}{|h|} \leq C|h|^{\alpha-1}$$

if we take $y-x=h$. Then, $f'(x)=0$ if $h\to 0$. Therefore, f is constant function.

3. Main results

Let p be in $1 < p < \infty$, q the conjugate exponent of p, and $0 < \lambda < 1$. Also let $L^p(\mathbb{T})$ be the usual L^p-space on the unit circle $\mathbb{T}$ with respect to the normalized Haar measure. The Morrey spaces $L^{p,\lambda}(\mathbb{T})$ are defined by

$$L^{p,\lambda}(\mathbb{T}) = \left\{ f \;\middle|\; ||f||_{p,\lambda} = \sup_{\substack{I \subset \mathbb{T} = [-\pi,\pi) \\ I \neq \emptyset : \text{interval}}} \left(\frac{1}{|I|^\lambda} \int_I |f|^p dx \right)^{1/p} < \infty \right\},$$

and $L_0^{p,\lambda}(\mathbb{T})$ the closure of $C(\mathbb{T})$ in $L^{p,\lambda}(\mathbb{T})$, where $C(\mathbb{T})$ is the set of all continuous functions on $\mathbb{T}$. Then it is easy to see that $L^{p,\lambda}(\mathbb{T})$ is a Banach space (cf. Kufner [**37**], Torchinsky [**53**, p.215]). Also $Z^{q,\lambda}(\mathbb{T})$ $(1/p + 1/q = 1)$ are defined by $\{f \mid ||f||_{Z^{q,\lambda}} < \infty\}$, where

$$||f||_{Z^{q,\lambda}} = \inf \left\{ \sum_{k=1}^{\infty} |c_k| \;\middle|\; f(x) = \sum_{k=1}^{\infty} c_k a_k(x),\ c_k \in \mathbb{C},\ a_k(x) : (q,\lambda)\text{-block} \right\},$$

where $a_k(x)$ is called (q,λ)-block, if

(1) $\operatorname{supp} a_k \subset I$

(2) $||a_k||_q \leq \frac{1}{|I|^{\lambda/p}}$, where $1/p + 1/q = 1$,

for some interval I. In particular, $a_k(x)$ is called (q,λ)-atom, if a_k satisfies $\int_I a_k(x)dx = 0$, which is called cancellation property. $Z^{q,\lambda}(\mathbb{T})$ is a Banach space with the norm $||\cdot||_{Z^{q,\lambda}}$. Zorko [**55**] introduced the space $Z^{q,\lambda}(\mathbb{T})$, and proved that $Z^{q,\lambda}(\mathbb{T})$ is the predual of $L^{p,\lambda}(\mathbb{T})$. Also she [**55**] defined $L_0^{p,\lambda}(\mathbb{T})$, and remarked some properties. Adams-Xiao [**3**] pointed out that $L_0^{p,\lambda}(\mathbb{T})$ is the predual of $Z^{q,\lambda}(\mathbb{T})$, but they did not give the reason why they insisted that the proof is akin to that of H^1-VMO in Stein [**51**] (cf. [**53**]). Like Adams-Xiao [**3**], we think that $L^{p,\lambda}(\mathbb{T}), Z^{q,\lambda}(\mathbb{T}), L_0^{p,\lambda}(\mathbb{T})$ are similar to $BMO(\mathbb{T}), H^1(\mathbb{T}), VMO(\mathbb{T})$, respectively.

In the rest of this chapter, we show some properties of $L_0^{p,\lambda}(\mathbb{T})$, which is similar to that of $VMO(\mathbb{T})$. Next we give a detailed proof of the fact that $L_0^{p,\lambda}(\mathbb{T})$ is the predual of $Z^{q,\lambda}(\mathbb{T})$, by the method of Coifman-Weiss [**10**]. We expect that our proofs in the case of $\mathbb{T}$ may be available to Euclidean case $\mathbb{R}^n$.

Our results are as follows:

THEOREM 1.22. *Let $1 \leq p < \infty$, and $0 < \lambda < 1$. Also let ϕ be an infinitely differentiable function such that supp $\phi \subset [-1,1]$, $\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(x)dx = 1$ and $\phi \geq 0$, and let $\phi_j(x) = j\phi(jx)$ $(j = 1, 2, \cdots)$. Then, the following properties are equivalent:*

(1) $f \in L_0^{p,\lambda}(\mathbb{T})$

(2) $f \in L^{p,\lambda}(\mathbb{T})$ *and* $||\tau_y f - f||_{p,\lambda} \to 0$ $(y \to 0)$,

where $\tau_y f(x) = f(x-y)$

(3) $f \in L^{p,\lambda}(\mathbb{T})$ *and* $||f - f * \phi_j||_{p,\lambda} \to 0$ $(j \to \infty)$

(4) $\lim_{\delta\to 0}\sup_{|I|\leq\delta, I\subset\mathbb{T}:\text{interval}}\frac{1}{|I|^{\lambda}}\int_I |f(x)|^p dx = 0$

THEOREM 1.23. *Let $1 < p < \infty$, and $0 < \lambda < 1$. Then $L_0^{p,\lambda}(\mathbb{T})$ is the predual of $Z^{q,\lambda}(\mathbb{T})$, where $1/p + 1/q = 1$.*

Throughout the rest of this chapter, the dual space of a Banach space X is denoted by X^*. For an interval I, $|I|$ denotes the measure of I with respect to the normalized Haar measure of $\mathbb{T}$. Also the letter C stands for a constant not necessarily the same at each occurrence. $A \sim B$ stands for $C^{-1}A \leq B \leq CA$ for some $C > 0$.

4. Proofs of Main Theorems

4.1. Proof of Theorem 1.22.

Proof. According to Zorko [**55**], it is easy to prove that (1), (2) and (3) are equivalent. Then, we omit their proofs. We show (4), when we assume (1). By the definition, for $f \in L_0^{p,\lambda}(\mathbb{T})$ and for any $\eta > 0$ there exists $g \in C(\mathbb{T})$ such that $||f - g||_{p,\lambda} < \eta$. Then for an interval $I \subset \mathbb{T}$ with $|I| \leq \delta$, we have

$$\begin{aligned}
&\left(\frac{1}{|I|^\lambda}\int_I |f(x)|^p dx\right)^{1/p} \\
&\leq \left(\frac{1}{|I|^\lambda}\int_I |f(x) - g(x)|^p dx\right)^{1/p} + \left(\frac{1}{|I|^\lambda}\int_I |g(x)|^p dx\right)^{1/p} \\
&\leq \eta + \left(\frac{1}{|I|^\lambda}\int_I |g(x)|^p dx\right)^{1/p} \\
&\leq \eta + |I|^{\frac{1-\lambda}{p}} ||g||_{C(\mathbb{T})} \\
&\leq \eta + \delta^{\frac{1-\lambda}{p}} ||g||_{C(\mathbb{T})},
\end{aligned}$$

and

$$\lim_{\delta \to 0} \sup_{|I| \leq \delta, I:\text{interval}} \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx \leq \eta^p.$$

So we obtain (4). Next we show (3), when we assume (4). For any $\eta > 0$, there exists $\delta_0 > 0$ such that

$$\sup_{|I| \leq \delta_0, I:\text{interval}} \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx < \eta^p.$$

Then for $|I| \leq \delta_0$, we have

$$\begin{aligned}
\frac{1}{|I|^\lambda}\int_I |f * \phi_j(x)|^p dx &\leq \frac{1}{|I|^\lambda}\int_I \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y)|^p \phi_j(y) dy\right) dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_j(y) \frac{1}{|I|^\lambda}\int_I |f(x-y)|^p dx dy \\
&\leq \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx \\
&< \eta^p
\end{aligned}$$

by the Hölder inequality. Hence, for an interval $I \subset \mathbb{T}$ with $|I| \leq \delta_0$, we have

$$\begin{aligned}
&\left(\frac{1}{|I|^\lambda}\int_I |f(x) - f * \phi_j(x)|^p dx\right)^{1/p} \\
&\leq \left(\frac{1}{|I|^\lambda}\int_I |f(x)|^p dx\right)^{1/p} + \left(\frac{1}{|I|^\lambda}\int_I |f * \phi_j(x)|^p dx\right)^{1/p} \\
&\leq 2\left(\sup_{|I|\leq\delta_0, I:\text{interval}} \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx\right)^{1/p} \\
&< 2\eta.
\end{aligned}$$

On the other hand, for an interval $I \subset \mathbb{T}$ with $|I| > \delta_0$, we have

$$\begin{aligned}
\frac{1}{|I|^\lambda}\int_I |f(x) - f * \phi_j(x)|^p dx &\leq \frac{2\pi}{\delta_0^\lambda}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f * \phi_j(x)|^p dx \\
&= \frac{2\pi}{\delta_0^\lambda}||f - f * \phi_j||_p^p.
\end{aligned}$$

After all, we obtain

$$\sup_{I\subset\mathbb{T}:\text{interval}} \frac{1}{|I|^\lambda}\int_I |f(x) - f * \phi_j(x)|^p dx < (2\eta)^p + \frac{2\pi}{\delta_0^\lambda}||f - f * \phi_j||_p^p.$$

Therefore, we have

$$\lim_{j\to\infty} ||f - f * \phi_j||_{p,\lambda} = 0.$$

□

REMARK 1.24. Let f be in $Z^{q,\lambda}(\mathbb{T})$ such that $f = \sum_{k=1}^\infty c_k a_k$, where $\sum_k |c_k| < \infty$, a_k:(q,λ)-block. Then $f = \sum_k c_k a_k$ converges in $L^1(\mathbb{T})$ by the definition of $Z^{q,\lambda}(\mathbb{T})$ and Hölder's inequality.

4.2. Proof of Theorem 1.23.

For the proof, we give some lemmas.

LEMMA 1.25 ([**55**]). *Let $1 < p < \infty, 0 < \lambda < 1$ and q the conjugate exponent of p. Then the dual space of $Z^{q,\lambda}(\mathbb{T})$ is $L^{p,\lambda}(\mathbb{T})$.*

LEMMA 1.26. *Let $1 < p < \infty$ and q be the conjugate exponent. Also let $0 < \lambda < 1$. Then every $f \in Z^{q,\lambda}(\mathbb{T})$ can be decomposed into a sum of block and atoms:*

$$f = c_0 a_0 + \sum_{k=1}^{\infty} c_k a_k,$$

where $c_k \in \mathbb{C}$ and $|c_0| + \sum_{k=1}^{\infty} |c_k| \le C||f||_{Z^{q,\lambda}}$, a_0 is a (q,λ)-block with supp $a_0 \subset \mathbb{T}$, $a_k's$ are (q,λ)-atoms such that supp $a_k \subset I_k$ satisfying $|I_k| \le \frac{1}{4}$.

PROOF. Let $\mathbb{T} = [0, 2\pi)$, and $f \in Z^{q,\lambda}(\mathbb{T})$. Then, f is decomposed so that

$$f = \sum_{k=0}^{\infty} c'_k b_k,$$

where $c'_k \in \mathbb{C}$, $\sum |c'_k| \le 2\|f\|_{Z^{q,\lambda}}$, and $\{b_k\}_{k=0}^{\infty}$ are (q,λ)-blocks. Let $b(x)$ be $b_k(x)$ for any $k \ge 0$, and A a set of functions defined by

$$A := \left\{ b_k \;\middle|\; \text{supp } b_k \subset I,\ ||b_k||_q \le \frac{1}{|I|^{\lambda/p}},\ \text{and } |I| > \frac{1}{4} \right\}.$$

In the case of $|I| \le \frac{1}{4}$, we define b_1^1, b_2^1, I_1 by

$$\begin{aligned} b_1^1(x) &= \frac{b(x) - b(x - |I|)}{2^{\frac{\lambda-1}{p}+1}}, \\ b_2^1(x) &= \frac{b(x) + b(x - |I|)}{2^{\frac{\lambda-1}{p}+1}}, \\ I_1 &= I \cup (I + |I|). \end{aligned}$$

Then, we have supp $b_j^1 \subset I_1$ $(j = 1, 2)$ and

$$\begin{aligned} \left(\int_{I_1} |b_j^1(x)|^q dx \right)^{1/q} &= \left(2 \int_I |b(x)|^q dx \right)^{1/q} 2^{-\frac{\lambda-1}{p}-1} \\ &\le 2^{\frac{1}{q}-\frac{\lambda-1}{p}-1} \frac{1}{|I|^{\lambda/p}} \\ &= 2^{-\lambda/p} \frac{1}{|I|^{\lambda/p}} = \frac{1}{|I_1|^{\lambda/p}} \qquad (j = 1, 2), \end{aligned}$$

which shows that b_j^1 is a (q,λ)-block $(j=1,2)$. We also have

$$\int_0^{2\pi} b_1^1(x)\,dx = 0,$$

$$2^{\frac{\lambda-1}{p}} b_1^1(x) + 2^{\frac{\lambda-1}{p}} b_2^1(x) = \frac{b(x)-b(x-|I|)}{2} + \frac{b(x)+b(x-|I|)}{2} = b(x).$$

So, b_1^1 is a (q,λ)-atom. When we set $\alpha = 2^{\frac{\lambda-1}{p}}$ and $a_k^1(x) = b_1^1(x)$, we have $b_k(x) = \alpha a_k^1(x) + \alpha b_2^1(x)$. Next, if we have $|I_1| \le \frac{1}{4}$, there exists a natural number $\ell \ge 3$ such that $\frac{1}{2^\ell} < |I_1| \le \frac{1}{2^{\ell-1}}$. So, we decompose $b_2^1(x)$ like $b(x)$ and define a_k^2, b_2^2, I_2 by

$$a_k^2(x) = \frac{b_2^1(x) - b_2^1(x-|I_1|)}{2^{\frac{\lambda-1}{p}+1}},$$

$$b_2^2(x) = \frac{b_2^1(x) + b_2^1(x-|I_1|)}{2^{\frac{\lambda-1}{p}+1}},$$

$$I_2 = I_1 \cup (I_1 + |I_1|).$$

Then we have

$$\int_0^{2\pi} a_k^2(x)dx = 0,$$

$$b_2^1(x) = \alpha a_k^2(x) + \alpha b_2^2(x),$$

$$\begin{aligned} b_k(x) &= \alpha a_k^1(x) + \alpha b_2^1(x) \\ &= \alpha a_k^1(x) + \alpha^2 a_k^2(x) + \alpha^2 b_2^2(x), \end{aligned}$$

and hence, we see that a_k^1, a_k^2 are (q,λ)-atoms and b_2^2 is a (q,λ)-block. In fact,

$$\left(\int_{I_2} |b_2^2(x)|^q dx\right)^{1/q} \le 2^{-\lambda/p}|I_1|^{-\lambda/p} = |I_2|^{-\lambda/p}.$$

We repeat this process ℓ times until we have $|I_\ell| > \frac{1}{4}$. After all, we get

$$b_k(x) = \sum_{j=1}^{\ell} \alpha^j a_k^j(x) + \alpha^\ell b_2^\ell(x),$$

where $\alpha = 2^{\frac{\lambda-1}{p}}$, a_k^j $(j = 1, \cdots, \ell)$: (q, λ)-atoms with supp $a_k^j \subset I_j$, and b_2^ℓ : (q, λ)-block with supp $b_k^\ell \subset I_\ell$. When we set $\ell_k = \ell$, we have

$$b_k(x) = \sum_{j=1}^{\ell_k} \alpha^j a_k^j(x) + \alpha^{\ell_k} b_2^{\ell_k}(x).$$

After we repeat this process for b_k, we obtain

$$f(x) = \sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} c'_k \alpha^\ell a_k^\ell(x) + \sum_{b_k \notin A} c'_k \alpha^{\ell_k} b_2^{\ell_k}(x) + \sum_{b_k \in A} c'_k b_k(x).$$

Noting $0 < \alpha < 1$, we have

$$\sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} |c'_k| \alpha^\ell + \sum_{b_k \notin A} |c'_k| \alpha^{\ell_k} + \sum_{b_k \in A} |c'_k| \leq \left(\frac{1}{1-\alpha} + \alpha + 1 \right) \sum_{k=0}^{\infty} |c'_k|.$$

Also when we define

$$a_0(x) = \frac{\sum_{b_k \notin A} c'_k \alpha^{\ell_k} b_2^{\ell_k}(x) + \sum_{b_k \in A} c'_k b_k(x)}{4^{\lambda/p} \left(\sum_{b_k \notin A} |c'_k| \alpha^{\ell_k} + \sum_{b_k \in A} |c'_k| \right)},$$

we have that $||a_0||_q \leq 1$, supp $a_0 \subset \mathbb{T} = [0, 2\pi)$ and a_0 : (q, λ)-block, since

$$\begin{aligned} &\left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{b_k \notin A} c'_k \alpha^{\ell_k} b_2^{\ell_k}(x) + \sum_{b_k \in A} c'_k b_k(x) \right|^q dx \right)^{1/q} \\ &\leq 4^{\lambda/p} \left(\sum_{b_k \notin A} |c'_k| \alpha^{\ell_k} + \sum_{b_k \in A} |c'_k| \right). \end{aligned}$$

Moreover, we obtain

$$f(x) = 4^{\lambda/p} \left(\sum_{b_k \notin A} |c'_k| \alpha^{\ell_k} + \sum_{b_k \in A} |c'_k| \right) a_0(x) + \sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} c'_k \alpha^\ell a_k^\ell(x)$$

and

$$4^{\lambda/p} \left(\sum_{b_k \notin A} |c'_k| \alpha^{\ell_k} + \sum_{b_k \in A} |c'_k| \right) + \sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} |c'_k| \alpha^\ell \leq 2 \left(4^{\lambda/p} + \frac{1}{1-\alpha} \right) ||f||_{Z^{q,\lambda}}.$$

$\square$

LEMMA 1.27. *Let n be any positive integer, $B_j^n = [\frac{j-1}{3^n}2\pi, \frac{j}{3^n}2\pi)$ $(j = 1, \cdots, 3^n)$, and $\tilde{B}_j^n = 3B_j^n$, where the center of $\tilde{B}_j^n$ is the same as the center of B_j^n, and $|\tilde{B}_j^n| = 3|B_j^n|$. Also let $B^0 = B_1^0 = [0, 2\pi)$, and $\tilde{B}^0 = \tilde{B}_1^0 = [0, 2\pi)$. Then, $f \in Z^{q,\lambda}(\mathbb{T})$ has the representation*

$$f(x) = \lambda_0 a_0(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n a_j^n(x),$$

where $a_0 : (q, \lambda)$-block, $a_j^n : (q, \lambda)$-atoms, supp $a_0 \subset \mathbb{T}$, supp $a_j^n \subset \tilde{B}_j^n$, and $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \leq C||f||_{Z^{q,\lambda}}$.

PROOF. By Lemma 1.26, $f \in Z^{q,\lambda}(\mathbb{T})$ can be decomposed into a sum of block and atoms:

$$f = c_0 b_0 + \sum_{k=1}^{\infty} c_k b_k,$$

where $c_k \in \mathbb{C}$, $|c_0| + \sum_{k=1}^{\infty} |c_k| \leq C||f||_{Z^{q,\lambda}}$, and b_0 is a (q, λ)-block with supp $b_0 \subset \mathbb{T}$, and b_k's are (q, λ)-atoms such that supp $b_k \subset I_k$ satisfying $|I_k| \leq \frac{1}{4}$. For I_k with $\frac{1}{3^2} < |I_k| \leq \frac{1}{3}$, there exists $j \in \{1, 2, 3\}$ such that $I_k \cap B_j^1 \neq \emptyset$. For B_1^1 we let Λ_1^1 be the index set $k \in \mathbb{N}$, determined by those b_k with $\frac{1}{3^2} < |I_k| \leq \frac{1}{3}$ and $I_k \cap B_1^1 \neq \emptyset$. Then, we see that $I_k \subset \tilde{B}_1^1$ for $k \in \Lambda_1^1$ and

$$\Big\|\sum_{k\in\Lambda_1^1} c_k b_k\Big\|_q \leq \sum_{k\in\Lambda_1^1} |c_k| \; ||b_k||_q \leq \sum_{k\in\Lambda_1^1} |c_k| \; |\tilde{B}_1^1|^{-\lambda/p} 3^{2\lambda/p}.$$

So, when we define

$$a_1^1 = \frac{\sum_{k\in\Lambda_1^1} c_k b_k}{3^{2\lambda/p} \sum_{k\in\Lambda_1^1} |c_k|} \text{ and } \lambda_1^1 = \sum_{k\in\Lambda_1^1} |c_k| 3^{2\lambda/p},$$

we have supp $a_1^1 \subset \tilde{B}_1^1$, $||a_1^1||_q \leq \frac{1}{|\tilde{B}_1^1|^{\lambda/p}}$, and a_1^1 satisfies the cancellation property, that is, a_1^1 is a (q, λ)-atom supported by $\tilde{B}_1^1$, and

$$\lambda_1^1 a_1^1 = \sum_{k\in\Lambda_1^1} c_k b_k.$$

Next for B_2^1 we let Λ_2^1 be the index set determined by b_k in $\{b_j\}$ with $\frac{1}{3^2} < |I_k| \leq \frac{1}{3}$ and $I_k \cap B_2^1 \neq \emptyset$, excluding b_k which we have already chosen before. We construct (q, λ)-atom a_2^1 in the same way as for B_1^1. Similarly we construct (q, λ)-atom a_3^1 for B_3^1. We do this process for b_k with $\frac{1}{3^3} < |I_k| \leq \frac{1}{3^2}$, and obtain the index set Λ_j^2, (q, λ)-atoms a_j^2 with supp $a_j^2 \subset \tilde{B}_j^2$, and numbers λ_j^2 $(j = 1, \cdots, 3^2)$, satisfying

$$\lambda_j^2 a_j^2 = \sum_{k \in \Lambda_j^2} c_k b_k.$$

After that, we repeat this process. In the n-th step, for b_k with $\frac{1}{3^{n+1}} < |I_k| \leq \frac{1}{3^n}$ we obtain the index set Λ_j^n, (q, λ)-atoms a_j^n with supp $a_j^n \subset \tilde{B}_j^n$, and numbers λ_j^n $(j = 1, \cdots, 3^n)$, satisfying

$$\lambda_j^n a_j^n = \sum_{k \in \Lambda_j^n} c_k b_k.$$

By the construction of a_j^n and λ_j^n, we have

$$f(x) = \lambda_0 a_0(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n a_j^n(x),$$

where $a_0 = b_0$: (q, λ)-block, $\lambda_0 = c_0$, a_j^n : (q, λ)-atoms, supp $a_0 \subset \mathbb{T}$, supp $a_j^n \subset \tilde{B}_j^n$, and $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \leq 2 \cdot 3^{2\lambda/p} ||f||_{Z^{q,\lambda}}$. $\square$

Lemma 1.28. *Suppose $||f_k||_{Z^{q,\lambda}} \leq 1, k = 1, 2, \cdots$. Then there exist $f \in Z^{q,\lambda}(\mathbb{T})$ and a subsequence $\{f_{k_j}\}$ such that*

$$\lim_{j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f_{k_j}(x) v(x) dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) v(x) dx$$

for all $v \in C(\mathbb{T})$.

Proof. By Lemma 1.27, we may assume that $f_k \in Z^{q,\lambda}(\mathbb{T})$ has the representation

$$f_k(x) = \lambda_0(k) a_0(k)(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n(k) a_j^n(k)(x),$$

where $a_0(k)$: (q,λ)-block, $a_j^n(k)$: (q,λ)-atoms, supp $a_0(k) \subset \mathbb{T}$, supp $a_j^n(k) \subset \tilde{B}_j^n$, and $|\lambda_0(k)| + \sum_{j,n} |\lambda_j^n(k)| \le C$. Also we may assume that $\lambda_0(k)$, $\lambda_j^n(k) \ge 0$, $||a_j^n(k)||_q \le |\tilde{B}_j^n|^{-\lambda/p}$, and that there exist λ_0, λ_j^n such that $\lim_{k\to\infty} \lambda_0(k) = \lambda_0$, $\lim_{k\to\infty} \lambda_j^n(k) = \lambda_j^n$ $(j,n \ge 1)$, and $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \le C$. Let $L^q(\tilde{B}_j^n) = (L^p(\tilde{B}_j^n))^*$ be the dual space of $L^p(\tilde{B}_j^n)$ (L^p-space on $\tilde{B}_j^n$). By $a_j^n(k) \in L^q(\tilde{B}_j^n)$ and the diagonal argument, there exists an increasing sequence of natural numbers, $k_1 < k_2 < \cdots < k_n < \cdots$ and $a_0 \in L^q(\tilde{B}^0)$, $a_j^n \in L^q(\tilde{B}_j^n)$ such that for $\phi \in L^p(\mathbb{T})$

$$\lim_{\ell\to\infty} \frac{1}{2\pi} \int_0^{2\pi} a_j^n(k_\ell)(x)\phi(x)dx = \frac{1}{2\pi} \int_0^{2\pi} a_j^n(x)\phi(x)dx$$

and

$$\lim_{\ell\to\infty} \frac{1}{2\pi} \int_0^{2\pi} a_0(k_\ell)(x)\phi(x)dx = \frac{1}{2\pi} \int_0^{2\pi} a_0(x)\phi(x)dx,$$

that is, $a_j^n(k_\ell) \to a_j^n$ $(\ell \to \infty)$ in the weak*-topology of $\sigma(L^q(\tilde{B}_j^n), L^p(\tilde{B}_j^n))$ $(j,n \ge 1)$ and $a_0(k_\ell) \to a_0$ $(\ell \to \infty)$ in the weak*-topology of $\sigma(L^q(\tilde{B}^0), L^p(\tilde{B}^0))$. Here, we define f by

$$f(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n a_j^n(x),$$

where $a_1^0 = a_0$ and $\lambda_1^0 = \lambda_0$. Then f is in $Z^{q,\lambda}(\mathbb{T})$ and a_j^n are (q,λ)-atoms, since supp $a_j^n \subset \tilde{B}_j^n$, $||a_j^n||_q \le |\tilde{B}_j^n|^{-\lambda/p}$, $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \le C$, and $\int_{\tilde{B}_j^n} a_j^n(x)dx = 0$. Let $v \in C(\mathbb{T})$, and $a_1^0(k_\ell) = a_0(k_\ell), \lambda_1^0(k_\ell) = \lambda_0(k_\ell)$. We define

$$J_{k_\ell} = \frac{1}{2\pi} \int_0^{2\pi} f_{k_\ell}(x)v(x)dx = \sum_{n=0}^{\infty} \sum_j \lambda_j^n(k_\ell) \frac{1}{2\pi} \int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx,$$

and

$$J = \frac{1}{2\pi}\int_0^{2\pi} f(x)v(x)dx = \sum_{n=0}^{\infty}\sum_j \lambda_j^n \frac{1}{2\pi}\int_0^{2\pi} a_j^n(x)v(x)dx.$$

Also, for any integer N we define

$$\begin{aligned}
J_{k_\ell}^N &= \sum_{n=0}^{N}\sum_j \lambda_j^n(k_\ell)\frac{1}{2\pi}\int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx,\\
J_{k_\ell}^{N,\infty} &= \sum_{n=N+1}^{\infty}\sum_j \lambda_j^n(k_\ell)\frac{1}{2\pi}\int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx,\\
J^N &= \sum_{n=0}^{N}\sum_j \lambda_j^n \frac{1}{2\pi}\int_0^{2\pi} a_j^n(x)v(x)dx,
\end{aligned}$$

and

$$J^{N,\infty} = \sum_{n=N+1}^{\infty}\sum_j \lambda_j^n \frac{1}{2\pi}\int_0^{2\pi} a_j^n(x)v(x)dx.$$

Moreover, when the center of $\tilde{B}_j^n$ $(j, n \geq 1)$ is denoted by x_j^n, we have

$$J_{k_\ell}^{N,\infty} = \sum_{n=N+1}^{\infty}\sum_j \lambda_j^n(k_\ell)\frac{1}{2\pi}\int_{\tilde{B}_j^n} a_j^n(k_\ell)(x)(v(x) - v(x_j^n))dx,$$

since $a_j^n(k)$ $(j, n \geq 1)$ are (q, λ)-atoms. Here, we remark that v is uniformly continuous on $\mathbb{T}$. Hence, for any $\varepsilon > 0$ there exists N_0 such that

$$|J_{k_\ell}^{N_0,\infty}| \leq \varepsilon \sum_{n=N_0+1}^{\infty}\sum_j \lambda_j^n(k_\ell)|\tilde{B}_j^n|^{\frac{1-\lambda}{p}} \leq C\varepsilon.$$

The same conclusion can be drawn for $J^{N_0,\infty}$, since a_j^n are (q,λ)-atoms. Also we have

$$\begin{aligned}
&\left|\sum_{n=0}^{N_0}\sum_{j=1}^{3^n}\left(\lambda_j^n(k_\ell)\frac{1}{2\pi}\int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx - \lambda_j^n\frac{1}{2\pi}\int_0^{2\pi} a_j^n(x)v(x)dx\right)\right| \\
&\leq \sum_{n=0}^{N_0}\sum_{j=1}^{3^n}\left\{\lambda_j^n(k_\ell)\left|\frac{1}{2\pi}\int_0^{2\pi}(a_j^n(k_\ell)(x)-a_j^n(x))v(x)dx\right|\right. \\
&\quad \left. +|\lambda_j^n(k_\ell)-\lambda_j^n|\left|\frac{1}{2\pi}\int_0^{2\pi} a_j^n(x)v(x)dx\right|\right\} \\
&\quad \to 0,
\end{aligned}$$

as $\ell\to\infty$. Then, we obtain

$$J_{k_\ell}-J=(J_{k_\ell}^{N_0}-J^{N_0})+(J_{k_\ell}^{N_0,\infty}-J^{N_0,\infty}),$$

$$\begin{aligned}
|J_{k_\ell}^{N_0,\infty}-J^{N_0,\infty}| &\leq |J_{k_\ell}^{N_0,\infty}|+|J^{N_0,\infty}| \\
&\leq 2C\varepsilon.
\end{aligned}$$

Hence, we have $\limsup_{\ell\to\infty}|J_{k_\ell}-J|\leq 2C\varepsilon$, and $\lim_{\ell\to\infty}J_{k_\ell}=J$. Therefore, we get the result:

$$\lim_{\ell\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f_{k_\ell}(x)v(x)dx=\frac{1}{2\pi}\int_0^{2\pi} f(x)v(x)dx \ (v\in C(\mathbb{T})).$$

$\square$

LEMMA 1.29. *Let f be in $Z^{q,\lambda}(\mathbb{T})$. Then we have*

$$||f||_{Z^{q,\lambda}}\sim ||f||_{(L_0^{p,\lambda})^*}.$$

PROOF. Let $A=||f||_{Z^{q,\lambda}}>0$. Then there exists $g\in L^{p,\lambda}(\mathbb{T})$ such that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx\right|\geq\frac{A}{2},\ ||g||_{p,\lambda}\leq 1.$$

By $f \in Z^{q,\lambda}(\mathbb{T})$, we may assume that

$$f(x) = \sum_{k=0}^{\infty} c_k a_k(x),$$

where a_k : (q,λ)-block, supp $a_k \subset B_k$ for some interval B_k, and $\sum_{k=0}^{\infty} |c_k| \leq 2||f||_{Z^{q,\lambda}}$. Also for any $\varepsilon > 0$ let $\phi_\varepsilon(x) = \frac{1}{|I_\varepsilon|}\chi_{I_\varepsilon}(x)$, where $I_\varepsilon = [-\varepsilon, \varepsilon]$ and χ_E denotes the characteristic function of E. When we define $g_\varepsilon(x) = g * \phi_\varepsilon(x)$ for $g \in L^{p,\lambda}(\mathbb{T})$, it is easy to see $g_\varepsilon \in C(\mathbb{T})$ and $||g_\varepsilon||_{p,\lambda} \leq ||g||_{p,\lambda}$. Now for any integer $N \geq 1$ and $g \in L^{p,\lambda}(\mathbb{T})$, we define

$$I_\varepsilon^N = \sum_{k=0}^{N} c_k \frac{1}{2\pi} \int_0^{2\pi} a_k(x)(g(x) - g_\varepsilon(x))dx,$$

and

$$II_\varepsilon^N = \sum_{k=N+1}^{\infty} c_k \frac{1}{2\pi} \int_0^{2\pi} a_k(x)(g(x) - g_\varepsilon(x))dx.$$

Then, we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} f(x)(g(x) - g_\varepsilon(x))dx &= \sum_{k=0}^{\infty} c_k \frac{1}{2\pi}\int_0^{2\pi} a_k(x)(g(x) - g_\varepsilon(x))dx \\ &= I_\varepsilon^N + II_\varepsilon^N. \end{aligned}$$

By $||g_\varepsilon||_{p,\lambda} \leq ||g||_{p,\lambda}$, we obtain

$$\begin{aligned} |II_\varepsilon^N| &\leq \sum_{k=N+1}^{\infty} |c_k| \; ||a_k||_{Z^{q,\lambda}} ||g - g_\varepsilon||_{p,\lambda} \\ &\leq 2 \sum_{k=N+1}^{\infty} |c_k|. \end{aligned}$$

Also for any $\eta > 0$, there exists N_0 a positive integer such that $\sum_{k=N_0+1}^{\infty} |c_k| < \frac{\eta}{2}$. Hence, we have $|II_\varepsilon^{N_0}| < \eta$ for all $\varepsilon > 0$. Moreover, we have

$$
\begin{aligned}
|I_\varepsilon^{N_0}| &\le \sum_{k=0}^{N_0} |c_k|\ ||a_k||_q ||g - g_\varepsilon||_p \\
&= \sum_{k=0}^{N_0} |c_k|\ ||a_k||_q ||g - g * \phi_\varepsilon||_p \\
&\to 0,
\end{aligned}
$$

as $\varepsilon \to 0$. Therefore, we get

$$\limsup_{\varepsilon\to 0} \left| \frac{1}{2\pi} \int_0^{2\pi} f(x) g_\varepsilon(x) dx - \frac{1}{2\pi} \int_0^{2\pi} f(x) g(x) dx \right| \le \eta,$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_0^{2\pi} f(x) g_\varepsilon(x) dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) g(x) dx.$$

Hence, there exists $\varepsilon_0 > 0$ such that $|\frac{1}{2\pi} \int_0^{2\pi} f(x) g_{\varepsilon_0}(x) dx| \ge \frac{A}{3}$. So we obtain

$$\sup_{||g||_{p,\lambda} \le 1, g \in L_0^{p,\lambda}} \left| \frac{1}{2\pi} \int_0^{2\pi} f(x) g(x) dx \right| \ge \frac{A}{3}.$$

Therefore, we have $||f||_{Z^{q,\lambda}} \le 3||f||_{(L_0^{p,\lambda})^*}$. Since the converse is trivial, we get the desired result. $\square$

Now we are ready to prove Theorem 1.23.

Proof of Theorem 1.23. First we have $Z^{q,\lambda}(\mathbb{T}) \subset (L_0^{p,\lambda}(\mathbb{T}))^*$ by Lemma 1.25. Since $\left(Z^{q,\lambda}(\mathbb{T})\right)^* = L^{p,\lambda}(\mathbb{T}) \supset L_0^{p,\lambda}(\mathbb{T})$, we see that the annihilator of $Z^{q,\lambda}(\mathbb{T})$ is $\{0\}$, and hence $Z^{q,\lambda}(\mathbb{T})$ is weak*-dense in $(L_0^{p,\lambda}(\mathbb{T}))^*$ (see Theorem 4.7 (b) in Rudin [**46**]). By the Banach-Alaoglu theorem and the separability of $L_0^{p,\lambda}(\mathbb{T})$ we see that the unit ball of $(L_0^{p,\lambda}(\mathbb{T}))^*$ is weak*-compact and metrizable (see Theorem 3.16 in Rudin [**46**]). Thus, if T is in $(L_0^{p,\lambda}(\mathbb{T}))^*$ with $||T||_{(L_0^{p,\lambda}(\mathbb{T}))^*} \le 1$, then there exists a sequence $\{f_k\} \subset Z^{q,\lambda}(\mathbb{T})$ with $||f_k||_{(L_0^{p,\lambda}(\mathbb{T}))^*} \le 1$ such that $f_k \to T$ in the weak*-topology of $(L_0^{p,\lambda}(\mathbb{T}))^*$. Here, we may assume $||f_k||_{Z^{q,\lambda}(\mathbb{T})} \leq 1$ by Lemma 1.29. Hence, by Lemma 1.28, there exist $f \in Z^{q,\lambda}(\mathbb{T})$ and a subsequence $\{f_{k_j}\}$ ($k_1 < k_2 < \cdots$) such that $||f_{k_j}||_{Z^{q,\lambda}} \leq 1$ and

$$\lim_{j\to\infty} \frac{1}{2\pi} \int_0^{2\pi} f_{k_j}(x)g(x)dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)dx$$

for all $g \in C(\mathbb{T})$. Hence, we have

$$\langle T, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)dx$$

for all $g \in C(\mathbb{T})$. Therefore we get the desired result. $\square$

CHAPTER 2

Fourier multipliers from L^p-spaces to Morrey spaces on the unit circle

1. Fourier multiplier and main results

Let $1 \le p \le \infty$ and $0 \le \lambda \le 1$. Then $L^p(\mathbb{T})$ denotes the L^p-spaces on the unit circle $\mathbb{T}$ and $L^{p,\lambda}(\mathbb{T})$ denotes Morrey spaces defined by

$$L^{p,\lambda}(\mathbb{T}) = \left\{ f \;\middle|\; ||f||_{p,\lambda} := \sup_{\substack{I \subset \mathbb{T} = [-\pi,\pi) \\ I \neq \phi:\text{interval}}} \left(\frac{1}{|I|^\lambda} \int_I |f|^p \frac{dx}{2\pi} \right)^{\frac{1}{p}} < \infty \right\}.$$

We note $L^{p,0}(\mathbb{T}) = L^p(\mathbb{T})$, $L^{p,1}(\mathbb{T}) = L^\infty(\mathbb{T})$ and $L^{p,\lambda}(\mathbb{T})$ is a Banach space (cf. [**37**], [**53**, p.215]). We remark $L^{p,\lambda}(\mathbb{T}) \neq L^p(\mathbb{T})$ for $0 < \lambda < 1$ ([**55**]).

For Banach spaces X and Y which are translation invariant function spaces contained in $L^1(\mathbb{T})$, we denote by $M(X,Y)$ the set of all operators which are translation invariant bounded linear operators from X to Y. We note $M(X,Y)$ is a Banach space with respect to the operator norm $||\cdot||_{M(X,Y)}$. An element of $M(X,Y)$ is called a Fourier multiplier (operator). When $X = L^p$ and $Y = L^q$, an element of $M(L^p, L^q) \cap M(\mathbb{T})$ for $1 \le p < q$ is called an L^p-improving measure ([**25**] cf. [**22**], [**26**]), where $M(\mathbb{T})$ is the set of all bounded regular Borel measures on $\mathbb{T}$. Let μ be a non-negative measure on $\mathbb{T}$. For $0 < \alpha < 1$, we denote $\mu \in Lip_\alpha(M(\mathbb{T}))$, if there exists a positive constant C such that $\mu(I) \le C|I|^\alpha$ for any non-empty interval $I \subset \mathbb{T}$. μ_f is called that the distribution function of μ_f satisfies the Lipschitz condition, if $\mu_f \in Lip_\alpha(M(\mathbb{T}))$ for some $0 < \alpha < 1$, where $\mu_f(E) = \int_E f(x)\frac{dx}{2\pi}$ for a measurable set E on $\mathbb{T}$ and a nonnegative function $f \in L^1(\mathbb{T})$. For $M(L^p, L^q)$ and $Lip_\alpha(M(\mathbb{T}))$, the following results are known.

Theorem A. ([**16**] cf. [**17**], [**38**]) *Let $1 < p < q \le 2$. Then we have*

$$M(L^p, L^p) \neq M(L^q, L^q).$$

Theorem B. ([**21**]) *There exists* $f \in L^1(\mathbb{T})$ *with* $f \geq 0$ *such that*

$$T_f \notin \bigcup_{1 \leq p < q < \infty} M(L^p, L^q), \ \mu_f \in \bigcap_{0 < \alpha < 1} Lip_\alpha(M(\mathbb{T})).$$

Then we study those results in Morrey spaces.

Our main results are as follows:

Theorem 2.1. *Let* $1 \leq p, q < \infty$ *and* $0 < \lambda, \nu < 1$. *Suppose* $\frac{\lambda}{p} \neq \frac{\nu}{q}$. *Then we have*

$$M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu}).$$

Theorem 2.2. *Let* $0 < \lambda, \nu < 1$. *Also let* p, q *be positive numbers with* $1 + \lambda < p < q$ *and* $\frac{1}{p} + \frac{1}{q} < 1$. *Suppose* $\frac{\lambda}{p} = \frac{\nu}{q}$. *Then we have*

$$M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu}).$$

Theorem 2.3. *Let* $f \in L^1(\mathbb{T})$ *be a non-negative function. Then we have that* μ_f *is in* $Lip_\alpha(M(\mathbb{T}))$ *for some* $0 < \alpha < 1$, *if and only if* $T_f \in M(L^p, L^{p,\lambda})$ *for some* $1 < p < \infty$ *and* $0 < \lambda < 1$, *where* $T_f g = f * g$.

The chapter is organized as follows: In §2, we investigate the inclusion relation between $L^p(\mathbb{T})$ and $L^{p,\lambda}(\mathbb{T})$. In §3, we prove Theorem 2.1 by the norm estimate of the Dirichlet kernel in $M(L^p, L^{p,\lambda})$. In §4, we prove Theorem 2.2 by using the norm estimate of the Rudin-Shapiro polynomials in $M(L^p, L^{p,\lambda})$. In §5, we prove Theorem 2.3. Throughout this chapter, we denote by $|E|$ the normalized Haar measure of $E \subset \mathbb{T}$.

The letter C stands for a constant not necessarily the same at each occurrence. $A \sim B$ stands for $C^{-1}A \leq B \leq CA$ for some $C > 0$.

2. $L^p(\mathbb{T})$ and $L^{p,\lambda}(\mathbb{T})$

In this section, we will consider the inclusion relation between the L^p-spaces and Morrey spaces on $\mathbb{T}$.

Proposition 2.4. (cf. [**28**, Proposition 5.1], [**48**, Lemma 1.3]) *Let $1 \leq r, p < \infty$ and $0 < \lambda < 1$. Then, we have the following:*

(1) $L^{p,\lambda}(\mathbb{T}) \subsetneq L^r(\mathbb{T})$ *if* $1 \leq r \leq p < \infty$;

(2) $L^{p,\lambda}(\mathbb{T}) \not\subset L^r(\mathbb{T})$ *and* $L^r(\mathbb{T}) \not\subset L^{p,\lambda}(\mathbb{T})$ *if* $p < r < \frac{p}{1-\lambda}$;

(3) $L^r(\mathbb{T}) \subsetneq L^{p,\lambda}(\mathbb{T})$ *if* $r \geq \frac{p}{1-\lambda}$.

Proof. (1) Since $L^{p,\lambda}(\mathbb{T}) \subsetneq L^p(\mathbb{T})$ (see [**55**, p.587]), we get the desired result.

(2) By the assumption on r, we can choose $0 < \lambda_0 < \lambda$ as $r = \frac{p}{1-\lambda_0}$, and $\mu > 0$ such that $\frac{1-\lambda}{p} < \mu < \frac{1}{r}$. Set $f(x) = \chi_{(0,1)}(x)x^{-\mu} \in L^r(\mathbb{T})$. Then we have $f \notin L^{p,\lambda}(\mathbb{T})$. Let $I = (a,b)$ for $0 < a < b < 1$. By the mean value theorem, we have

$$\begin{aligned} \frac{1}{|I|^\lambda}\int_I |f|^p \frac{dx}{2\pi} &= (b-a)^{-\lambda}\int_a^b x^{-p\mu}\frac{dx}{2\pi} \\ &= C(b-a)^{1-\lambda}(a+\theta(b-a))^{-p\mu} \\ &\geq C(b-a)^{1-\lambda}b^{-p\mu} \end{aligned}$$

for some $0 < \theta < 1$. So, putting $a = \frac{b}{2}$, we have

$$\frac{1}{|I|^\lambda}\int_I |f|^p \frac{dx}{2\pi} \geq Cb^{1-\lambda-p\mu}$$

for all $0 < b < 1$. Since $\mu > \frac{1-\lambda}{p}$, we have $f \notin L^{p,\lambda}(\mathbb{T})$. Therefore, we get $f \in L^r(\mathbb{T})$ and $f \notin L^{p,\lambda}(\mathbb{T})$.

Next we show $L^{p,\lambda}(\mathbb{T}) \not\subset L^r(\mathbb{T})$ for all $\lambda_0 < \lambda < 1$. Suppose $L^{p,\lambda}(\mathbb{T}) \subset L^r(\mathbb{T})$. By the closed graph theorem, there exists a constant C such that

$$||f||_r \quad \leq \quad C||f||_{p,\lambda}$$

for all $f \in L^{p,\lambda}(\mathbb{T})$. Now let δ be in $0 < \delta < \frac{1}{10}$, and $N \in \mathbb{N}$. Also we denote $I(k,\delta) = \{x \in (0,1) | \frac{k}{N} - \frac{\delta}{2} < x < \frac{k}{N} + \frac{\delta}{2}\}$ for $k = 1, \cdots, N-1$, $I(N,\delta) = \{x \in (0,1) | 1 - \frac{\delta}{2} < x < 1\}$, and $E = \cup_{k=1}^{N} I(k,\delta)$. Then we choose a natural number N such that $\delta N \sim \delta^{1-\lambda}$. Hence, we have $|E| \sim \delta N \sim \delta^{1-\lambda}$. When we define $g_\delta = \delta^{-\frac{1}{r}}\chi_E$. For any non-empty interval $I \subset \mathbb{T}$, we have

$$\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \leq |I|^{-\lambda}\delta^{-\frac{p}{r}}|E \cap I|.$$

Here, we investigate the left-hand sides of the inequality for $k = Card\{\ell | I(\ell,\delta) \cap (E \cap I) \neq \phi\} \geq 4$. Since $\frac{k}{2N} \leq |I| \leq \frac{k+1}{N}$ and $(k-2)\delta \leq |E \cap I| \leq k\delta$, we have

$$|I|^{-\lambda}\delta^{-\frac{p}{r}}|E \cap I| \leq |I|^{-\lambda}\delta^{-\frac{p}{r}}k\delta \leq |I|^{-\lambda}\delta^{-\frac{p}{r}}(2N|I|)\delta \leq C\delta^{\lambda_0-\lambda},$$

and

$$\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \leq C\delta^{\lambda_0-\lambda}.$$

Next we estimate $\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi}$ for $k = Card\{\ell | I(\ell,\delta) \cap (E \cap I) \neq \phi\} \leq 3$. Since $|E \cap I| \leq C\min\{3\delta, |I|\}$, we have

$$\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \leq C\min\{|I|^{1-\lambda}\delta^{-\frac{p}{r}}, |I|^{-\lambda}\delta^{1-\frac{p}{r}}\}.$$

Hence, we have $\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \leq C\delta^{1-\lambda-\frac{p}{r}}$ by using the case $|I| \leq \delta$ or $|I| > \delta$. Thus, we obtain $||g_\delta||_{p,\lambda} \leq C\delta^{\frac{\lambda_0-\lambda}{p}}$ for sufficiently small $\delta > 0$.

By the assumption $L^{p,\lambda}(\mathbb{T}) \subset L^r(\mathbb{T})$, we have

$$\delta^{-\frac{\lambda}{r}} \sim ||g_\delta||_r \le C||g_\delta||_{p,\lambda} \le C\delta^{\frac{\lambda_0-\lambda}{p}}.$$

This contradicts $\delta^{\frac{\lambda-\lambda_0}{p}-\frac{\lambda}{r}} \le C$ with $\frac{\lambda-\lambda_0}{p} - \frac{\lambda}{r} = \frac{\lambda_0}{p}(\lambda-1) < 0$ for $0 < \lambda < 1$. Hence we have $L^{p,\lambda}(\mathbb{T}) \not\subset L^r(\mathbb{T})$.

(3) By the Hölder inequality, we have $||f||_{p,\lambda} \le C||f||_r$ for all $f \in L^r(\mathbb{T})$, and thus $L^r(\mathbb{T}) \subset L^{p,\lambda}(\mathbb{T})$. Suppose $r_0 = \frac{p}{1-\lambda}$. When we define $f(x) = \chi_{(0,1)}(x)x^{-\frac{1}{r_0}}$, it is easy to show $f \notin L^{r_0}(\mathbb{T})$ and $f \in L^{p,\lambda}(\mathbb{T})$ similar to (1). Thus, we have $L^r(\mathbb{T}) \subsetneq L^{p,\lambda}(\mathbb{T})$ for $r \ge \frac{p}{1-\lambda}$. □

Corollary 2.5. *Let D_N be the Dirichlet kernel $D_N(x) = \sum_{k=-N}^{N} e^{ikx}$ of degree N. Then, we have*

$$||D_N||_{p,\lambda} \sim N^{\frac{\lambda}{p}+\frac{1}{p'}}$$

for any $1 \le p < \infty$ and $0 < \lambda < 1$.

Proof. Since we have $L^r(\mathbb{T}) \subset L^{p,\lambda}(\mathbb{T})$ for $r = \frac{p}{1-\lambda}$ by Proposition 2.4 (3), there exists a constant $C > 0$ such that $||D_N||_{p,\lambda} \le C||D_N||_r$. By Edwards [**14**, Exercise 7.5], we have

$$||D_N||_{p,\lambda} \le C||D_N||_r \sim N^{\frac{1}{r'}} = N^{\frac{\lambda}{p}+\frac{1}{p'}}.$$

For the interval $I_N = [-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]$, we have

$$|I_N|^{-\lambda}\int_{I_N} |D_N|^p \frac{dx}{2\pi} \ge |I_N|^{-\lambda}\int_0^{\frac{\pi}{2N+1}} \left(\frac{(N+\frac{1}{2})x\frac{2}{\pi}}{\frac{x}{2}}\right)^p \frac{dx}{2\pi} \sim N^{p+\lambda-1},$$

and $||D_N||_{p,\lambda} \ge CN^{\frac{\lambda}{p}+\frac{1}{p'}}$. Therefore, we get the desired result. □

Remark 2.6. Similarly, for the Poisson kernel $P_r(x) = \frac{1-r^2}{1-2r\cos x+r^2}$ $(0 < r < 1)$, we have

$$||P_r||_{p,\lambda} \sim ((1-r)^{-1})^{\frac{\lambda}{p}+\frac{1}{p'}}.$$

3. $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$ $(\frac{\lambda}{p} \neq \frac{\nu}{q})$

In this section, we consider between $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$.

First we obtain the following:

LEMMA 2.7. *Let $0 < \lambda < 1$ and $1 \leq p, q < \infty$. Suppose $q > p(1-\lambda)$. We define the operator $T \in M(L^p, L^{q,\lambda})$ such that $Tf = D_N * f$. Then, we have*

$$||D_N||_{M(L^p,L^{q,\lambda})} = ||T||_{M(L^p,L^{q,\lambda})} \sim N^{\frac{1}{p}-\frac{1-\lambda}{q}}.$$

In particular, $||D_N||_{M(L^p,L^{p,\lambda})} \sim N^{\frac{\lambda}{p}}$.

PROOF. Since we have $L^r(\mathbb{T}) \subset L^{q,\lambda}(\mathbb{T})$ for $r = \frac{q}{1-\lambda}$ and $L^r(\mathbb{T}) \subset L^p(\mathbb{T})$ by the assumption, we obtain $||T||_{M(L^p,L^{q,\lambda})} \leq ||T||_{M(L^p,L^r)}$. By the norm estimate of D_N in $M(L^p, L^r)$ (cf. [**14**]), we get

$$||T||_{M(L^p,L^r)} \leq CN^{\frac{1}{p}-\frac{1}{r}}.$$

Conversely, we have $||T||_{M(L^p,L^{q,\lambda})} \geq CN^{\frac{1}{p}-\frac{1-\lambda}{q}}$, by $||D_N||_{q,\lambda} \leq ||T||_{M(L^p,L^{q,\lambda})}||D_N||_p$ and Corollary 2.5. Hence, we obtain

$$||D_N||_{M(L^p,L^{q,\lambda})} = ||T||_{M(L^p,L^{q,\lambda})} \sim N^{\frac{1}{p}-\frac{1-\lambda}{q}},$$

and we get the desired result. □

Now we can prove Theorem 2.1.

PROOF OF THEOREM 2.1. Let $0 < \lambda, \nu < 1$, $1 \leq p, q < \infty$, and $\frac{\lambda}{p} \neq \frac{\nu}{q}$. By Lemma 2.7, we have $||D_N||_{M(L^p,L^{p,\lambda})} \sim N^{\frac{\lambda}{p}}$. Thus, we obtain $M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu})$. □

COROLLARY 2.8. *Let $0 < \lambda, \nu < 1$ and $1 \leq p, q < \infty$. Suppose $\frac{\lambda}{p} > \frac{\nu}{q}$. Then there exists $f \in L^1(\mathbb{T})$ such that $T_f \in M(L^q, L^{q,\nu})$ and $T_f \notin M(L^p, L^{p,\lambda})$, where $T_f g = f * g$.*

PROOF. Let a be a positive number with $\frac{\nu}{q} < a < \frac{\lambda}{p}$. Also we define $k_n = 2^{n+4}$. Then, we have $k_n + 2^n < k_{n+1} - 2^{n+1}$ $(n \geq 1)$. When we define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^{an}} D_{2^n}(x) e^{ik_n x},$$

we show that T_f satisfies the desired conditions. When we choose r such that $\frac{1}{r'} < \frac{\nu}{q}$ with $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$\begin{aligned} ||f||_r &\leq C \sum_{n=1}^{\infty} \frac{1}{2^{an}} ||D_{2^n}(x) e^{ik_n x}||_r \\ &\leq C \sum_{n=1}^{\infty} 2^{n(-a+\frac{1}{r'})} < \infty, \end{aligned}$$

and $f \in L^r(\mathbb{T}) \subset L^1(\mathbb{T})$. Also we obtain $T_f \in M(L^q, L^{q,\nu})$, since

$$\begin{aligned} ||f * g||_{q,\nu} &\leq C \sum_{n=1}^{\infty} \frac{1}{2^{an}} ||D_{2^n}(x) e^{ik_n x} * g||_{q,\nu} \\ &\leq C ||g||_q \end{aligned}$$

by Lemma 2.7 and $a > \frac{\nu}{q}$. Similarly, since $T_f(D_{2^n}(x)e^{ik_n x}) = 2^{-an} D_{2^n}(x) e^{ik_n x}$, we have $T_f \notin M(L^p, L^{p,\lambda})$. Thus, we get the desired result. $\square$

REMARK 2.9. We have $M(L^p, L^{p,\lambda}) = M(L^p, L_0^{p,\lambda})$ $(1 \leq p < \infty,\ 0 < \lambda < 1)$, where $L_0^{p,\lambda}(\mathbb{T})$ is the closure of $C(\mathbb{T})$ in $L^{p,\lambda}(\mathbb{T})$.

REMARK 2.10. We remark $M(L^1, L^{p,\lambda}) = L^{p,\lambda}(\mathbb{T})$ $(1 < p < \infty,\ 0 < \lambda < 1)$. In fact, let f_0 be in $L^{p,\lambda}(\mathbb{T})$, and g in $L^1(\mathbb{T})$. Then we have $||f_0 * g||_{p,\lambda} \leq ||f_0||_{p,\lambda} ||g||_1$ by the Hölder inequality, and $L^{p,\lambda}(\mathbb{T}) \subset M(L^1, L^{p,\lambda})$. Conversely, let T be in $M(L^1, L^{p,\lambda})$, and $K_N(x) = \sum_{j=-N}^{N} (1 - \frac{|j|}{N+1}) e^{ijx}$ the Fejér kernel of degree N. Then we obtain $TK_N \in L^{p,\lambda}(\mathbb{T})$ and $||TK_N||_{p,\lambda} \leq ||T||_{M(L^1, L^{p,\lambda})}$ $(N \geq 1)$. Hence, there exists $\{TK_{N_j}\}_j$, a subsequence of $\{TK_N\}_N$, such that TK_{N_j} converges in the weak*-topology of $L^{p,\lambda}(\mathbb{T})$ for some $f \in L^{p,\lambda}(\mathbb{T})$. By the Banach-Alaoglu theorem, since we have the predual of $L^{p,\lambda}(\mathbb{T})$ ([**55**]), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Tg(x)h(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f * g(x)h(x)dx$$

for all $h \in C(\mathbb{T})$ and any trigonometric polynomial g. Therefore, we obtain $Tg = f * g$ $(g \in L^1(\mathbb{T}))$. Then we get $M(L^1, L^{p,\lambda}) = L^{p,\lambda}(\mathbb{T})$.

PROPOSITION 2.11. *Let $0 < \lambda, \nu < 1$ and $1 < p, q < \infty$. Suppose $2 < p < q$ or $q < p \leq 2$. For $\lambda = \frac{p-2}{q-2}\nu$, we have*

$$M(L^q, L^{q,\nu}) \subsetneq M(L^p, L^{p,\lambda}).$$

PROOF. Since $L^{q,\nu}(\mathbb{T}) \subset L^q(\mathbb{T})$, we have $M(L^q, L^{q,\nu}) \subset M(L^2, L^2)$. First let $2 < p < q$, and $T \in M(L^q, L^{q,\nu})$. Since T is bounded from $L^q(\mathbb{T})$ to $L^{q,\nu}(\mathbb{T})$ and from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$, we obtain that T is bounded from $L^p(\mathbb{T})$ to $L^{p,\kappa}(\mathbb{T})$ by the Peetre interpolation theorem [**45**, Theorem 4.1], where p and κ are defined by $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}$ and $\frac{\kappa}{p} = \frac{\theta}{q}\nu + \frac{1-\theta}{2}0$. Then an arithmetic shows $\kappa = \frac{p-2}{q-2}\nu$. Since $\frac{\lambda}{p} \neq \frac{\nu}{q}$, we have $M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu})$. $\square$

4. $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$ $(\frac{\lambda}{p} = \frac{\nu}{q})$

In this section, we consider the inclusion relation between $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$ for $\frac{\lambda}{p} = \frac{\nu}{q}$, and $0 < \lambda, \nu < 1$, $1 < p < q < \infty$. For this, we recall the Rudin-Shapiro polynomials (cf. [**34**], [**53**]).

DEFINITION 2.12. Let m be a non-negative integer. We define trigonometric polynomials $P_m(x)$, $Q_m(x)$ such that

(1) $P_0(x) = Q_0(x) = 1$;

(2) $P_{m+1}(x) = P_m(x) + e^{i2^m x}Q_m(x)$, $Q_{m+1}(x) = P_m(x) - e^{i2^m x}Q_m(x)$.

We prepare the following lemmas which will be used in the proof of Theorem 2.2.

LEMMA 2.13. (cf. [**34**], [**53**]) *The Rudin-Shapiro polynomials P_m, Q_m have the following properties:*

(1) $P_m(x) = \sum_{k=0}^{2^m-1} \varepsilon_k e^{ikx}$, $Q_m(x) = \sum_{k=0}^{2^m-1} \eta_k e^{ikx}$ *for some* $\varepsilon_k, \eta_k \in \{-1, 1\}$;

(2) $|P_m(x)| \le C(2^m)^{\frac{1}{2}}$ $(x \in \mathbb{T})$;

(3) $||T_m||_{M(L^q,L^q)} \sim (2^m)^{|\frac{1}{2}-\frac{1}{q}|}$ $(1 < q < \infty)$, *where* $T_m f = P_m * f$.

By Lemma 2.13 and the Peetre interpolation theorem [**45**], we obtain the following:

LEMMA 2.14. *Let* $0 < \lambda < 1$, *and* $p > 1 + \lambda$. *Then we have the estimates:*

$$||T_m||_{M(L^p,L^{p,\lambda})} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}} \quad (p \ge 2);$$

$$||T_m||_{M(L^p,L^{p,\lambda})} \le C(2^m)^{\frac{\lambda}{p}+\frac{1}{p}-\frac{1}{2}} \quad (1 + \lambda < p < 2);$$

$$||T_m||_{M(L^p,L^{p,\lambda})} \ge C(2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}} \quad (1 + \lambda < p < 2),$$

where $T_m f = P_m * f$.

PROOF. Step 1. We show $||T_m||_{M(L^2,L^{2,\lambda})} \sim (2^m)^{\frac{\lambda}{2}}$. Let P be a trigonometric polynomial such that $P(x) = \sum_{k=-n}^{n} a_k e^{ikx}$ for any positive integer n. Since $P_m * P(x) = \sum_{k=0}^{\min(2^m-1,n)} \varepsilon_k a_k e^{ikx}$, we have $|P_m * P(x)|^2 \le C2^m ||P||_2^2$ by the Schwarz inequality. Then for any interval I with $|I| < 2^{-m}$, we have

$$\frac{1}{|I|^\lambda} \int_I |P_m * P|^2 \frac{dx}{2\pi} \le C 2^{m\lambda} ||P||_2^2$$

by the Parseval inequality. When $|I| \geq 2^{-m}$, we obtain

$$
\begin{aligned}
\frac{1}{|I|^\lambda}\int_I |P_m * P|^2 \frac{dx}{2\pi} &\leq \frac{1}{|I|^\lambda}\int_{-\pi}^{\pi} |P_m * P|^2 \frac{dx}{2\pi} \\
&\leq \frac{1}{|I|^\lambda}\sum_{k=0}^{2^m-1} |a_k|^2 \\
&\leq C2^{m\lambda}||P||_2^2
\end{aligned}
$$

by the Parseval inequality. Hence, we get $||T_m P||_{2,\lambda} \leq C(2^m)^{\frac{\lambda}{2}}||P||_2$, and $||T_m||_{M(L^2,L^{2,\lambda})} \leq C(2^m)^{\frac{\lambda}{2}}$. On the other hand, since

$$
\begin{aligned}
||P_m * P_m||_{2,\lambda} &\leq ||T_m||_{M(L^2,L^{2,\lambda})}||P_m||_2 \\
&\leq C||T_m||_{M(L^2,L^{2,\lambda})}(2^m)^{\frac{1}{2}}
\end{aligned}
$$

and $||P_m * P_m||_{2,\lambda} \sim (2^m)^{\frac{\lambda}{2}+\frac{1}{2}}$ by Lemma 2.13, we obtain $||T_m||_{M(L^2,L^{2,\lambda})} \sim (2^m)^{\frac{\lambda}{2}}$.

Step 2. When $p > 2$ and $0 < \lambda < 1$, we have

$$||T_m||_{M(L^p,L^{p,\lambda})} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}.$$

In fact, let $r > 2$ and $0 < \theta, \kappa < 1$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r}$ and $\frac{\lambda}{p} = \frac{\theta}{2}\kappa$. By Lemma 2.13, we have $||T_m||_{M(L^r,L^r)} \sim (2^m)^{\frac{1}{2}-\frac{1}{r}}$. Applying Step 1 and the Peetre interpolation theorem, we have

$$||T_m||_{M(L^p,L^{p,\lambda})} \leq C(2^m)^{\frac{\theta\kappa}{2}}(2^m)^{(\frac{1}{2}-\frac{1}{r})(1-\theta)}.$$

Hence, we obtain $||T_m||_{M(L^p,L^{p,\lambda})} \leq C(2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$. Conversely, we get

$$||T_m||_{M(L^p,L^{p,\lambda})} \geq C(2^m)^{\frac{\lambda}{p}+\frac{1}{p'}-\frac{1}{2}} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$$

by Corollary 2.5 and Lemma 2.13. Therefore we have $||T_m||_{M(L^p,L^{p,\lambda})} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$.

Step 3. We show $||T_m||_{M(L^p,L^{p,\lambda})} \le C(2^m)^{\frac{\lambda}{p}+\frac{1}{p}-\frac{1}{2}}$ for $1+\lambda < p < 2$. First, we choose $1 < r < p$ and $0 < \theta, \kappa < 1$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r}$ and $\frac{\lambda}{p} = \frac{\theta}{2}\kappa$. Then, we can show that

$$\begin{aligned} ||T_m||_{M(L^p,L^{p,\lambda})} &\le C||T_m||^{\theta}_{M(L^2,L^{2,\lambda})}||T_m||^{1-\theta}_{M(L^r,L^r)} \\ &\le C(2^m)^{\frac{\lambda}{p}+\frac{1}{p}-\frac{1}{2}} \end{aligned}$$

by applying the Peetre interpolation theorem. On the other hand, by $||T_m(P_m)||_{p,\lambda} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{p'}}$ we have $||T_m||_{M(L^p,L^{p,\lambda})} \ge C(2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$, similarly in Step 2. After all, we get the desired result. $\square$

Proof of Theorem 2.2. By the assumption, we have $q > 2$, and $||T_m||_{M(L^q,L^{q,\nu})} \sim (2^m)^{\frac{\lambda}{q}+\frac{1}{2}-\frac{1}{q}}$ for m. If we have $M(L^p, L^{p,\lambda}) = M(L^q, L^{q,\nu})$, we obtain the contradiction to $p < q$ for $p > 2$. For $1 + \lambda < p \le 2$, we have $M(L^p, L^{p,\lambda}) \ne M(L^q, L^{q,\nu})$ by the estimate in Lemma 2.14. Then we get the desired result. $\square$

Corollary 2.15. *Let $0 < \lambda, \nu < 1, 1 + \lambda < p < q$, and $\frac{1}{p} + \frac{1}{q} < 1$. Suppose $\frac{\lambda}{p} = \frac{\nu}{q}$. Then there exists $T \in M(L^p, L^{p,\lambda})$ such that $T \notin M(L^q, L^{q,\nu})$.*

Proof. Let $2 < p < q$. Then there exists a in $\frac{\lambda}{p} + \frac{1}{2} - \frac{1}{p} < a < \frac{\nu}{q} + \frac{1}{2} - \frac{1}{q}$. Also we define $k_n = 2^{n+4}$. Then, we have $k_n + 2^{n+1} < k_{n+1} - 2^{n+2}$. We define

$$S_N(x) = \sum_{m=1}^{N} \frac{1}{2^{am}} P_m(x) e^{ik_m x}$$

for any $N \in \mathbb{N}$. Then, $\{S_N\}_N$ is a cauchy sequence in $M(L^p, L^{p,\lambda})$ by the choice of a and Lemma 2.14, and there exists $S \in M(L^p, L^{p,\lambda})$ such that $||S_N - S||_{M(L^p,L^{p,\lambda})} \to 0$ as $N \to \infty$. Also let g be a function such that $g(x) = P_m(x)e^{ik_m x}$. We consider $\{S_N * g\}_{N>m}$. Then we can prove $S \notin M(L^q, L^{q,\nu})$ by the way similar to Corollary 2.8 in view of the choice of a. In case of $p \leq 2 \leq q$, we omit the details, since the proof is similar to it of the case $2 < p < q$. $\square$

5. $M(L^p, L^{p,\lambda})$ and the Lipschitz conditions

Definition 2.16. Let μ be in $M(\mathbb{T})$ and $0 < \alpha < 1$. We say that $\mu \in Lip_\alpha(M(\mathbb{T}))$ for $\mu \in M(\mathbb{T})$ with $\mu \geq 0$ if for any interval $I = [x, x+h]$,

$$\mu(I) \leq C|I|^\alpha = C|h|^\alpha$$

for some constant $C > 0$ independent of I. For $f \in L^1(\mathbb{T})$ with $f \geq 0$, we denote $\mu_f(E) = \frac{1}{2\pi}\int_E f(x)dx$ for any measurable set $E \subset \mathbb{T}$.

It is easy to prove the following:

Proposition 2.17. *Let f be in $L^1(\mathbb{T})$ with $f \geq 0$. Then we have that μ_f is in $Lip_\alpha(M(\mathbb{T}))$ if and only if $f \in L^{1,\alpha}(\mathbb{T})$.*

By applying Proposition 2.17, we can show the following:

Proposition 2.18. *Suppose $f \in L^1(\mathbb{T})$ and $f \geq 0$. Then we have the following:*

(1) *If $\mu_f \in Lip_\alpha(M(\mathbb{T}))$ for all $0 < \alpha < 1$, then $T_f \in M(L^p, L^{p,\lambda})$ for all $1 < p < \infty$.*
(2) *If $T_f \in M(L^p, L^{p,\lambda})$ for some $1 < p < \infty$ and $0 < \lambda < 1$, then $\mu_f \in Lip_{\frac{\lambda}{p}}(M(\mathbb{T}))$.*

Proof. (1) Since $\mu_f \in Lip_\alpha(M(\mathbb{T}))$ for all $0 < \alpha < 1$, we get $f \in L^{1,\alpha}(\mathbb{T})$ by Proposition 2.17. Let $I \subset \mathbb{T}$ be a nonempty interval.

For $g \in L^p(\mathbb{T})$, we have

$$\begin{aligned}
& \frac{1}{|I|^\alpha}\int_I \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy \right|^p \frac{dx}{2\pi} \\
\leq\ & \frac{1}{|I|^\alpha}\int_I \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(y)|^p |f(x-y)|dy \right) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-y)|dy \right)^{\frac{p}{p'}} \frac{dx}{2\pi} \\
\leq\ & ||f||_{1,\alpha}^p ||g||_p^p
\end{aligned}$$

by the Hölder inequality. Hence, we obtain $||f * g||_{p,\alpha} \leq ||f||_{1,\alpha}||g||_p$ and $T_f \in M(L^p, L^{p,\alpha})$.

(2) Let f be in $L^1(\mathbb{T})$ with $f \geq 0$, and $T_f \in M(L^p, L^{p,\lambda})$. Now, let $I_\delta = [-\delta, \delta]$ $(0 < \delta < 1)$ and $g = \chi_{I_\delta}$. It is sufficient to show $\mu_f(I_\eta) \leq C|I_\eta|^{\frac{\lambda}{p}}$ for sufficiently small $\eta > 0$. First we remark

$$f * g(x) \ = \ \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-y)f(y)dy = \mu_f(I_\delta + x),$$

and $I_{\frac{\delta}{2}} \subset I_\delta + x$ for $x \in I_{\frac{\delta}{2}}$. Hence, we obtain

$$\frac{1}{|I_\delta|^\lambda}\int_{I_\delta} |f * g|^p \frac{dx}{2\pi} \geq \frac{1}{|I_\delta|^\lambda}\int_{I_{\frac{\delta}{2}}} \mu_f(I_{\frac{\delta}{2}})^p \frac{dx}{2\pi},$$

and

$$\begin{aligned}
|I_\delta|^{-\lambda}\mu_f(I_{\frac{\delta}{2}})^p |I_{\frac{\delta}{2}}| \ &\leq\ |I_\delta|^{-\lambda}\int_{I_\delta} |f * g|^p \frac{dx}{2\pi} \\
&\leq\ ||f * g||_{p,\lambda}^p \\
&\leq\ ||T_f||_{M(L^p, L^{p,\lambda})}^p ||g||_p^p \\
&\leq\ C|I_\delta|.
\end{aligned}$$

Therefore, we get $\mu_f(I_{\frac{\delta}{2}}) \leq C|I_{\frac{\delta}{2}}|^{\frac{\lambda}{p}}$, and the desired result. □

As a corollary of Proposition 2.18, we have Theorem 2.3.

Moreover, by Theorem B and Proposition 2.18, we conclude that $M(L^r, L^{r,\lambda})$ are different from $M(L^p, L^q)$ $(1 \leq p < q < \infty)$. Precisely, we obtain the following corollary:

Corollary 2.19. *Let* $1 \leq p < q < \infty,\ 1 \leq r < \infty$, *and* $0 < \lambda < 1$. *Then we have*

$$M(L^p, L^q) \neq M(L^r, L^{r,\lambda}).$$

CHAPTER 3

The fractional integral operators on weighted Morrey spaces

1. A preliminary

Throughout this chapter, we will use the following notation: For $E \subset \mathbb{R}^n$, we denote the Lebesgue measure of E by $|E|$. We call a nonnegative locally integrable function w on $\mathbb{R}^n$ a weight function and define $w(E) = \int_E w(x)dx$. For a ball Q, $2Q$ denotes the ball with the same center as Q whose radius is twice as large. For $1 < p < \infty$, p' is defined by the conjugate index which satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. Also, the letter C stands for a constant not necessarily the same at each occurrence.

First, we introduce some definitions.

Definition 3.1. Let $0 < \alpha < n$. Then, the fractional integral operator I_α is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Definition 3.2. Let $1 < p < \infty$, $0 \leq \lambda < 1$, and u, v are weight. Then, weighted Morrey space $L^{p,\lambda}(u,v)(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(u,v)(\mathbb{R}^n) := \Bigg\{ f \in L^1_{loc}(u)(\mathbb{R}^n) : \\ ||f||_{L^{p,\lambda}(u,v)} = \sup_{Q \subset \mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{v(Q)^\lambda} \int_Q |f(y)|^p u(y) dy \right)^{\frac{1}{p}} < \infty \Bigg\}.$$

When $u = v = 1$ in Definition 3.2, then it is classical Morrey space, that is,

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f : ||f||_{L^{p,\lambda}} = \sup_{Q \subset \mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{|Q|^\lambda} \int_Q |f(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}.$$

Definition 3.3. Let $1 < p, q < \infty$. We say that a weight w belongs to $A_{p,q}(\mathbb{R}^n)$ if

$$\sup_{Q \subset \mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{|Q|} \int_Q w^q(y)dy \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w^{-p'}(y)dy \right)^{\frac{1}{p'}} < \infty.$$

Definition 3.4. (1) We say that a weight w satisfies the doubling condition if there exists $K_1 > 0$ such that

$$w(2Q) \le K_1 w(Q)$$

for all balls Q.

(2) We say that a weight w satisfies the reverse doubling condition if there exists $K_2 > 1$ such that

$$w(2Q) \ge K_2 w(Q)$$

for all balls Q.

Remark 3.5. If $w \in A_{p,q}(\mathbb{R}^n)$ for $1 < p, q < \infty$, then w^q and $w^{-p'}$ satisfy both the doubling condition and the reverse doubling condition, respectively.

Komori and Shirai [**35**] proved a weighted estimate (cf. [**8**]).

Theorem C ([**35**, Theorem 3.6]). *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 \le \lambda < \frac{p}{q_1}$, and $w \in A_{p,q_1}(\mathbb{R}^n)$. Then, the fractional integral operator I_α is bounded from $L^{p,\lambda}(w^p, w^{q_1})$ to $L^{q_1, \frac{\lambda q_1}{p}}(w^{q_1}, w^{q_1})$, where $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$.*

Remark 3.6 ([**43**]). When $\lambda = 0$ in Theorem C, then it is the Muchenhoupt-Wheeden inequality:

$$||I_\alpha f||_{L^{q_1}(w^{q_1})} \le C||f||_{L^p(w^p)}.$$

We improve Theorem C in the next section.

2. Main result

Our result is as follows:

THEOREM 3.7. *Let* $0 < \alpha < n$, $1 < p < \frac{n(1-\lambda)}{\alpha}$, $0 \le \lambda < \frac{p}{q_1}$, *and* $w \in A_{p,q_1}(\mathbb{R}^n)$. *Then, the fractional integral operator* I_α *is bounded from* $L^{p,\lambda}(w^p, w^{q_1})$ *to* $L^{q_2,\lambda}(w^{q_1}, w^{q_1})$, *where* $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$, *and* $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$.

From this theorem, we can see the following:

REMARK 3.8. (1) Since $L^{q_2,\lambda}(w^{q_1}, w^{q_1})(\mathbb{R}^n) \subset L^{q_1,\frac{\lambda q_1}{p}}(w^{q_1}, w^{q_1})(\mathbb{R}^n)$, Theorem 3.7 improves Theorem C. In fact, since $\frac{1-\lambda}{q_2} = \frac{1}{q_1} - \frac{\lambda}{p}$, we have

$$\begin{aligned}
&\frac{1}{w^{q_1}(Q)^{\frac{\lambda q_1}{p}}} \int_Q |f(x)|^{q_1} w^{q_1}(x)dx \\
&\le \frac{1}{w^{q_1}(Q)^{\frac{\lambda q_1}{p}}} \left(\int_Q |f(x)|^{q_2} w^{q_1}(x)dx \right)^{\frac{q_1}{q_2}} w^{q_1}(Q)^{1-\frac{q_1}{q_2}} \\
&\le ||f||^{q_1}_{L^{q_2,\lambda}(w^{q_1}, w^{q_1})}
\end{aligned}$$

by the Hölder inequality. When $w = 1$, we note $L^{q_2,\lambda}(\mathbb{R}^n) \subsetneqq L^{q_1,\frac{\lambda q_1}{p}}(\mathbb{R}^n)$. It is easy to check this fact by the method of [**28**, Proposition 5.1] (cf. [**44**]).

(2) ([**2**], [**43**]) When $\lambda = 0$ in Theorem 3.7, then it is the Muckenhoupt-Wheeden inequality. When $w = 1$, then we have the Adams inequality:

$$||I_\alpha f||_{L^{q_2,\lambda}} \le C||f||_{L^{p,\lambda}}.$$

For the proof of Theorem 3.7, we need measures on $\mathbb{R}^n$.

DEFINITION 3.9. Let μ be a positive measure on $\mathbb{R}^n$. We say that μ is a doubling measure if there exists $C > 0$ such that

$$\mu(2Q) \le C\mu(Q)$$

for all balls Q, where $\mu(Q) = \int_Q d\mu$.

Throughout the rest of this section, we assume that μ is a doubling measure.

DEFINITION 3.10. Let $1 < p < \infty$ and $0 \le \lambda < 1$. Then, $L^{p,\lambda}(\mu)(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mu)(\mathbb{R}^n) := \Bigg\{ f \in L^1_{loc}(\mu)(\mathbb{R}^n) :$$
$$||f||_{L^{p,\lambda}(\mu)} = \sup_{Q\subset\mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{\mu(Q)^\lambda} \int_Q |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} < \infty \Bigg\}.$$

DEFINITION 3.11. The Hardy-Littlewood maximal operator M_μ is defined by

$$M_\mu f(x) := \sup_{x\in Q\subset\mathbb{R}^n, Q:\text{ball}} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y).$$

Next, we give some lemmas. It is easy to see Lemma 3.12.

LEMMA 3.12 (cf. [**6**], [**8**]). *Let $1 < p < \infty$ and $0 \le \lambda < 1$. Then, the Hardy-Littlewood maximal operator M_μ is bounded on $L^{p,\lambda}(\mu)$.*

LEMMA 3.13. *If $w \in A_{p,q_1}(\mathbb{R}^n)$, then there exists p_0 such that $1 < p_0 < p$ and*

$$\sup_{Q\subset\mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{|Q|} \int_Q w^{q_1}(y) dy \right)^{\frac{1}{q_1}} \left(\frac{1}{|Q|} \int_Q w^{-p_0'}(y) dy \right)^{\frac{1}{p_0'}} < \infty.$$

PROOF. Since $w \in A_{p,q_1}(\mathbb{R}^n)$ if and only if $w^{-p'} \in A_{1+\frac{p'}{q_1}}(\mathbb{R}^n)$ ([**35**, Remark 2.11]), this lemma is proved by the reverse Hölder inequality.

□

By Lemma 3.13, we get the following:

LEMMA 3.14. *If $w \in A_{p,q_1}(\mathbb{R}^n)$, then there exists r such that $1 < r < p$ and*

$$\sup_{Q\subset\mathbb{R}^n, Q:\text{ball}} \left(\frac{1}{|Q|}\int_Q w^{q_1}(y)dy\right)^{\frac{1}{r}-\frac{\alpha}{n}} \left(\frac{1}{|Q|}\int_Q w^{-q_1 r'(\frac{1}{r}-\frac{\alpha}{n})}(y)dy\right)^{\frac{1}{r'}} < \infty,$$

where $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$.

PROOF. As for p_0 in Lemma 3.13, we define r as $r'(\frac{1}{r} - \frac{\alpha}{n})q_1 = p_0'$. Then, we obtain the desired result by applying Lemma 3.13. □

Now, we define $m_Q f = \frac{1}{|Q|}\int_Q f(y)dy$. Then, we estimate $|Q|^{\frac{\alpha}{n}} m_Q|f|$ in two different ways. First, we have the following:

LEMMA 3.15. *If $w \in A_{p,q_1}(\mathbb{R}^n)$, then we have*

$$|Q|^{\frac{\alpha}{n}} m_Q|f| \le \frac{C||f||_{L^{p,\lambda}(w^p,w^{q_1})}}{w^{q_1}(Q)^d},$$

where $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$, and $d = \frac{1-\lambda}{q_2}$.

PROOF. By $w \in A_{p,q_1}(\mathbb{R}^n)$ and the Hölder inequality, we have

$$\begin{aligned} |Q|^{\frac{\alpha}{n}} m_Q|f| &\le |Q|^{\frac{\alpha}{n}-\frac{1}{p}} \left(\int_Q |f(x)|^p w^p(y)dy\right)^{\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q w^{-p'}(y)dy\right)^{\frac{1}{p'}} \\ &\le C||f||_{L^{p,\lambda}(w^p,w^{q_1})} w^{q_1}(Q)^{\frac{\lambda}{p}-\frac{1}{q_1}}, \end{aligned}$$

and get the desired inequality by the choice of d. □

Next we estimate $|Q|^{\frac{\alpha}{n}} m_Q|f|$ in terms of M_μ.

LEMMA 3.16. *If $w \in A_{p,q_1}(\mathbb{R}^n)$ and $d\mu(y) = w^{q_1}(y)dy$, then we have*

$$|Q|^{\frac{\alpha}{n}} m_Q|f| \le C w^{q_1}(Q)^{\frac{\alpha}{n}} M_\mu(|fw^{-a}|^r)(x)^{\frac{1}{r}}$$

for all $x \in Q$, where $a = \frac{q_1}{p} - 1$, and r is a number chosen in Lemma 3.14.

PROOF. Let $c = q_1\left(\frac{1}{r} - \frac{1}{p}\right) + 1$. Then, by the Hölder inequality and Lemma 3.14, we obtain

$$
\begin{aligned}
&|Q|^{\frac{\alpha}{n}} m_Q|f| \\
&\le |Q|^{\frac{\alpha}{n}} \left(\frac{1}{|Q|}\int_Q |f(y)|^r w^{cr}(y)dy\right)^{\frac{1}{r}} \left(\frac{1}{|Q|}\int_Q w^{-cr'}(y)dy\right)^{\frac{1}{r'}} \\
&= |Q|^{\frac{\alpha}{n}} \left(\frac{1}{|Q|}\int_Q |f(y)|^r w^{-ar}(y)d\mu(y)\right)^{\frac{1}{r}} \left(\frac{1}{|Q|}\int_Q w^{-cr'}(y)dy\right)^{\frac{1}{r'}} \\
&\le \left(\frac{1}{|Q|}\right)^{\frac{1}{r}-\frac{\alpha}{n}} M_\mu(|fw^{-a}|^r)(x)^{\frac{1}{r}} \mu(Q)^{\frac{1}{r}} \left(\frac{1}{|Q|}\int_Q w^{-cr'}(y)dy\right)^{\frac{1}{r'}} \\
&\le C w^{q_1}(Q)^{\frac{\alpha}{n}} M_\mu(|fw^{-a}|^r)(x)^{\frac{1}{r}}.
\end{aligned}
$$

□

By using these lemmas, we prove Theorem 3.7.

PROOF OF THEOREM 3.7. Let r be a number chosen in Lemma 3.14, and d, a be in Lemma 3.15 and Lemma 3.16, respectively. First, we obtain

$$
\begin{aligned}
|I_\alpha f(x)| &\le \sum_{j=-\infty}^{\infty} \int_{2^{j-1}<|x-y|\le 2^j} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \\
&\le \sum_{j=-\infty}^{\infty} \frac{1}{2^{(j-1)(n-\alpha)}} \int_{|x-y|\le 2^j} |f(y)|dy \\
&\le C \sum_{j=-\infty}^{\infty} 2^{(j+1)(\alpha-n)} \int_{Q_j} |f(y)|dy \\
&\le C \sum_{j=-\infty}^{\infty} |Q_j|^{\frac{\alpha}{n}} m_{Q_j}|f|,
\end{aligned}
$$

where $Q_j = Q_j(x) = \{y : |x-y| \le 2^j\}$. By Lemma 3.15 and Lemma 3.16, we have

$$|I_\alpha f(x)| \le C\left(\sum_{j=-\infty}^{J} w^{q_1}(Q_j)^{\frac{\alpha}{n}} M_\mu(|fw^{-a}|^r)(x)^{\frac{1}{r}} + \sum_{j=J}^{\infty} \frac{||f||_{L^{p,\lambda}(w^p,w^{q_1})}}{w^{q_1}(Q_j)^d}\right)$$

for all $J \in \mathbb{Z}$. Since w^{q_1} satisfies both the doubling condition and the reverse doubling condition, there exist constants K_1 and K_2 such that $1 < K_1 \le \frac{w^{q_1}(Q_{j+1})}{w^{q_1}(Q_j)} \le K_2 < \infty$ for all $j \in \mathbb{Z}$. Therefore, we get

$$|I_\alpha f(x)| \le C\left\{w^{q_1}(Q_J)^{\frac{\alpha}{n}} M_\mu(|fw^{-a}|^r)(x)^{\frac{1}{r}} + \frac{||f||_{L^{p,\lambda}(w^p,w^{q_1})}}{w^{q_1}(Q_J)^d}\right\}$$

for all $J \in \mathbb{Z}$. Now, we take J such that

$$w^{q_1}(Q_J)^{d+\frac{\alpha}{n}} \le \frac{||f||_{L^{p,\lambda}(w^p,w^{q_1})}}{M_\mu(|fw^{-a}|^r)(x)^{\frac{1}{r}}} \le K_2^{d+\frac{\alpha}{n}} w^{q_1}(Q_J)^{d+\frac{\alpha}{n}}$$

for all $x \in Q$, and we have

$$|I_\alpha f(x)| \le C||f||_{L^{p,\lambda}(w^p,w^{q_1})}^{\frac{\alpha}{nd+\alpha}} M_\mu(|fw^{-a}|^r)(x)^{\frac{nd}{r(nd+\alpha)}}.$$

By the choice of q_2 and d, we have

$$\begin{aligned}
&\left(\frac{1}{w^{q_1}(Q)^\lambda}\int_Q |I_\alpha f(x)|^{q_2} w^{q_1}(x)dx\right)^{\frac{1}{q_2}} \\
&\le C||f||_{L^{p,\lambda}(w^p,w^{q_1})}^{\frac{\alpha}{nd+\alpha}} \left(\frac{1}{w^{q_1}(Q)^\lambda}\int_Q M_\mu(|fw^{-a}|^r)(x)^{\frac{q_2 nd}{r(nd+\alpha)}} w^{q_1}(x)dx\right)^{\frac{1}{q_2}} \\
&= C||f||_{L^{p,\lambda}(w^p,w^{q_1})}^{\frac{\alpha}{nd+\alpha}} \left(\frac{1}{w^{q_1}(Q)^\lambda}\int_Q M_\mu(|fw^{-a}|^r)(x)^{\frac{p}{r}} w^{q_1}(x)dx\right)^{\frac{1}{q_2}}.
\end{aligned}$$

Since $p > r$, we can use Lemma 3.12. By the choice of a, we have

$$\begin{aligned}
&\frac{1}{w^{q_1}(Q)^\lambda}\int_Q M_\mu(|fw^{-a}|^r)(x)^{\frac{p}{r}} w^{q_1}(x)dx \\
&= \left(\frac{1}{\mu(Q)^\lambda}\int_Q M_\mu(|fw^{-a}|^r)(x)^{\frac{p}{r}} d\mu(x)\right)^{\frac{1}{(p/r)}\frac{p}{r}} \\
&\le ||M_\mu(|fw^{-a}|^r)||^{\frac{p}{r}}_{L^{\frac{p}{r},\lambda}(\mu)} \\
&\le C||\ |fw^{-a}|^r||^{\frac{p}{r}}_{L^{\frac{p}{r},\lambda}(\mu)} \\
&= C||f||^p_{L^{p,\lambda}(w^p,w^{q_1})}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\left(\frac{1}{w^{q_1}(Q)^\lambda}\int_Q |I_\alpha f(x)|^{q_2} w^{q_1}(x)dx\right)^{\frac{1}{q_2}} &\le C||f||^{\frac{\alpha}{nd+\alpha}}_{L^{p,\lambda}(w^p,w^{q_1})}||f||^{\frac{p}{q_2}}_{L^{p,\lambda}(w^p,w^{q_1})} \\
&= C||f||_{L^{p,\lambda}(w^p,w^{q_1})}.
\end{aligned}$$

□

3. A remark

We show a multilinear version of Theorem 3.7.

Definition 3.17. Let $0 < \alpha < n$, $\theta_i \neq 0$ $(1 \le i \le m)$, and θ_i are all distinct. Then, the multilinear fractional integral operator $I^m_{\alpha,\theta}$ is defined by

$$I^m_{\alpha,\theta}(f_1,\dots,f_m)(x) := \int_{\mathbb{R}^n} \frac{\prod_{i=1}^m f_i(x-\theta_i y)}{|y|^{n-\alpha}}dy,$$

where $\theta = (\theta_1,\dots,\theta_m)$.

We give a remark as a corollary of Theorem 3.7.

Proposition 3.18. *Let $0 < \alpha < n$, $1 < p < \frac{n(1-\lambda)}{\alpha}$, $0 \le \lambda < \frac{p}{q_1}$, and $w \in A_{p,q_1}(\mathbb{R}^n)$. Then, the multilinear fractional integral operator*

$I^m_{\alpha,\theta}$ is bounded from $\prod_{i=1}^m L^{p_i,\lambda}(w^p, w^{q_1})$ to $L^{q_2,\lambda}(w^{q_1}, w^{q_1})$, where $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$, and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$.

Proof. First, we remark

$$\left|I^m_{\alpha,\theta}(f_1,\ldots,f_m)(x)\right| \leq C\prod_{i=1}^m \left(I_\alpha\left(|f_i|^{\frac{p_i}{p}}\right)(x)\right)^{\frac{p}{p_i}}$$

by the Hölder inequality (cf. [**27**]). From this fact and Theorem 3.7, we have

$$\begin{aligned}
&\left(\frac{1}{w^{q_1}(Q)^\lambda}\int_Q |I^m_{\alpha,\theta}(f_1,\ldots,f_m)(x)|^{q_2} w^{q_1}(x)dx\right)^{\frac{1}{q_2}}\\
&\leq C\left(\frac{1}{w^{q_1}(Q)^\lambda}\int_Q \left(\prod_{i=1}^m \left(I_\alpha\left(|f_i|^{\frac{p_i}{p}}\right)(x)\right)^{\frac{p}{p_i}}\right)^{q_2} w^{q_1}(x)dx\right)^{\frac{1}{q_2}}\\
&= C\left(\int_Q \prod_{i=1}^m \left(I_\alpha\left(|f_i|^{\frac{p_i}{p}}\right)(x)^{q_2} w^{q_1}(x) w^{q_1}(Q)^{-\lambda}\right)^{\frac{p}{p_i}} dx\right)^{\frac{1}{q_2}}\\
&\leq C\prod_{i=1}^m \left(\frac{1}{w^{q_1}(Q)^\lambda}\int_Q I_\alpha\left(|f_i|^{\frac{p_i}{p}}\right)(x)^{q_2} w^{q_1}(x)dx\right)^{\frac{1}{q_2}\frac{p}{p_i}}\\
&\leq C\prod_{i=1}^m \left|\left|I_\alpha\left(|f_i|^{\frac{p_i}{p}}\right)\right|\right|^{\frac{p}{p_i}}_{L^{q_2,\lambda}(w^{q_1},w^{q_1})}\\
&\leq C\prod_{i=1}^m \left|\left||f_i|^{\frac{p_i}{p}}\right|\right|^{\frac{p}{p_i}}_{L^{p,\lambda}(w^{p},w^{q_1})}\\
&= C\prod_{i=1}^m ||f_i||_{L^{p_i,\lambda}(w^p,w^{q_1})}.
\end{aligned}$$

Therefore, we obtain the desired result. □

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