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FOUNDATIONS OF LINEAR ALGEBRAIC GROUPS

Part II

by

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To Masahide, Eri and Emiko

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Preface

Algebraic group theory is one of the basic subjects of graduate level algebra. However, most of graduate programs of algebra do not teach this important theory. This is because graduate students are expected to have understood the theory before they entered the graduate programs. But it is often the case that they have not acquired the basic knowledge of the algebraic group theory. Furthermore, there are a few appropriate textbooks with which they can learn it by themselves.

The objective of these notes is to provide graduate students with completely self–contained lectures with which they can learn the basic theory of algebra. I explained most of the proofs of the theorems from commutative algebras to algebraic geometry (Chapters 1 and 2). These would help them understand the basic concepts of algebraic groups (Chapter 3) and construct homogeneous spaces of linear algebraic groups (Chapter 5). Also I attempted to relate a particular theory of this topics to other subjects of algebra with which graduate students may be familiar.

The original lectures started in 1980 when I was a Humboldt–fellow at the University of Essen and continued sporadically at Sophia University since then. The manuscript was completed in 1988, one year after the second visit to the University of Essen as a Humboldt and DFG–fellow.

I am very grateful to my colleagues who were involved in this project, especially Prof. Dr. Gerhard Michler, who gave me a chance of giving the lectures at the University of Essen and invited me again in 1987. Sections 21, 22 and 23 are the result of seminars with Dr. Klaus Timmerscheidt in 1987. Although only I am the person who is responsible to these notes, I should say that these sections are the joint work with him.

I am also grateful to Prof. Dr. Charles W. Curtis, who kindly gave me his informal lecture notes on linear algebraic groups which were very useful for preparing Chapter 1. Finally I should like to thank Sophia University for granting me the study leave twice and Frau Sabine Weber for her beautiful and careful typing.

Sophia University, November 1989 and March 1991 Hideki Sawada

CHAPTER III

BASIC CONCEPTS OF ALGEBRAIC GROUPS

In this chapter we define algebraic groups and explain the related basic concepts such as subgroups, morphisms of algebraic groups, connectedness, abelian varieties and linearization of affine algebraic groups.

14. Definition of Algebraic Groups

(14.1) Definition. An algebraic group $(G, \mathscr{O}_G)$ over K is a variety over K which has a group structure, and the operations

$$\begin{aligned} m : G \times G &\longrightarrow G \\ (x,y) &\longrightarrow xy \end{aligned}$$

and

$$\begin{aligned} \tau : G &\longrightarrow G \\ x &\longrightarrow x^{-1} \end{aligned}$$

are morphisms of varieties.

(14.2) Example. Let n be a positive integer. $GL(n,K)$ is an affine algebraic group with coordinate ring $K[M(n,K)]_\Delta$ where $M(n,K)$ is the set of all $n{\times}n$ matrices with coefficients in K and $\begin{aligned} \Delta : M(n,K) &\longrightarrow K \\ z &\longrightarrow \det z \end{aligned}$ (we are considering $GL(n,K)$ as a principal open set in $M(n,K)$ defined by Δ).

Proof. Let $X_{ij} \in M(M(n,K),K)$ be a map which takes each matrix $z \in M(n,K)$ to its (i,j)th coefficient, where $1 \le i,j \le n$. Then $(M(n,K),A) \in \mathscr{A}(K)$ where A is the K–subalgebra of $M(M(n,K),K)$ generated by $\{X_{ij} \mid 1 \le i,j \le n\}$ (see Example 1.2).

Now let Δ be a map of $M(n,K)$ into K such that

$$\begin{aligned} \Delta : M(n,K) &\longrightarrow K \quad , \\ z &\longrightarrow \det z \end{aligned}$$

then $\Delta \in A$.

Since $M(n,K)_\Delta = \{z \in M(n,K) \mid \Delta(z) \neq 0\} = GL(n,K)$, $(GL(n,K), A_\Delta)$ is an affine variety over K according to Proposition 2.8. Let $G = GL(n,K)$, $f_{ij} = X_{ij}|_G$ $(1 \le i,j \le n)$ and

$$\begin{aligned} \delta : G &\longrightarrow K \quad , \\ z &\longrightarrow \frac{1}{\det z} \end{aligned}$$

then $A_\Delta = K[f_{ij}, \delta \mid 1 \le i,j \le n]$. Since

$$m^*(\delta) = \delta \otimes \delta \,, \quad m^*(f_{ij}) = \sum_{k=1}^{n} f_{ik} \otimes f_{kj} \,,$$

$\tau^*(\delta) = \Delta|_G$ and $\tau^*(f_{ij}) = (-1)^{i+j} \{\det(A_{ji})\}\delta$, where $\det(A_{ji}) : \begin{array}{l} G \longrightarrow K \\ x \longrightarrow \det(A_{ji}(x)) \end{array}$ and $A_{ji}(x)$ is the (j,i)th minor of $x \in G$ $(1 \leq i,j \leq n)$, $m : \begin{array}{l} G \times G \longrightarrow G \\ (x,y) \longrightarrow xy \end{array}$ and $\tau : \begin{array}{l} G \longrightarrow G \\ x \longrightarrow x^{-1} \end{array}$ are morphisms. Hence $(GL(n,K),A_\Delta)$ is an affine algebraic group.

Q.E.D.

__Exercise 49__. Show that $GL(n,K)$ is irreducible and $\dim GL(n,K) = n^2$.

__(14.3) Remark__. Let $(G, \mathscr{O}_G)$ be an algebraic group over K . If $\dim G = 0$, then G is a finite group and also a topological group. If $\dim G > 0$, then G is an infinite group but not a topological group.

__Proof__. It is clear that G is finite if $\dim G = 0$. Since G has a discrete topology, the Zariski topology on $G \times G$ is also the product topology of G . Hence G is a topological group.

Assume that $\dim G > 0$ and G is also a topological group. We show the contradiction that G is a Hausdorff space. Let $a,b \in G$ such that $a \neq b$. Since $\{a\}$ is closed in G , there exists an open neighbourhood W of b which does not contain a . Since Wb^{-1} is an open neighbourhood of 1 and the map

$$\begin{array}{l} G \times G \longrightarrow G \\ (x,y) \longrightarrow xy^{-1} \end{array}$$

is continuous, there exists an open set O of G which contains 1 and $OO^{-1} \subset Wb^{-1}$. Let $p \in \overline{O}$, then $(pO) \cap O \neq \emptyset$ and there exist $x,y \in O$ such that $px = y$. Hence $p = yx^{-1} \in Wb^{-1}$ for any $p \in \overline{O}$. Thus $\overline{O} \subset Wb^{-i}$ and $Ob \subset \overline{O}b \subset W$. Since $Ob \cap (\overline{O}b)^c = \emptyset$ and Ob and $(\overline{O}b)^c$ are open neighbourhoods of b and a , respectively, G becomes a Hausdorff space. Q.E.D.

__(14.4) Definition__. A __subgroup of an algebraic group__ $(G, \mathscr{O}_G)$ always means just a non–empty subset of G closed under the group operations.

__(14.5) Definition__. Let $(G, \mathscr{O}_G)$ and $(H, \mathscr{O}_H)$ be algebraic groups over K . A map $\varphi : G \to H$ is said to be a __morphism of algebraic groups__ if

(i) φ is a morphism of varieties and

(ii) φ is a group homomorphism.

For example, $\det : \begin{array}{l} GL(n,K) \longrightarrow GL(1,K) \\ \quad x \longrightarrow \det x \end{array}$ is a morphism of algebraic groups.

(14.6) Lemma. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and $u \in G$, then the following maps are morphisms of varieties.

(i) $\begin{array}{l} G \longrightarrow G \\ x \longrightarrow xu \end{array}$,

(ii) $\begin{array}{l} G \longrightarrow G \\ x \longrightarrow ux \end{array}$,

(iii) $\begin{array}{l} G \longrightarrow G \\ x \longrightarrow u^{-1}xu \end{array}$ and

(iv) $\begin{array}{l} G \longrightarrow G \\ x \longrightarrow x^{-1}ux \end{array}$.

Proof. (i) and (ii). Since the map $\rho : \begin{array}{l} G \longrightarrow G \times G \\ x \longrightarrow (x,u) \end{array}$ is a morphism of varieties from the commutative diagram

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ G & & G \ni & u, \\ & G \ni x \arrow[ul, "1_G"] \arrow[uu, "\exists 1_\rho"'] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

$\begin{array}{l} G \longrightarrow G \times G \longrightarrow G \\ x \longrightarrow (x,u) \longrightarrow xu \end{array}$ is a morphism.

(iii) is clear from (i) and (ii).

(iv) Since the map $\mu : \begin{array}{l} G \longrightarrow G \times G \\ y \longrightarrow (y, uy^{-1}) \end{array}$ is a morphism of varieties from the following commutative diagram:

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ G & & G \ni & uy^{-1}, \\ & G \ni y \arrow[ul, "1_G"] \arrow[uu, "\exists 1_\mu"'] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

$\begin{array}{l} G \to G \longrightarrow G \longrightarrow G \times G \longrightarrow G \\ x \to x^{-1} \to x^{-1}u \longrightarrow (x^{-1}u, x) \to x^{-1}ux \end{array}$ is a morphism. Q.E.D.

<u>(14.7) Corollary</u>. Every irreducible algebraic group G is smooth.

<u>Proof</u>. Let $\{U_i \mid i = 1,2,\dots,m\}$ be an affine open covering of G. Since U_i's are irreducible, G has certainly a non–singular point a from Theorem 7.18. Let $x_0 \in G$ and $u = a^{-1}x_0$, then $\begin{array}{l} \varphi : G \longrightarrow G \\ \quad\;\; x \longrightarrow xu \end{array}$ is an isomorphism of varieties which takes a to x_0. Hence x_0 is also non–singular, and G is smooth. Q.E.D.

<u>(14.8) Lemma</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and H a closed subgroup of G. Let $\iota : H \to G$ be an inclusion map. Then (H, S_H) (see Examples 10.11) is an algebraic group and ι is a morphism of algebraic groups.

<u>Proof</u>. Let $\{U_i \mid i = 1,2,\dots,m\}$ be an affine open covering of $(G, \mathscr{O}_G)$, then $\{U_i \cap H \mid i = 1,2,\dots,m\}$ is an affine open covering of $(H, \mathscr{O}_H)$ (see Proposition 10.4). Since $\iota(U_i \cap H) \subset U_i$ and $f \circ (\iota|_{U_i \cap H}) \in \mathscr{O}_H(U_i \cap H) = K[U_i \cap H]$ for any $f \in \mathscr{O}_G(U_i) = K[U_i]$, where $i = 1,2,\dots,m$, ι is a morphism of varieties from Proposition 10.7. Since

$$H \times H \subset G \times G \xrightarrow{m} G \quad \text{and} \quad H \subset G \xrightarrow{\tau} G$$

are morphisms of varieties (see Exercise 42.1 on p.118),

$$\begin{array}{ccc} H \times H \longrightarrow H & \text{and} & H \longrightarrow H \\ (x,y) \longrightarrow xy & & x \longrightarrow x^{-1} \end{array}$$

are also morphisms from Exercise 40 on p.113. Hence $(H, \mathscr{O}_H)$ is an algebraic group.

Q.E.D.

<u>(14.9) Examples of algebraic groups</u> (see e.g. Humphreys [2, p.52]).

(i) The special linear group: $SL\,(n+1,K) = \mathscr{V}_{GL(n+1,K)}\,(K\,[GL\,(n+1,K)]\,(\Delta-1))$ where $\begin{array}{l} \Delta : GL\,(n+1,K) \longrightarrow K \\ \qquad\qquad\quad g \longrightarrow \det g\,. \end{array}$

(ii) $(K, K[X])$ is an affine algebraic group by addition.

$$\begin{array}{ccc} m : K \times K \longrightarrow K & \text{and} & \tau : K \longrightarrow K \\ (\alpha,\beta) \longrightarrow \alpha+\beta & & \alpha \longrightarrow -\alpha\,. \end{array}$$

(iii) Let $(G, \mathscr{O}_G)$ and $(H, \mathscr{O}_H)$ be algebraic groups over K, then $(G\times H, \mathscr{O}_{G\times H})$ (see Theorem 10.8 and Example 10.11) is also an algebraic group over K by the usual direct product of groups.

<u>(14.10) Proposition</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and H a subgroup of G. Then

(i) if H contains a non-empty open subset of G, then H is open,
(ii) if H is open, then H is closed,
(iii) if H is closed and $[G:H]$ is finite, then H is open,
(iv) $\overline{H}$, the closure of H, is a subgroup of G,
(v) $N_G(S) = \{x \in G \mid x^{-1} Sx = S\}$ is closed for any closed subset S of G, and
(vi) $C_G(S) = \{x \in G \mid x^{-1} sx = s \text{ for } \forall\, s \in S\}$ is closed for any subset S of G.

<u>Proof</u>. (i) Let O be a non-empty open subset of G contained in H. Since $O \subset H$, we have $H = \bigcup_{x \in H} xO$. From Lemma 14.6 xO is a homeomorphic image of O and hence xO is open. Thus H is also open.

(ii) Since H is open, $\bigcup_{x \in G-H} xH$ is also open from the same argument of (i). Since $H = G - \left(\bigcup_{x \in G-H} xH\right)$, H is closed.

(iii) also follows from the similar argument as above.

(iv) Since $H \subset \overline{H}$, $Hx = H \subset \overline{H} \cap (\overline{H}x)$ for any $x \in H$. Thus we have $\overline{H} \subset \overline{H} \cap (\overline{H}x) \subset \overline{H}x$, because $\overline{H}x$ is closed from Lemma 14.6. Hence $\overline{H}H \subset \overline{H}$. Now let $y \in \overline{H}$, then $yH \subset \overline{H}$. Since $\overline{yH} = y\overline{H}$ from Lemma 14.6, we have $y\overline{H} \subset \overline{H}$, i.e., $\overline{H}\,\overline{H} \subset \overline{H}$.
Since the inversion is a homeomorphism, we have $\overline{H}^{-1} = \overline{H^{-1}}$. Hence $\overline{H}$ is a subgroup of G.

(v) Since $N_G(S) = \{x \in G \mid x^{-1} Sx \subset S\} \cap \{x \in G \mid xSx^{-1} \subset S\}$ and $\{x \in G \mid x^{-1} Sx \subset S\} = \bigcap_{s \in S} \{x \in G \mid x^{-1} sx \in S\}$, we only have to show that $\{x \in G \mid x^{-1} sx \in S\}$ is closed for any fixed $s \in S$. Let $\varphi_s : G \to G$ be a map which takes $x \in G$ to $x^{-1} sx \in G$, then φ_s is a morphism from Lemma 14.6. Since $\varphi_s^{-1}(S) = \{x \in G \mid x^{-1} sx \in S\}$ and S is closed, $\{x \in G \mid x^{-1} sx \in S\}$ is closed as expected.

(vi) is clear from (v), because a point is closed in a variety. Q.E.D.

15. Connectedness and Irreducible Components of Algebraic Groups

In this section we study irreducible components of algebraic groups and show an algebraic group is irreducible if and only if it is connected, i.e., it is not a union of any pair of non–empty disjoint closed subsets S_1 and S_2 .

(15.1) Theorem. Let $(G, \mathscr{O}_G)$ be an algebraic group over K . Then

(i) the irreducible components of G are disjoint;

(ii) let G^0 be the irreducible component of G which contains 1 , then G^0 is a closed normal subgroup of G of finite index and the irreducible components of G are the cosets of G^0 ;

(iii) G^0 is open and closed in G ;

(iv) G is connected if and only if G is irreducible;

(v) any closed subgroup of G of finite index contains G^0 .

Proof. (i) Assume that there exist two irreducible components Z_1 and Z_2 of G such that $Z_1 \cap Z_2 \neq \emptyset$. Let $z \in Z_1 \cap Z_2$, then for any $x \in G$ we have $x \in (xz^{-1} Z_1) \cap (xz^{-1} Z_2)$. Thus every element of G is contained in two different irreducible components, because $xz^{-1} Z_1$ and $xz^{-1} Z_2$ are homeomorphic images of irreducible components (see Lemma 14.6).
Now let $X_1, X_2, \ldots, X_n$ be the irreducible components of G , then we have $X_1 \subset X_2 \cup \ldots \cup X_n$, because every element of X_1 is also contained in another irreducible component different from X_1 . Thus we have $X_1 \subset X_i$ for some $i > 1$, a contradiction. Hence the irreducible components of G are disjoint.

(ii) From Lemma 14.6 xG^0 is an irreducible component for any $x \in G^0$. Since $(xG^0) \cap G^0 \ni x$, we have $xG^0 = G^0$ from (i). Hence $G^0 G^0 = G^0$. Similarly $(G^0)^{-1} = \{x^{-1} \mid x \in G^0\}$ is also an irreducible component containing 1 , because $\tau : \begin{array}{l} G \longrightarrow G \\ x \longmapsto x^{-1} \end{array}$ is a homeomorphism. Thus we have $G^0 = (G^0)^{-1}$. Hence G^0 is a closed subgroup of G .
Let x be any element of G . Since $x^{-1} G^0 x$ is a homeomorphic image of the conjugation by x , $x^{-1} G^0 x$ is an irreducible component of G containing 1 . Hence $x^{-1} G^0 x = G^0$ for any $x \in G$, i.e., G^0 is normal in G .

Now let $X_1 = G^0$, $X_2,\dots,X_n$ be the irreducible components of G, then for any $x_i \in X_i$ we have $x_i \in X_i \cap (x_i G^0)$. Thus $X_i = x_i G^0$ $(1 \leq i \leq n)$. Hence irreducible components are the cosets of G^0 and G^0 is of finite index.

(iii) is clear from (ii) and Proposition 14.10.

(iv) follows from (iii).

(v) Let H be a closed subgroup of finite index. Since each coset of H in G is also closed, G^0 is contained in one of the cosets of H. Hence $H \supset G^0$.

Q.E.D.

Exercise 50. Let $(G, \mathscr{O}_G)$ be an algebraic group over K. Show that

(1) $T(G)_1 \cong T(G^0)_1$ as K–spaces;

(ii) $\dim G = \dim_K T(G)_1$.

(15.2) Lemma. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and U,V be dense open subsets of G. Then

$$G = UV.$$

Proof. Since $\tau : \begin{array}{l} G \longrightarrow G \\ x \longrightarrow x^{-1} \end{array}$ is a homeomorphism, V^{-1} is open and dense. Let $x \in G$, and assume that $U \cap (xV^{-1}) = \emptyset$. Then $U \subset G - xV^{-1}$. Since xV^{-1} is open, $G - xV^{-1}$ is closed and $\overline{U} \subset G - xV^{-1}$. Hence $xV^{-1} = \emptyset$, a contradiction. Thus there exists $u \in U \cap (xV^{-1})$ and we have $u = xv^{-1}$ for some $v \in V$, which shows $x = uv \in UV$. Therefore

$$G = UV. \qquad \text{Q.E.D.}$$

(15.3) Proposition. Let H be a subgroup of an algebraic group $(G, \mathscr{O}_G)$ over K (see Definition 14.4). If H is a constructible set, then H is closed.

Proof. First we assume that $\overline{H}$ is irreducible (see Proposition 14.10). Since $H = C_1 \cup C_2 \cup \dots \cup C_m$ where $C_{i's}$ are locally closed sets, we have $\overline{H} = \overline{C}_1 \cup \overline{C}_2 \cup \dots \cup \overline{C}_m$. Since $\overline{H}$ is irreducible, $\overline{H} = \overline{C}_i$ for some i. From Lemma 15.2 we have $\overline{H} = C_i C_i$. Thus $\overline{H} \subset H$ and H is closed.

In general $\overline{H}$ is a finite union of cosets of $(\overline{H})^0$. Since $H/H\cap(\overline{H})^0 \cong H(\overline{H})^0/(\overline{H})^0$, $H \cap (\overline{H})^0$ is also of finite index in H. Thus we have

$$H = x_1 (H \cap (\overline{H})^0) \cup \ldots \cup x_n (H \cap (\overline{H})^0),$$

and also
$$\overline{H} = x_1 \overline{(H \cap (\overline{H})^0)} \cup \ldots \cup x_n \overline{(H \cap (\overline{H})^0)}.$$

From Theorem 15.1.v we have $(\overline{H})^0 \subset \overline{H \cap (\overline{H})^0}$. Thus we have $(\overline{H})^0 = \overline{H \cap (\overline{H})^0}$, because $H \cap (\overline{H})^0 \subset (\overline{H})^0$ and $(\overline{H})^0$ is closed. Since $H \cap (\overline{H})^0$ is constructible and $\overline{H \cap (\overline{H})^0}$ is irreducible, $H \cap (\overline{H})^0$ is closed. Hence $(\overline{H})^0 = H \cap (\overline{H})^0$. Since $\overline{H} \supset H \supset (\overline{H})^0$, H is a union of finite number of closed cosets of $(\overline{H})^0$. Hence $\overline{H} = H$, i.e., H is closed. Q.E.D.

<u>(15.4) Theorem</u>. Let $(G, \mathscr{O}_G)$, $(H, \mathscr{O}_H)$ be algebraic groups over K and $\varphi : G \to H$ be a morphism of algebraic groups. Then

(i) Ker φ is a closed normal subgroup;

(ii) $\varphi(G)$ is a closed subgroup of H;

(iii) $\varphi(G^0) = \varphi(G)^0$;

(iv) $\dim G = \dim \operatorname{Ker} \varphi + \dim \varphi(G)$.

<u>Proof</u>. (i) Since $\{1\}$ is closed in H, $\operatorname{Ker} \varphi = \varphi^{-1}(\{1\})$ is closed.

(ii) From Theorem 8.6 and the proof of Proposition 10.7 $\varphi(G) \cap O_l$ is constructible in each $(O_l, K[O_l])$ where $\{O_l \mid l = 1,2,\ldots,t\}$ is a finite affine open covering of H. Hence $\varphi(G)$ is constructible in H. Therefore, $\varphi(G)$ is closed from Proposition 15.3.

(iii) Since $\varphi(G^0)$ is irreducible from Lemma 5.6, we have $\varphi(G^0) \subset \varphi(G)^0$. Since $\varphi(G^0)$ is of finite index and closed in $\varphi(G)$, we also have $\varphi(G^0) \supset \varphi(G)^0$ from Theorem 15.1.v. Hence $\varphi(G^0) = \varphi(G)^0$.

(iv) Since $\varphi(G^0) = \varphi(G)^0$, $\varphi|_{G^0}$ is a dominant morphism of G^0 into $\varphi(G)^0$ (see Lemma 14.8 and Exercise 40 on p.113). Let $\varphi_0 = \varphi|_{G^0} : G^0 \to \varphi(G)^0$ and $U \neq \emptyset$ be an open subset of G^0 which satisfies the conditions of Theorem 13.14. Let $u \in U$. Since $u \operatorname{Ker} \varphi_0 = \varphi_0^{-1}(\varphi_0(u))$, we have

$$\dim \operatorname{Ker} \varphi_0 = \dim G^0 - \dim \varphi(G)^0$$

from Theorem 13.14. Hence

$$\dim G = \dim \operatorname{Ker} \varphi_0 + \dim \varphi(G) .$$

Since $(\operatorname{Ker} \varphi_0)^0 = (\operatorname{Ker} \varphi)^0$ and $\dim \operatorname{Ker} \varphi = \dim(\operatorname{Ker} \varphi)^0$, we have

$$\dim G = \dim \operatorname{Ker} \varphi + \dim \varphi(G) . \qquad \text{Q.E.D.}$$

Next we shall show a proposition which is useful to construct connected algebraic groups.

<u>(15.5) Proposition</u>. Let $(G, \mathcal{O}_G)$ be an algebraic group over K and I be an index set. Let $\{f_i : X_i \to G\}_{i \in I}$ be a family of morphisms of irreducible varieties $(X_i, \mathcal{O}_{X_i})$ over K into $(G, \mathcal{O}_G)$ such that $Y_i = f_i(X_i) \ni 1$ for all $i \in I$. Write $\mathcal{G}(\bigcup_{i \in I} Y_i)$ for the smallest closed subgroup of G containing $\bigcup_{i \in I} Y_i$, i.e., the intersection of all closed subgroup of G containing $\bigcup_{i \in I} Y_i$. Then

(i) $\mathcal{G}(\bigcup_{i \in I} Y_i)$ is connected;

(ii) there exists a finite sequence $b = \{b(1),\dots,b(l)\}$ in I such that $\mathcal{G}(\bigcup_{i \in I} Y_i) = Y_{b(1)}^{e_1} \cdots Y_{b(l)}^{e_l}$, where $e_j = \pm 1$ for all $1 \leq j \leq l$.

<u>Proof</u>. It is enough to prove the proposition with the enlarged index set which includes all the morphisms

$$\begin{aligned} X_i &\longrightarrow G \\ x &\longrightarrow f_i(x)^{-1} . \end{aligned}$$

We denote by Y_a the subset $Y_{a(1)} Y_{a(2)} \dots Y_{a(n)}$ of G for a finite sequence $a = \{a(1),\dots,a(n)\}$ of elements from I . Let f_a be a map of $X_{a(1)} \times \dots \times X_{a(n)}$ into G such that

$$\begin{aligned} f_a : X_{a(1)} \times \dots \times X_{a(n)} &\longrightarrow G \\ (x_1, x_2, \dots, x_n) &\longrightarrow f_{a(1)}(x_1) \cdot f_{a(2)}(x_2) \dots f_{a(n)}(x_n) , \end{aligned}$$

then f_a is a morphism of varieties and the image of f_a is the subset $Y_{a(1)} \cdot Y_{a(2)} \dots Y_{a(n)}$ of G and irreducible from Lemma 5.6 and Exercise 44 on p.118.

Since $\dim \overline{Y}_a \leq \dim G$, we can choose $a_0 = \{a_0(1),\dots,a_0(n_0)\}$ such that $\dim \overline{Y}_{a_0}$ is maximal. It is clear that $\overline{Y}_{a_0}$ is maximal among the subsets $\overline{Y}_b = \overline{Y_{b(1)} \dots Y_{b(l)}}$, where $b = \{b(1),\dots,b(l)\}$ is a finite sequence in I (see Exercise 48 on p.140).

Now let $c = \{c(1),...,c(s)\}$ and $d = \{d(1),...,d(t)\}$ be any two sequences in I , then we write (c,d) for the sequence $\{c(1),...,c(s), d(1),...,d(t)\}$. Since $Y_c y \subset Y_{(c,d)}$ for any $y \in Y_d$, we have $\overline{Y_c y} \subset \overline{Y_{(c,d)}}$. Thus $\overline{Y}_c\, Y_d \subset \overline{Y_{(c,d)}}$. Similarly we have

$$\overline{Y}_c\, \overline{Y}_d \subset \overline{Y}_{(c,d)} ,$$

because $xY_d \subset \overline{Y}_{(c,d)}$ for any $x \in \overline{Y}_c$.

(i) From the above argument we have $\overline{Y}_{a_0}\, \overline{Y}_d \subset \overline{Y}_{(a_0,d)}$ for any sequence d . Since $1 \in Y_d$, $\overline{Y}_{a_0} \subset \overline{Y}_{(a_0,d)}$. Hence $\overline{Y}_{a_0} = \overline{Y}_{(a_0,d)} \supset \overline{Y}_{a_0}\, \overline{Y}_d \supset \overline{Y}_d$ for any sequence d . Thus we have

$$\overline{Y}_{a_0} \supset \bigcup_{i \in I} Y_i \text{ and } \overline{Y}_{a_0}\, \overline{Y}_{a_0} \subset \overline{Y}_{a_0} .$$

From the assumption of the enlarged index set which contains all morphisms

$$\begin{aligned} X_i &\longrightarrow G \\ x &\longrightarrow f_i(x)^{-1} , \end{aligned}$$

there exists a sequence d_0 in I such that $Y_{d_0} = Y_{a_0}^{-1}$. Hence $\overline{Y_{a_0}^{-1}} = \overline{Y}_{a_0}^{-1} = \overline{Y}_{d_0} \subset \overline{Y}_{a_0}$. Thus $\overline{Y}_{a_0}$ is a closed irreducible subgroup of G containing $\bigcup_{i \in I} Y_i$. It is clear that

$$\overline{Y}_{a_0} = \mathscr{G}(\bigcup_{i \in I} Y_i) .$$

(ii) Let $\{O_j \mid j = 1,...,m\}$ be a finite affine open covering of $(G, \mathscr{O}_G)$, then $f_{a_0}(X_{a_0(1)} \times ... \times X_{a_0(n_0)}) \cap O_j = Y_{a_0} \cap O_j$ is constructible in each $(O_j, K[O_j])$ from Theorem 8.6 and the proof of Proposition 10.7. Hence Y_{a_0} is constructible in G . Let $Y_{a_0} = C_1 \cup ... \cup C_r$ be the union of locally closed sets $C_1, C_2, ..., C_r$. Since $\overline{Y}_{a_0} = \overline{C}_1 \cup ... \cup \overline{C}_r$ and $\overline{Y}_{a_0}$ is irreducible, we have $\overline{Y}_{a_0} = \overline{C}_q$ for some $1 \le q \le r$. Since C_q is an open and dense subset of an algebraic group $\overline{Y}_{a_0}$, $\overline{Y}_{a_0} = C_q \cdot C_q$ from Lemma 15.2. Thus we have

$$\overline{Y}_{a_0} = C_q \cdot C_q \subset Y_{a_0} Y_{a_0} = Y_{(a_0,a_0)} \subset \overline{Y}_{a_0} .$$

Hence if we take $b = (a_0,a_0)$, we have $\mathscr{G}(\bigcup_{i \in I} Y_i) = Y_b$. Q.E.D.

<u>(15.6) Corollary</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and $\{H_i\}_{i \in I}$ be closed connected subgroups. Then the subgroup of G generated by $\{H_i\}_{i \in I}$ is closed

and connected.

Proof. Take $\{H_i \subset G\}_{i \in I}$ (see Lemma 14.8). Q.E.D.

(15.7) Example. Let $G = SL(n,K)$, then G is a closed subgroup of $GL(n,K)$ (see Example 14.9). Let

$$x_{ij}(\alpha) = \begin{array}{cc} \begin{array}{ccccccc} & & & & & j & \end{array} & \\ \begin{bmatrix} 1 & & & & & \vdots & \\ & 1 & & & 0 & \vdots & 0 \\ \cdots & \cdots & 1 & \cdots & \cdots & \alpha & \cdots \\ & & & \ddots & & \vdots & 0 \\ & 0 & & & \ddots & \vdots & \\ & & & & & \vdots \ddots & \\ & & & & & \vdots & 1 \end{bmatrix} & \cdots i \end{array}$$

be an element of G , where $1 \leq i \neq j \leq n$ and $\alpha \in K$. Then for each $1 \leq i \neq j \leq n$ we have

(i) $x_{ij}(\alpha)\, x_{ij}(\beta) = x_{ij}(\alpha+\beta)$ for any $\alpha, \beta \in K$;

(ii) let $U_{ij} = \langle x_{ij}(\alpha) \mid \alpha \in K \rangle$ be the subgroup of G generated by $\{x_{ij}(\alpha) \mid \alpha \in K\}$, then U_{ij} is a closed subgroup of G ;

(iii) let $\begin{array}{rl} x_{ij} : K & \longrightarrow U_{ij} \\ \alpha & \longrightarrow x_{ij}(\alpha) \end{array}$ be a group homomorphism of the additive group K into U_{ij} which takes each $\alpha \in K$ to $x_{ij}(\alpha) \in U_{ij}$, then x_{ij} is a morphism of affine varieties of $(K, K[X])$ into $(U_{ij}, K[U_{ij}])$.

Since each U_{ij} is a morphic image of the irreducible affine variety $(K, K[X])$, U_{ij}'s are irreducible. Hence $G = \langle U_{ij} \mid 1 \leq i \neq j \leq n \rangle$ is irreducible from Corollary 15.6.

16. A Remark on Rigidity Lemma and Abelian Varieties

In this section we show that a connected algebraic group whose underlying variety is complete is abelian.

(16.1) Rigidity Lemma. Let X be an irreducible complete variety over K, and Y and Z be any varieties over K. Assume that Y is irreducible. Let $f : X \times Y \to Z$ be a morphism of varieties such that $|f(X \times \{y_0\})| = 1$ for some $y_0 \in Y$. Then there exists a morphism of varieties g of Y into Z such that $f = g \circ \pi_2$ where $\pi_2 : X \times Y \to Y$ is the projection.

$$\begin{tikzcd} X \times Y \arrow[dd, "f"'] \arrow[dr, "\pi_2"] & \\ & Y \arrow[dl, "\exists g"] \\ Z & \end{tikzcd}$$

Proof (see Mumford [1, p.43]). Let x_0 be any fixed point of X and g be a map of Y into Z such that

$$\begin{aligned} g : Y &\longrightarrow Z \\ y &\longrightarrow f(x_0, y) . \end{aligned}$$

It is clear that g is a morphism of varieties, because

$$\begin{aligned} g : Y &\longrightarrow X \times Y \xrightarrow{f} Z \\ y &\longrightarrow (x_0, y) \longrightarrow f(x_0, y) . \end{aligned}$$

We shall show that $f = g \circ \pi_2$.

Let $\{z_0\} = f(X \times \{y_0\})$ and U be an affine open set of Z which contains z_0. Let $F = Z - U$, then $f^{-1}(F)$ is closed in $X \times Y$. Since X is complete, π_2 is a closed map (see Definition 12.1) and $\pi_2(f^{-1}(F))$ is closed in Y. Since $f(X \times \{y_0\}) = \{z_0\}$ and $z_0 \notin F$, we have $y_0 \notin \pi_2(f^{-1}(F))$. Hence $V = Y - \pi_2(f^{-1}(F)) \ni y_0$ is a non-empty open subset of Y.

Now let $y \in V$. Since $y \notin \pi_2(f^{-1}(F))$, $(x,y) \notin f^{-1}(F)$ for any $x \in X$. Hence $f(X \times \{y\}) \subset U$ for any $y \in V$. Since $X \times \{y\}$ is a closed subvariety of $X \times Y$ and is also the product variety of complete varieties X and $\{y\}$ (see Exercise 42.2 on p. 118), $X \times \{y\}$ is complete (see Proposition 12.2). Since $X \times \{y\}$ is irreducible, we have $|f(X\times\{y\})| = 1$ from Corollary 12.3. Hence for any $x \in X$ and $y \in V$ we have $f(x,y) = f(x_0,y) = g\circ\pi_2(x,y)$.

Since $\{(x,y) \in X\times Y \mid f(x,y) = g\circ\pi_2(x,y)\}$ is closed from Remark 10.10 and $X\times Y$ is irreducible, $\{(x,y) \in X\times Y \mid f(x,y) = g\circ\pi_2(x,y)\} \supset \overline{X\times V} = X\times Y$. Therefore, $f = g\circ\pi_2$ on $X\times Y$. Q.E.D.

<u>(16.2) Corollary</u>. (i) Let X be an irreducible complete algebraic group over K and Y be an algebraic group over K . If f is a morphism of varieties of X into Y such that

$$f(1) = 1 ,$$

then f is a group homomorphism.

(ii) Any irreducible complete algebraic group is commutative.

<u>Proof</u>. (i) Let φ be a map of $X\times X$ into Y which takes $(x,y) \in X\times X$ to $f(xy)(f(x)\,f(y))^{-1} \in Y$. It is clear that φ is a morphism of varieties, because

$$\begin{array}{ccccc} X\times X & \longrightarrow & X & \longrightarrow & Y \\ (x\,,y) & \longrightarrow & xy & \longrightarrow & f(xy) \end{array}$$

and

$$\begin{array}{ccccc} X\times X & \longrightarrow & Y\times Y & \longrightarrow & Y \\ (x\,,y) & \rightarrow & (f(x)\,,f(y)) & \rightarrow & (f(x)\,f(y))^{-1} \end{array}$$

are morphisms. Since $\varphi(X\times\{1\}) = \{1\}$ and $\varphi(\{1\}\times X) = \{1\}$, there exist morphisms of varieties $g_i : X \to Y$ $(i = 1,2)$ as follows.

$$\begin{tikzcd} & X\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\varphi"] & \\ X \arrow[dr, "g_1"'] & & X \arrow[dl, "g_2"] \\ & Y & \end{tikzcd}$$

(commutative diagram)

Hence $\varphi(x,y) = \varphi(x',y')$ for any $(x,y), (x',y') \in X\times X$. Therefore, $\varphi(x,y) = 1$ on $X\times X$, which implies $f(xy) = f(x)\,f(y)$ for any $x,y \in X$.

(ii) From (i) $\begin{array}{l} \tau : X \longrightarrow X \\ \quad\;\; x \longrightarrow x^{-1} \end{array}$ is a group homomorphism. Hence $(xy)^{-1} = x^{-1}\,y^{-1}$, i.e., $xy = yx$ for any $x,y \in X$. Q.E.D.

<u>(16.3) Definition</u>. We call an irreducible complete algebraic group over K an <u>abelian variety</u>.

17. Operations of Algebraic Groups on Varieties

In this section we study morphic operations of algebraic groups on varieties and show the existence of closed orbits.

(17.1) Definition. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and $(X, \mathscr{O}_X)$ be a prevariety over K. We say that G operates on X morphically if there is a morphism of prevarieties $\varphi : G \times X \to X$ such that

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \text{for any } g_1, g_2 \in G \text{ and } x \in X ;$$
$$1 \cdot x = x \quad \text{for all } x \in X ,$$

where we write $\varphi(g,x) = g \cdot x$ for brevity. We call X a G–prevariety over K. A (prevariety) morphism $f : X \to Y$ of two G–prevarietes is said to be a G–morphism if

$$f(g \cdot x) = g \cdot f(x) \quad \text{for any } g \in G \text{ and } x \in X .$$

(17.2) Remark. Let X be a G–prevariety over K.

(i) Let $g \in G$ and T_g be a map of X into itself such that

$$T_g : X \longrightarrow X \; , \quad x \longmapsto g \cdot x$$

then T_g is a morphism of prevarieties.

(ii) Let $x \in X$ and φ_x be a map of G into X such that

$$\varphi_x : G \longrightarrow X \; , \quad g \longmapsto g \cdot x$$

then φ_x is a morphism of prevarieties.

Exercise 51. Prove the above Remarks.

(17.3) Proposition. Let X be a G–prevariety over K and Y and Z be subsets of X. Assume that Z is closed. Then

(i) the set of transporters $\mathrm{Tran}_G(Y,Z) = \{g \in G \mid g \cdot Y \subset Z\}$ is closed in G ;

(ii) for each $x \in X$, the isotropy group $G_x = \{g \in G \mid g \cdot x = x\}$ of x is closed, in particular the centralizer $C_G(X) = \bigcap_{x \in X} G_x$ of X in G is closed;

(iii) if G is connected, G stabilizes each irreducible component of X. Hence G acts trivially on any finite set.

Proof (see Humphreys [2, Proposition 8.2]). (i) Since $\varphi_x : \begin{array}{l} G \longrightarrow X \\ g \longrightarrow g\cdot x \end{array}$ is a morphism of prevarieties from Remark 17.2, $\varphi_x^{-1}(Z)$ is closed for any $x \in X$. Hence

$$\mathrm{Tran}_G(Y,Z) = \bigcap_{y \in Y} \varphi_y^{-1}(Z)$$

is closed.

(ii) Since $G_x = \mathrm{Tran}_G(\{x\}, \{x\})$, G_x is closed. Hence $C_G(X) = \bigcap_{x \in X} G_x$ is also closed.

(iii) Let X_0 be an irreducible component of X. Since X_0 is closed, $H = \mathrm{Tran}_G(X_0, X_0)$ is a closed subset of G from (i). Let $g \in G$, T_g is a homeomorphism from Remark 17.2, gX_0 is also an irreducible component of X. Hence $gX_0 = X_0$ for any $g \in H$. Therefore, H is a closed subgroup of G. Since G operates on the set of irreducible components of X, which is finite, H is of finite index in G. Hence $H = G$ from Theorem 15.1. Q.E.D.

(17.4) Proposition. Let X be a G–variety over K, then the fixed point set $\{x \in X \mid g\cdot x = x\}$ of $g \in G$ is closed, in particular,

$$X^G = \{x \in X \mid g\cdot x = x \ \text{ for all } \ g \in G\}$$

is closed.

Proof. Let $g \in G$ and f be a map of X into $X\times X$ such that

$$f : \begin{array}{l} X \longrightarrow X\times X \\ x \longrightarrow (x, g\cdot x) \end{array},$$

then f is a morphism of varieties.

$$\begin{tikzcd} & X\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \ni g\cdot x \\ & X \ni x \arrow[ul, "1_X"] \arrow[uu, "f"] \arrow[ur] & \end{tikzcd}$$ (commutative diagram)

Since $\Delta(X)$ is closed in $X\times X$ (see Definition 10.9), $f^{-1}(\Delta(X)) = \{x \in X \mid g\cdot x = x\}$ is closed. Hence

$$X^G = \bigcap_{g \in G} \{x \in X \mid g \cdot x = x\}$$

is also closed Q.E.D.

<u>(17.5) Proposition</u>. Let G be a connected algebraic group over K and X be a G–variety. Let Y be a G–orbit of X. Then

(i) Y is irreducible;

(ii) Y is locally closed, i.e., Y is open in $\overline{Y}$;

(iii) $\dim(\overline{Y}-Y) < \dim \overline{Y}$;

(iv) $\overline{Y}-Y$ is G–stable, i.e., $g\cdot(\overline{Y}-Y) \subset \overline{Y}-Y$ for any $g \in G$.

<u>Proof</u> (see Springer [1, Lemma 4.3.1]). (i) and (ii) Let $x \in X$ and $Y = G\cdot x$. Since $\varphi_x : \begin{array}{l} G \longrightarrow X \\ g \longrightarrow g\cdot x \end{array}$ is a morphism of varieties, $\overline{\varphi}_x : \begin{array}{l} G \longrightarrow \overline{Y} \\ g \longrightarrow g\cdot x \end{array}$ is also a morphism of varieties from Exercise 40 on p.113. It is clear that $\overline{Y}$ is irreducible from Proposition 5.4 and Lemma 5.6. From Theorem 13.14 $\overline{Y}$ has a non–empty open set U_0 such that $U_0 \subset \overline{\varphi}_x(G) = Y$. Therefore,

$$Y = \bigcup_{g \in G} gU_0$$

is open in $\overline{Y}$, because G also operates on $\overline{Y}$ morphically.

(iii) Since $\overline{Y}-Y$ is a proper closed subset of $\overline{Y}$, we have $\dim(\overline{Y}-Y) < \dim \overline{Y}$ from Exercise 48 on p.140.

(iv) Since $g\cdot Y = Y$ and $g\cdot\overline{Y} = \overline{g\cdot Y} = \overline{Y}$ for any $g \in G$, we have $g\cdot(\overline{Y}-Y) \subset \overline{Y}-Y$.
Q.E.D.

<u>(17.6) Definition</u>. Let G be a connected algebraic group over K and X be a G–variety. Let Y be a G–orbit of X. We define $\dim Y$ to be the <u>dimension of Y</u> as open subvariety of $\overline{Y}$. Therefore $\dim Y = \dim_X \overline{Y}$.

<u>(17.7) Corollary</u>. Let G and X be as in Proposition 17.5. Then the orbits of minimal dimension are closed.

<u>Proof</u>. Let Y be an orbit of minimal dimension. Since $\dim(\overline{Y}-Y) < \dim \overline{Y} = \dim Y$ and $\overline{Y}-Y$ is the union of other G–orbits from (iv), $\overline{Y}-Y = \emptyset$. Hence Y is closed. Q.E.D.

Independently of Proposition 17.5 we get the following proposition.

(17.8) Proposition. Let G be an algebraic group over K and X be a G–variety, then any G–orbit Y of X is locally closed.

Proof. Let $Y = G{\cdot}y$ for some $y \in Y$. Since $\begin{array}{rl}\varphi_y : G & \longrightarrow X \\ g & \longrightarrow g{\cdot}y\end{array}$ is a morphism of varieties, $\begin{array}{rl}\overline{\varphi}_y : G & \longrightarrow \overline{Y} \\ g & \longrightarrow g{\cdot}y\end{array}$ is also a morphism of varieties from Exercise 40 on p.113. From Theorem 8.6 and the proof of Proposition 10.7 $\overline{\varphi}_y(G) \cap O_l$ is constructible in each $(O_l, K[O_l])$ where $\{O_l \mid l = 1,2,\ldots,t\}$ is a finite affine open covering of $\overline{Y}$. Hence Y is constructible in $\overline{Y}$ and a union of finite irreducible sets. From Exercise 29 on p.88 Y contains an open dense subset of $\overline{Y}$. Since $Y = G{\cdot}y$, Y is open in $\overline{Y}$.

Q.E.D.

(17.9) Definition. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and X be a G–variety. We call X a homogeneous space of G if X has only one G–orbit.

(17.10) Remark. A homogeneous irreducible G–variety is smooth.

Proof. Let X be a given homogeneous irreducible G–variety. From Theorem 7.18 any affine open set O of X contains a simple point a in it. Since $\begin{array}{rl}T_g : X & \longrightarrow X \\ x & \longrightarrow gx\end{array}$ is an isomorphism of varieties, ga is also a simple point for any $g \in G$ (see Exercise 39 on p.112). Hence X is smooth.

(17.11) Lemma. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be irreducible varieties over K and $\varphi : X \to Y$ be a dominant morphism, i.e., φ is a morphism of varieties and $\overline{\varphi(X)} = Y$. Then there exist affine open sets U in X and O in Y and a positive integer r such that $\varphi(U) \subset O$ and $\varphi|_U$ can be factorized $\alpha = \pi_2 \circ \psi$ with $\psi : U \to K^r{\times}O$ surjective finite morphism and $\pi_2 : K^r{\times}O \to O$ the projection.

$$\varphi|_U : U \xrightarrow{\psi} K^r{\times}O \xrightarrow{\pi_2} O$$

Proof (see Steinberg [2, p.58]). Let O be an affine open set in Y , then there exists an affine open set U in X such that $\varphi(U) \subset O$.

$$\varphi|_U : U \longrightarrow O$$

is also a dominant morphism of varieties. Hence we can assume that X and Y are affine varieties (see Corollary 10.5).

Since φ is dominant, $\varphi^* : K[Y] \to K[X]$ is injective (see Lemma 8.3). We shall write $K[Y] = B$ and $K[X] = A$. Thus B can be considered as K–subalgebra of A . Let $Q(B)$ be the quotient field of B , then $Q(B)[A]$, the $Q(B)$–algebra generated by A , is also finitely generated over $Q(B)$. By Noether Normalization Theorem there exist algebraically independent elements $x_1,x_2,\dots,x_r$ (over $Q(B)$) in $Q(B)[A]$ such that $Q(B)[A]$ is integral over $Q(B)[x_1,\dots,x_r]$. Since B is a K–subalgebra of A , we can take $x_1,\dots,x_r$ in A . In the equations expressing the integrality over $Q(B)[x_1,\dots,x_r]$ for a finite generating set of $Q(B)[A]$, the coefficients are polynomials in $x_1,\dots,x_r$ with coefficients in $Q(B)$. Let $b \in B$ be a common denominator for all these coefficients. Then A_b , the ring of fractions of A by $\{b^n \mid n \in \mathbb{N}\}$ (see Lemma 2.7), is integral over $B_b[x_1,\dots,x_r]$, because $B_b[x_1,\dots,x_r]$ contains all such coefficients and the finite generating set of $Q(B)[A]$ is integral over $B_b[x_1,\dots,x_r]$ (see Proposition 6.5).

From the following commutative diagram

$$\begin{tikzcd}[column sep=small]
 & x\otimes y \in B_b \otimes_K K[x_1,\dots,x_r] \arrow[dr] & \\
(x,y) \in B_b\times K[x_1,\dots,x_r] \arrow[ur] \arrow[rr] & & B_b[x_1,\dots,x_r] \, , \\
(x,y) \arrow[rr, mapsto] & & xy
\end{tikzcd}$$

$B_b[x_1,\dots,x_r]$ is isomorphic to $B_b \otimes_K K[x_1,\dots,x_r]$ as K–algebras. Hence we have the following sequence of algebraic homomorphisms.

$$B_b \longrightarrow \varphi^*(B)_{\varphi^*(b)} \otimes_K K[x_1,\dots,x_r] \subset A_{\varphi^*(b)} \, .$$
$$a/b^n \longrightarrow \varphi^*(a)/\varphi^*(b)^n \otimes 1$$

Thus we have got the desired sequence of morphisms

$$\varphi|_{X_{\varphi^*(b)}} : X_{\varphi^*(b)} \xrightarrow{\psi} K^r\times Y_b \xrightarrow{\pi_2} Y_b$$

(see Lemma 2.4).

Finally we shall show the surjectivity of ψ . Since ψ is a closed map, i.e., φ maps a closed set to a closed set (see Proposition 8.4), $\psi(X_{\varphi^*(b)})$ is closed in $K^r\times Y_b$. Since

$$K\,[\psi(X_{\varphi^*(b)})] = \{f|_{\psi(X_{\varphi^*(b)})} \mid f \in K[K^r\times Y_b]\}$$

and

$$\begin{aligned} K\,[\psi(X_{\varphi^*(b)})] &\cong \psi^*(K[K^r\times Y_b]) \\ f|_{\psi(X_{\varphi^*(b)})} &\longmapsto f\circ\psi \end{aligned}$$

as K–algebras, we have $\dim \psi(X_{\varphi^*(b)}) = \dim(K^r\times Y_b)$. Hence $\psi(X_{\varphi^*(b)}) = K^r\times Y_b$ from Proposition 7.2. Q.E.D.

<u>(17.12) Lemma</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be homogeneous spaces of G (see Definition 17.9). Let $f : X \to Y$ be a G–morphism, then f is an open map.

<u>Proof</u> (see Steinberg [2, p.58]). It is enough to show that the map $\varphi_x : \begin{array}{l} G \longrightarrow X \\ g \longmapsto g\cdot x \end{array}$ is open for any fixed $x \in X$, because we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \ni g\cdot f(x) \\ & G \ni g \arrow[ul, "\varphi_x"] \arrow[u] \end{tikzcd}$$

Now let G^0 be the connected component of G containing 1 , then $G = \bigcup_{i=1}^{t} G^0 g_i$ (disjoint union of open subsets) and $X = G\cdot x = \bigcup_{i=1}^{t} G^0 g_i\cdot x = G^0 g_{i_1}\cdot x \cup\ldots\cup G^0 g_{i_s}\cdot x$ (disjoint union of G^0–orbits in X , $s \leq t$). Since $G^0 g_{i_1}\cdot x,\ldots,G^0 g_{i_s}\cdot x$ are homeomorphic, they are all closed from Corollary 17.7. Hence they are open and closed. Thus it is enough to prove that $\varphi_x | G^0 : G^0 \to G^0 x$ is an open map.

From now on we assume that G and X are irreducible. Let S be an open set in G and H be the isotropy group of x , i.e., $H = \{g \in G \mid g\cdot x = x\}$, then $\varphi_x^{-1}(S\cdot x) = SH$, $\varphi_x(SH) = \varphi_x(S)$ and $SH = \bigcup_{h\in H} Sh$ is open in G . Let U and O etc. be as in Lemma 17.11 such that $\varphi_x|_U : U \xrightarrow{\psi} K^r\times O \xrightarrow{\pi_2} O$.

Since $G = \bigcup_{g\in G} gU$, it is enough to show that $\varphi_x(SH \cap gU)$ is open for any g in order to see φ_x is an open map. Since

$$\varphi_x \circ T_{g^{-1}}\Big|_{gU} : gU \to U \xrightarrow{\psi} K^r \times O \xrightarrow{\pi_2} O$$

and
$$\begin{aligned}(g^{-1} SH) \cap U &= (\pi_2 \circ \psi)^{-1} \{((g^{-1} SH) \cap U)x\} \\ &= \psi^{-1} [\psi(U) \cap \pi_2{}^{-1} \{((g^{-1} SH) \cap U)x\}] ,\end{aligned}$$

$\varphi_x \circ T_{g^{-1}}(SH \cap gU) = \varphi_x((g^{-1} SH) \cap U) = ((g^{-1} SH) \cap U)x$ is open in X from Corollary to Proposition 8.4. Hence $g((g^{-1} SH) \cap U)x = (SH \cap gU)x = \varphi_x(SH \cap gU)$ is open for each $g \in G$. Q.E.D.

18. Rational Representations of Algebraic Groups

Now we define rational representations of algebraic groups and explain some related notions such as induced representations. We also prove that affine algebraic groups are faithfully (i.e. injectively) represented into some general linear groups.

Let G be a group. The group ring RG over a commutative ring R (with unity element 1) is the free R–module with basis $\{g \mid g \in G\}$ and with multiplication defined by

$$\Big[\sum_{x\in G} \alpha_x\, x\Big]\Big[\sum_{y\in G} \beta_y\, y\Big] = \sum_{x,\,y\in G} \alpha_x\, \beta_y\, xy\ .$$

Notice that almost all α_x's and β_y's are zero. It is clear that RG is an associative ring with unity element $1 \in G$. When R is a field, RG is of course an algebra over R.

Let M be an n–dimensional vector space over K. Assume that $\{m_1,m_2,\ldots,m_n\}$ is a K–basis of M, then with respect to this basis M has an affine variety structure which makes the follwing K–isomorphism ψ an isomorphism of varieties.

$$\begin{aligned} \psi : M &\longrightarrow K^n \\ x_1m_1+\ldots+x_nm_n &\to (x_1,x_2,\ldots,x_n) \end{aligned}$$

It can be easily shown that the variety structure of M does not depend on a given K–basis $\{m_1,\ldots,m_n\}$ of M.

(18.1) Definition. Let $(G,\mathscr{O}_G)$ be an algebraic group over K. A rational representation of G over K of degree n is a morphism of algebraic groups

$$\rho : G \longrightarrow GL(n,K) \ \text{ for some } n\ .$$

A finite dimensional KG–module M is said to be rational if for some K–basis $\{m_1,\ldots,m_n\}$ of M the map

$$\begin{aligned} \rho : G &\longrightarrow GL(n,K) \\ g &\longrightarrow (\rho_{ij}(g)) \end{aligned}$$

is a rational representation where

$$g(m_1,\ldots,m_n) = (m_1,\ldots,m_n)\begin{bmatrix} \rho_{11}(g), \rho_{12}(g),\ldots, \rho_{1n}(g) \\ \rho_{21}(g), \rho_{22}(g),\ldots, \rho_{2n}(g) \\ \vdots \qquad \ddots \qquad \vdots \\ \rho_{n1}(g),\ldots\ldots\ldots, \rho_{nn}(g) \end{bmatrix}$$

Clearly, the rationality of M does not depend on its given basis $\{m_1,\dots,m_n\}$.

(18.2) Proposition. Let G, M and ρ be as in Definition 18.1.

(i) A group homomorphism $\rho : G \longrightarrow GL(n,K),\ g \longmapsto (\rho_{ij}(g))$ is a rational representation over K if and only if all the functions $\{\rho_{ij}\}$ belong to $\mathscr{O}_G(G)$.

(ii) Let N be a KG–submodule of M , then N and M/N are rational KG–modules.

(iii) Let M_1 and M_2 be rational KG–modules, then the direct sum $M_1 \dotplus M_2$ and the tensor product $M_1 \otimes_K M_2$ is also a rational KG–module. We define $g(m\otimes n) = gm \otimes gn$, where $g \in G$, $m \in M_1$ and $n \in M_2$.

(iv) Let M be a rational KG–module, then $M^* = \mathrm{Hom}_K(M,K)$ is a rational KG–module by the following G–operation

$$\begin{aligned} G\times M^* &\longrightarrow M^* \\ (g\,,\,f\,) &\longrightarrow [\,g\cdot f : m \to f(g^{-1} m)] \quad (m \in M) . \end{aligned}$$

Further $\mathrm{End}_K(M)$ is also a rational KG–module by the following G–operation

$$\begin{aligned} G\times \mathrm{End}_K(M) &\to \mathrm{End}_K(M) \\ (g\,,\,f) &\longrightarrow [\,g\cdot f \ : \ m \to gf(g^{-1} m)] \quad (m \in M) . \end{aligned}$$

(v) Let M be a rational KG–module, then the map

$$\begin{aligned} \varphi :\ G\times M &\longrightarrow M \\ (g\,,m) &\longrightarrow g\,m \end{aligned}$$

is a morphism of varieties.

Proof. (i) Let $M(n,K)$ be the set of all. $n\times n$ matrices with coefficients in K and $\Delta : M(n,K) \to K,\ z \to \det z$, then $K[GL(n,K)] = K[M(n,K)]_\Delta$ (see Example 14.2). From the definition of morphisms of varieties all ρ_{ij}'s belong to $\mathscr{O}_G(G)$ if ρ is a rational representation.

Now assume that $\{\rho_{ij}\} \subset \mathscr{O}_G(G)$. From Proposition 10.7 it is enough to prove that $\delta\circ\rho \in \mathscr{O}_G(G)$ where

$$\begin{aligned} \delta : GL(n,K) &\longrightarrow K \\ z &\longrightarrow \frac{1}{\det z} . \end{aligned}$$

Since $\{\rho_{ij}\} \subset \mathscr{O}_G(G)$, $\Delta\circ\rho \in \mathscr{O}_G(G)$. Since $\tau : G \longrightarrow G,\ g \longrightarrow g^{-1}$ is a morphism, $\Delta\circ\rho\circ\tau \in \mathscr{O}_G(G)$. Hence $\Delta\circ\rho\circ\tau = \delta\circ\rho \in \mathscr{O}_G(G)$, because $\Delta\circ\rho\circ\tau(g) = \Delta(\rho(g^{-1})) = (\det(\rho(g)))^{-1} = \delta(\rho(g))$ for any $g \in G$.

(ii) Let $\{m_1, m_2, \ldots, m_n\}$ be a K–basis of M such that $N \doteq Km_1 \oplus \ldots \oplus Km_s$ $(s<n)$. Since

$$\rho(g) = \left[\begin{array}{c|c} \rho_1(g) & * \\ \hline 0 & \rho_2(g) \end{array}\right]$$

where ρ_1 and ρ_2 are the matrix representations afforded by N and M/N respectively, ρ_1 and ρ_2 are rational from (i).

(iii) Clearly, $M_1 \dotplus M_2$ is rational from (i). Let $M_1 = Km_1 \oplus \ldots \oplus Km_r$ and $M_2 = Kn_1 \oplus \ldots \oplus Kn_s$ where $\dim_K M_1 = r$ and $\dim_K M_2 = s$. Then $\{m_i \otimes n_j \mid 1 \leq i \leq r\ ,\ 1 \leq j \leq s\}$ form a K–basis of $M_1 \otimes_K M_2$. Let ρ and ν be the matrix representations afforded by M_1 and M_2 respectively. Since

$$g(m_i \otimes n_j) = gm_i \otimes gn_j$$

$$= \left[\sum_{l=1}^{r} \rho_{li}(g)m_l\right] \otimes \left[\sum_{k=1}^{s} \nu_{kj}(g)n_k\right] = \sum_{l=1}^{r}\sum_{k=1}^{s} \rho_{li}(g)\,\nu_{kj}(g)(m_l \otimes n_k)$$

and $\rho_{li} \cdot \nu_{kj} \in \mathscr{O}_G(G)$, $M_1 \otimes_K M_2$ is a rational KG–module.

(iv) Exercise.

(v) Let $\{m_1, \ldots, m_n\}$ be a K–basis of M . Let $m = \sum_{i=1}^{n} \alpha_i^m m_i$ be an element of M where $\alpha_1^m, \ldots, \alpha_n^m \in K$ and $gm_i = \sum_{k=1}^{n} \rho_{ki}(g)m_k$ $(i = 1,2,\ldots,n)$, then

$$gm = \sum_{i=1}^{n} \alpha_i^m \left[\sum_{k=1}^{n} \rho_{ki}(g)m_k\right] = \sum_{k=1}^{n} \left[\sum_{i=1}^{n} \alpha_i^m \rho_{ki}(g)\right] m_k .$$

Since $X_k \circ \varphi \in \mathscr{O}_{G\times M}(G\times M)$ for any $k = 1,2,\ldots,n$, where $X_k(m) = \alpha_k^m$ and $X_k \circ \varphi(g,m) = \sum_{i=1}^{n} \alpha_i^m \rho_{ki}(g)$, $\begin{array}{rl}\varphi : G\times M & \longrightarrow M \\ (g\,,m) & \longrightarrow g\,m\end{array}$ is a morphism of varieties from Proposition 10.7. Q.E.D.

<u>Exercise 52</u>. Prove Proposition 18.2.iv.

<u>(18.3) Definition</u>. Let G be an algebraic group over K and M be a KG–module. We call M a <u>locally finite rational KG–module</u>, for brevity, a <u>locally finite KG–module</u> if KGm is a finite dimensional rational KG–module for any $m \in M$.

We shall show that $\mathscr{O}_G(G)$ becomes a locally finite KG–module by right translation (see Corollary 18.6) if G is affine. We first define <u>left and right translations</u>.

Let G be a group and let
$$M(G,K) = \{f \mid f:G \to K \text{ , a map of } G \text{ into } K\}$$
(see §1).

Let $x,y \in G$ and let
$$R_x : M(G,K) \longrightarrow M(G,K)$$
and
$$L_y : M(G,K) \longrightarrow M(G,K)$$
be maps of $M(G,K)$ into itself defined by
$$R_x(f)(z) = f(zx)$$
and $L_y(f)(z) = f(yz)$, where $f \in M(G,K)$ and $z \in G$.

Then we call R_x right translation by x and L_y left translation by y . It is easy to check the following properties of right and left translations:

For any $x,y \in G$ and any $f_1,f_2 \in M(G,K)$.
$$R_xL_y = L_yR_x,\ \ R_{xy} = R_xR_y \text{ and } L_{xy} = L_yL_x \ .$$
$$R_x(f_1f_2) = R_x(f_1)R_x(f_2)$$
and $L_y(f_1f_2) = L_y(f_1)L_y(f_2)$.

<u>(18.4) Lemma</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K . Then

(i) $R_x(\mathscr{O}_G(G)) \subset \mathscr{O}_G(G)$ and $L_y(\mathscr{O}_G(G)) \subset \mathscr{O}_G(G)$ for any $x,y \in G$;

(ii) $\mathscr{O}_G(G)$ is a left KG–module with the following operation
$$KG \times \mathscr{O}_G(G) \longrightarrow \mathscr{O}_G(G)$$
$$\Big(\sum_{g\in G} \alpha_g g\, ,\ f\Big) \longrightarrow \sum_{g\in G} \alpha_g\, g{*}f\ ,$$
where $g{*}f := R_g(f)$;

(iii) $\mathscr{O}_G(G)$ is also a right KG–module with the following operation
$$\mathscr{O}_G(G) \times KG \longrightarrow \mathscr{O}_G(G)$$
$$\Big(f,\ \sum_{g\in G} \alpha_g g\Big) \longrightarrow \sum_{g\in G} \alpha_g\, f\Delta g\ ,$$
where $f\Delta g := L_g(f)$;

(iv) finally, for any x and $y \in G$, the maps

$$R_x : \mathscr{A}_G(G) \longrightarrow \mathscr{A}_G(G), \quad f \longmapsto x * f$$

and

$$L_y : \mathscr{A}_G(G) \longrightarrow \mathscr{A}_G(G), \quad f \longmapsto f \Delta y$$

are K–algebra automorphisms.

Proof. (i) Let $\{U_i \mid i = 1,2,\ldots,m\}$ be an affine open covering of G. Let $m : G\times G \longrightarrow G,\ (x,y) \longmapsto xy$, then we have $f\circ m \in \mathscr{A}_{G\times G}(G\times G)$ for any $f \in \mathscr{A}_G(G)$. Assume that $(z,x) \in U_i\times U_j$ and

$$f\circ m|_{U_i\times U_j} = \sum_{k=1}^{l} f_k \otimes f'_k \in K[U_i] \otimes K[U_j] ,$$

then $R_x(f)(z) = f(zx) = f\circ m(z,x) = (\sum_{k=1}^{l} f'_k(x)\, f_k)(z)$ for any $z \in U_i$. Since

$$R_x(f)|_{U_i} \in K[U_i] \text{ for any } 1 \leq i \leq m ,$$

$R_x(f) \in \mathscr{A}_G(G)$ for any $f \in \mathscr{A}_G(G)$.

(ii), (iii) and (iv) Exercise. Q.E.D.

Exercise 53. Prove Lemma 18.4.ii, iii and iv.

Henceforth left or right KG–module $\mathscr{A}_G(G)$ always means the left or right module defined in Lemma 18.4.

(18.5) Proposition. Let $(G,K[G])$ be an affine algebraic group over K.
(i) Let M be a finie dimensional left KG–submodule of $K[G]$, then M is rational.
(ii) Let N be a finite dimensional right KG–submodule of $K[G]$, then N is rational.

Proof. Let $\{m_1,m_2,\ldots,m_n\}$ be a K–basis of M. Let $g \in G$ and

$$g*(m_1,m_2,\ldots,m_n) = (m_1,\ldots,m_n) \begin{bmatrix} \rho_{11}(g), \rho_{12}(g),\ldots, \rho_{1n}(g) \\ \rho_{21}(g), \rho_{22}(g),\ldots, \rho_{2n}(g) \\ \vdots \qquad\qquad \vdots \\ \rho_{n1}(g),\ldots\ldots, \rho_{nn}(g) \end{bmatrix}$$

It is enough to prove that each $\rho_{ij} \in K[G]$.

Now let $\{m_1,\dots,m_n\} \cup \{w_\alpha\}_{\alpha\in\mathscr{A}}$ be a K–basis of $K[G]$, then

$$m^*(m_i) = \sum_{j=1}^{n} m_j \otimes \bar{f}_{ji} + \sum_{\alpha\in\mathscr{A}} w_\alpha \otimes \bar{f}_{\alpha i}$$

for some $\bar{f}_{ji}, \bar{f}_{\alpha i} \in K[G]$ where $m : \begin{array}{l} G\times G \longrightarrow G \\ (x,y) \longrightarrow xy \end{array}$. Thus for any $g \in G$ and $x \in G$ we have

$$(g*m_i)(x) = (R_g(m_i))(x) = m_i(xg) = m^*(m_i)(x,g)$$

$$= \sum_{j=1}^{n} \bar{f}_{ji}(g)\, m_j(x) + \sum_{\alpha\in\mathscr{A}} \bar{f}_{\alpha i}(g)\, w_\alpha(x) .$$

Since $\{m_1,\dots,m_n\} \cup \{w_\alpha\}_{\alpha\in\mathscr{A}}$ is a K–basis of $K[G]$, $\rho_{ji} = \bar{f}_{ji} \in K[G]$ for any $1 \le i,j \le n$. Q.E.D.

<u>Remark to Proposition 18.5</u>. $m^*(m_i) = \sum_{j=1}^{n} m_j \otimes \rho_{ji}$ for any $1 \le i \le n$.

<u>(18.6) Corollary</u>. Let $(G,K[G])$ be an affine algebraic group over K . Then the left (respectively right) KG–module $K[G]$ is locally finite.

<u>Proof</u>. Let $f \in K[G]$. Since $m^*(f) \in K[G] \otimes K[G]$, we have

$$m^*(f) = \sum_{i=1}^{l} f_i \otimes f'_i \quad \text{for some } f_i, f'_i \in K[G] .$$

Hence
$$(g*f)(x) = R_g(f)(x) = f(xg) = m^*(f)(x,g)$$

$$= \left[\sum_{i=1}^{l} f'_i(g)\, f_i\right](x) \quad \text{for any } g,x \in G .$$

Thus
$$(KG)*f \subset \sum_{i=1}^{l} Kf_i$$

and $(KG)*f$ is finite dimensional. Q.E.D.

Now we show that an affine algebraic group is isomorphic to a closed subgroup of a general linear group.

<u>(18.7) Theorem</u>. Let $(G,K[G])$ be an affine algebraic group over K, then there exists a morphism of affine algebraic groups ρ of G into $GL(n,K)$, for some n, such that $\rho(G)$ is closed in $GL(n,K)$ and ρ is an isomorphism of affine varieties of G onto $\rho(G)$.

<u>Proof</u>. Let $\{a_1,a_2,\dots,a_m\}$ be a set of generators of $K[G]$ as K–algebra. Let M be a finite dimensional left KG–submodule of $K[G]$ containing $\{a_1,a_2,\dots,a_m\}$ (see Corollary 18.6). Let $\{m_1,m_2,\dots,m_n\}$ be a K–basis of M, then with respect to this basis we can define a rational representation $\rho : G \longrightarrow GL(n,K)$, $g \longrightarrow (\rho_{ij}(g))$ such that

$$g*(m_1,\dots,m_n) = (m_1,\dots,m_n)\begin{bmatrix} \rho_{11}(g), \rho_{12}(g), \dots, & \rho_{1n}(g) \\ \rho_{21}(g), \rho_{22}(g), \dots, & \rho_{2n}(g) \\ \vdots \quad\quad \vdots \quad \ddots & \vdots \\ \rho_{n1}(g), \dots\dots\dots\dots, & \rho_{nn}(g) \end{bmatrix}$$

for any $g \in G$. From Proposition 2.5 it is enough to show that ρ^* is surjective in order to prove $\rho : G \to \rho(G)$ is an isomorphism of affine varieties. Since

$$m^*(m_i) = \sum_{j=1}^{n} m_j \otimes \rho_{ji} \quad \text{for any} \quad 1 \le i \le n$$

from Remark to Proposition 18.5, $m^*(m_i)(1,g) = m_i(g) = \sum_{j=1}^{n} \rho_{ji}(g)\, m_j(1)$ for all $g \in G$. Since $\rho_{ij} \in \operatorname{Im} \rho^*$ for all $1 \le i,j \le n$, we have $m_i = \sum_{j=1}^{n} m_j(1)\, \rho_{ji} \in \operatorname{Im} \rho^*$ for any $1 \le i \le n$. Hence $\operatorname{Im} \rho^* \supset M \supset \{a_1,\dots,a_m\}$, which implies $\operatorname{Im} \rho^* \supset K[G]$. Thus we have proved that $\operatorname{Im} \rho^* = K[G]$. Q.E.D.

Because of this fact we also call affine algebraic groups <u>linear algebraic groups</u>.

Now we shall define induced modules of algebraic groups.

<u>(18.8) Definition</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and M be a vector space over K. We define Map (G,M) to be the K–space of all maps f of G into M such that $f(G)$ spans a finite dimensional K–subspace N of M (we write $N = K\langle f(G)\rangle$) and $f : G \to N$ is a morphism of varieties.

<u>Exercise 54</u>. Justify Definition 18.8, i.e., Map (G,M) is a K–subspace of $\{f \mid f \text{ is a map of } G \text{ into } M\}$. The K–space operations of $\{f \mid f \text{ is a map of } G \text{ into } M\}$ are:

$$(f_1+f_2)(g) = f_1(g) + f_2(g)$$
$$(cf)(g) = cf(g) ,$$
where f_1, f_2 and f are maps of G into M and $g \in G$ and $c \in K$.

Exercise 55. Let G and M be as in Definition 18.8. Let f be a map of G into M such that $f(G)$ is contained in some finite dimensional K–subspace L of M and $f : G \to L$ is a morphism of varieties. Then $f \in \mathrm{Map}(G,M)$.

(18.9) Proposition. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and M be a vector space over K . Then

(i) Map (G,M) is a left KG–module by the following operation
$$G \times \mathrm{Map}(G,M) \longrightarrow \mathrm{Map}(G,M) ,$$
$$(g,f) \longmapsto g*f$$
where $(g*f)(x) = f(xg)$ for any $x \in G$.

(ii) If $M = K$, then $\mathrm{Map}(G,K) = \mathscr{O}_G(G)$.

(iii) $\mathscr{O}_G(G) \otimes_K M$ is a left KG–module by the following operation
$$G \times (\mathscr{O}_G(G) \otimes_K M) \longrightarrow \mathscr{O}_G(G) \otimes_K M ,$$
$$(g, f\otimes m) \longmapsto (g*f) \otimes m$$
and the map
$$\rho : \mathscr{O}_G(G) \otimes_K M \longrightarrow \mathrm{Map}(G,M) ,$$
$$f \otimes m \longmapsto \rho(f\otimes m)$$
where $\rho(f\otimes m)(g) = f(g)m$ $(g \in G)$, is a KG–isomorphism.

(iv) Map (G,M) is a locally finite KG–module if $\mathscr{O}_G(G)$ is locally finite as left module.

Proof. (i) Let $f \in \mathrm{Map}(G,M)$ and $N = K\langle f(G)\rangle$. Since $K\langle (g*f)(G)\rangle = N$ and $g*f : G \longrightarrow G \xrightarrow{f} N$, $x \longmapsto xg$ is a morphism, we have $g*f \in \mathrm{Map}(G,M)$. It is clear that $1*f = f$ and $(g_1 g_2)*f = g_1*(g_2*f)$ for any $f \in \mathrm{Map}(G,M)$ and $g_1, g_2 \in G$.

(ii) is clear from the definition.

(iii) Since $\mathscr{O}_G(G)$ is a (KG,K)–bimodule, $\mathscr{O}_G(G) \otimes_K M$ becomes a KG–module by the given operation. Since the map
$$\psi : \mathscr{O}_G(G) \times M \longrightarrow \mathrm{Map}(G,M) ,$$
$$(f,m) \longmapsto \psi(f,m)$$
where $\psi(f,m)(g) = f(g)m$ $(g \in G)$ is K–bilinear, ρ is a well–defined K–linear map.

Since

$$\rho(g(f\otimes m)) = \rho((g*f)\otimes m) : x \longrightarrow f(xg)m$$

and $$g*\rho\,(f\otimes m) : x \longrightarrow \rho(f\otimes m)(xg) = f(xg)m\ ,$$

where $g,x \in G$ and $f\otimes m \in \mathscr{A}_G(G)\otimes_K M$, ρ is a KG–homomorphism.

Now let $\{f_i \mid i \in I\}$ and $\{m_j \mid j \in J\}$ be K–basis of $\mathscr{A}_G(G)$ and M respectively, then $\{f_i \otimes m_j \mid i \in I$ and $j \in J\}$ is a K–basis of $\mathscr{A}_G(G)\otimes_K M$. Assume that $\rho(\sum_{i,j} c_{ij}\, f_i \otimes m_j)(x) = 0$ for any $x \in G$, where $\{c_{ij}\} \subset K$ and almost all c_{ij}'s are zero. Since $\rho(\sum_{i,j} c_{ij}\, f_i \otimes m_j)(x) = \sum_{i,j} c_{ij}\, f_i(x)m_j = \sum_j (\sum_i c_{ij}\, f_i(x))m_j = 0$ for any $x \in G$, $\sum_i c_{ij}\, f_i(x) = 0$ for any $x \in G$ and $j \in J$.

Hence all c_{ij}'s are zero and ρ is injective. Let $h \in \mathrm{Map}\,(G,M)$ and $\{m_1,\dots,m_n\}$ be a K–basis of $N = K\langle h(G)\rangle$. Since

$$h : G \to N = Km_1 \oplus \dots \oplus Km_n$$

is a morphism of varieties, each $X_i \circ h \in \mathscr{A}_G(G)$ where $X_i \in K[N]$ $(i = 1,2,\dots,n)$. Since

$$\rho\left[\sum_{i=1}^{n} (X_i \circ h)\otimes m_i\right](x) = h(x)$$

for any $x \in G$, ρ is surjective.

(iv) Since $\mathscr{A}_G(G)\otimes_K M \cong \mathrm{Map}\,(G,M)$, it is enough to prove that $\mathscr{A}_G(G)\otimes_K M$ is locally finite. Let $\sum_{i=1}^{l} f_i \otimes m_i \in \mathscr{A}_G(G)\otimes_K M$. Since

$$KG\left[\sum_{i=1}^{l} f_i \otimes m_i\right] \subset \sum_{i=1}^{l} KG\, f_i \otimes m_i$$

and each $KG\, f_i \otimes m_i$ is finite dimensional and rational, $KG(\sum_{i=1}^{l} f_i \otimes m_i)$ is also finite dimensional and rational from Proposition 18.2. Q.E.D.

<u>(18.10) Definition</u>. Let $(G,\mathscr{A}_G)$ be an algebraic group over K and H be a closed subgroup of G . Let V be a left KH–module, then we define the <u>induced KG–module</u> V_H^G induced from V to be the KG–module

$$V_H^G = \{\, f \in \mathrm{Map}\,(G,V) \mid f(hg) = hf(g) \text{ for all } h \in H \text{ and } g \in G\}\ .$$

(18.11) Remark.

(i) Let G, H and V be as in Definition 18.10. Let $\{1\}$ be the trivial subgroup of G , then

$$V^G_{\{1\}} = \mathrm{Map}\,(G,V)\ .$$

In particular $$K^G_{\{1\}} = \mathscr{A}_G(G)\ ,$$

where K is considered as the one–dimensional trivial module of $\{1\}$.

If Map (G,V) is locally finite, e.g., G is affine, then V^G_H is locally finite.

(ii) A Mackey's Lemma (see Mackey [1]). Let G be a group and H be a subgroup of G . Let L be a left kH–module where k is a field. We write $\hat{L}$ for the set of all maps $f : G \to L$ such that $f(hg) = hf(g)$ for any $h \in H$ and $g \in G$. Then

a) $\hat{L}$ becomes a left kG–module by the following operation

$(f_1 + f_2)(g) = f_1(g) + f_2(g)$ $(f_1, f_2 \in \hat{L},\ g \in G)$,

$(cf)(g) = cf(g)$ $(f \in \hat{L},\ g \in G,\ c \in k)$,

$(g*f)(x) = f(xg)$ $(f \in \hat{L},\ g,x \in G)$.

b) Let $G = \bigcup_{j\in J} Hg_j$ (disjoint union) and $L = \bigoplus_{i\in I} kl_i$ (direct sum). Let f_{ij} be the map of G into L such that $f_{ij}(hg_k) = \delta_{jk}\, hl_i$, then $f_{ij} \in \hat{L}$ for any $(i,j) \in I\times J$.

c) Let $L^G = kG \otimes_{kH} L$, then L^G has a k–basis $\{g_j{}^{-1} \otimes l_i \mid (i,j) \in I\times J\}$.

d) The map $\iota : L^G \to \hat{L}$ which takes each $g_j{}^{-1} \otimes l_i$ to f_{ij} , where $(i,j) \in I\times J$, is an injective kG–homomorphism.

Proof of (ii). (a), (b) and (c) are clear.

(d) Since $\{g_j{}^{-1} \otimes l_i\}$ forms a k–basis of L^G and f_{ij}'s are linearly independent, ι is an injective k–linear map. We show that $\iota(g(g_j{}^{-1} \otimes l_i)) = g*\iota(g_j{}^{-1} \otimes l_i)$ for any $g \in G$ and $(i,j) \in I\times J$. Since $f(hg) = hf(g)$ for all $f \in \hat{L}$, where $h \in H$ and $g \in G$, it is enough to show that

$$\iota(g(g_j{}^{-1} \otimes l_i))(g_k) = g*\iota(g_j{}^{-1} \otimes l_i)(g_k) \quad \text{for any } k \in J\ .$$

Assume that $gg_j{}^{-1} = g_s{}^{-1} h$ for some $h \in H$ and $s \in J$, then

$$\iota(g(g_j{}^{-1} \otimes l_i))(g_k) = \delta_{sk}\, hl_i\ .$$

If $g_k g = h' g_t$ for some $h' \in H$ and $t \in J$, then

$$g*\iota(g_j{}^{-1} \otimes l_i)(g_k) = f_{ij}\,(g_k g) = \delta_{jt}\, h'l_i\ .$$

Assume $j = t$, i.e., $\delta_{jt}\, h'l_i = h'l_i$ $(g_k g = h' g_j)$. Since $g_j g^{-1} = h^{-1} g_s = h'^{-1} g_k$, $s = k$ and $h = h'$. Conversely if $s = k$, then $j = t$ and $h = h'$. Hence we have shown that ι is a kG–homomorphism. Q.E.D.

<u>(18.12) Proposition</u>. Let $(G, \mathscr{A}_G)$ be an algebraic group over K and H be a closed subgroup of G . Let V be a KH–module.

(i) Let $\begin{array}{rl} \epsilon_V : V_H^G & \longrightarrow V \\ f & \longrightarrow f(1) \end{array}$, then ϵ_V is a KH–homomorphism.

(ii) For any locally finite rational KG–module M and KH–homomorphism φ of M into V , there exists a unique KG–homomorphism $\tilde{\varphi} : M \to V_H^G$ which makes the following diagram commutative.

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\exists_1\ \tilde{\varphi}"'] & & V \\ & V_H^G \arrow[ur, "\epsilon_V"'] & \end{tikzcd}$$

<u>Proof</u>. (i) $\epsilon_V(h*f) = f(h) = hf(1) = h\epsilon_V(f)$ for any $h \in H$ and $f \in V_H^G$.

(ii) Let $m \in M$, we define $\tilde{\varphi}(m)$ to be the map of G into V such that

$$\begin{array}{rl} \tilde{\varphi}(m) : G & \longrightarrow V \\ g & \longrightarrow \varphi(gm) . \end{array}$$

Since KGm is finite dimensional and $\begin{array}{rl} G & \longrightarrow KGm \\ g & \longrightarrow gm \end{array}$ is a morphism of varieties from Remark 17.2.ii and Proposition 18.2.v

$$\begin{array}{rl} \tilde{\varphi}(m) : G & \longrightarrow KGm \xrightarrow{\varphi} \varphi(KGm) \\ g & \longrightarrow gm \end{array}$$

is a morphism of varieties. Hence $\tilde{\varphi}(m) \in \mathrm{Map}(G,V)$. Clearly $\tilde{\varphi}(m)(hg) = \varphi(hgm) = h\varphi(gm) = h\tilde{\varphi}(m)(g)$ for any $h \in H$ and $g \in G$. Therefore, $\tilde{\varphi}(m) \in V_H^G$. Since

$$x*(\tilde{\varphi}(m))(g) = \tilde{\varphi}(m)(gx) = \varphi(gxm) = \tilde{\varphi}(xm)(g)$$

for any $x,g \in G$ and $m \in M$, $\tilde{\varphi}$ is a KG–homomorphism. Finally, let $f : M \to V_H^G$ be a KG–homomorphism such that $\epsilon_V \circ f = \varphi$. Since $\{f(m)\}(1) = \varphi(m)$ for any $m \in M$ and $\{f(m)\}(g) = g*\{f(m)\}(1) = f(gm)(1) = \varphi(gm)$, we have $f = \tilde{\varphi}$. Q.E.D.

<u>Corollary to Proposition 18.12 (Frobenius Reciprocity)</u>. Let M be a locally finite rational KG–module, then

$$\begin{array}{ccc} \mathrm{Hom}_{KH}(M,V) & \cong & \mathrm{Hom}_{KG}(M,V_H^G) \\ \varphi & \longrightarrow & \tilde{\varphi} \end{array}$$

as K–spaces where φ and $\tilde{\varphi}$ are as in (ii).

<u>(18.13) Proposition</u> (<u>Transitivity of Induction</u>). Let $(G, \mathscr{A}_G)$ be an algebraic group over K and $H \subset L$ be closed subgroups of G. Let V be a KH–module. Assume that the induced module V_H^G is locally finite, then

$$(V_H^L)_L^G \cong V_H^G .$$

<u>Proof.</u> Let $\begin{array}{rcl} \epsilon_1 : V_H^G & \longrightarrow & V \\ f & \longrightarrow & f(1) \end{array}$, $\begin{array}{rcl} \epsilon_2 : V_H^L & \longrightarrow & V \\ f & \longrightarrow & f(1) \end{array}$ and $\begin{array}{rcl} \epsilon_3 : (V_H^L)_L^G & \longrightarrow & V_H^L \\ f & \longrightarrow & f(1) \end{array}$. Then there exists a KL–homomorphism $\psi : V_H^G \to V_H^L$ and a KG–homomorphism $\tilde{\psi} : V_H^G \to (V_H^L)_L^G$ which make the following diagram commutative from Proposition 18.12.

$$\begin{tikzcd}
 & V & \\
V_H^G \arrow[ur, "\epsilon_1"] \arrow[rr, "\exists\psi"] \arrow[dr, "\exists\tilde{\psi}"'] & & V_H^L \arrow[ul, "\epsilon_2"'] \\
 & (V_H^L)_L^G \arrow[ur, "\epsilon_3"'] &
\end{tikzcd}$$

Notice that $\begin{array}{rcl} (\tilde{\psi}(f))(g) : L & \longrightarrow & V \\ l & \longrightarrow & f(lg) \end{array}$ for any $f \in V_H^G$ and $g \in G$. We write $\tilde{\psi}(f) = \tilde{f}$. Now let $\mu \in (V_H^L)_L^G$. We define $\bar{\mu}$ to be the map of G into V such that

$$\bar{\mu}(g) = (\mu(g))(1) \ \text{ for any } \ g \in G .$$

Since

$$\begin{array}{rcccl} \bar{\mu} : G & \longrightarrow & V_H^L & \xrightarrow{\epsilon_2} & V \\ g & \longrightarrow & \mu(g) & \longrightarrow & (\mu(g))(1) \end{array}$$

and $\bar{\mu}(hg) = (\mu(hg))(1) = (h*\mu(g))(1) = (\mu(g))(h) = h(\mu(g)(1)) = h(\bar{\mu}(g))$ for any $h \in H$ and $g \in G$, we have $\bar{\mu} \in V_H^G$. Since

$$\begin{array}{rcl} \tilde{\bar{\mu}}(g) : L & \longrightarrow & V \\ l & \longrightarrow & \bar{\mu}(lg) \end{array}$$

and $\bar{\mu}(lg) = (\mu(lg))(1) = (l*(\mu(g)))(1) = (\mu(g))(l)$ for any $l \in L$ and $g \in G$, we have $\tilde{\bar{\mu}} = \mu$. Hence $\tilde{\psi}$ is surjective. Injectivity of $\tilde{\psi}$ is clear. Q.E.D.

<u>(18.14) Lemma</u>. Let $(G, \mathscr{A}_G)$ be an algebraic group over K and M and N be locally finite rational KG–modules, then $M \otimes_K N$ becomes a locally finite rational KG–module under the following operation.

$$\begin{array}{rcl} G \times M \otimes_K N & \longrightarrow & M \otimes_K N \\ (g, \sum_i m_i \otimes n_i) & \longrightarrow & \sum_i gm_i \otimes gn_i . \end{array}$$

Proof. Clearly, $M \otimes_K N$ is a KG–module by the above operation. Let $\sum_i m_i \otimes n_i \in M \otimes_K N$. Since

$$KG \sum_i m_i \otimes n_i \subset \sum_i KG(m_i \otimes n_i)$$

and $KG(m_i \otimes n_i)$ is finite dimensional and rational for any i, $KG \sum_i m_i \otimes n_i$ is also finite dimensional and rational (see Proposition 18.2) Q.E.D.

(18.15) Proposition (Tensor Identity). Let $(G, \mathscr{O}_G)$ be an algebraic group over K and H be a closed subgroup of G. Let V be a KH–module and W be a locally finite rational KG–module. Then

$$\begin{aligned} V_H^G \otimes_K W &\overset{\rho}{\cong} (V \otimes_K W)_H^G \\ f \otimes w &\longrightarrow [\rho\ (f \otimes w) : g \longrightarrow f(g) \otimes gw] \end{aligned}$$

as KG–modules. KG or KH–module structure of $V_H^G \otimes_K W$ or $V \otimes_K W$ is as defined in Proposition 18.2.iii.

Proof (see Cline, Parshall & Scott [1]). We first show that the map

$$\begin{aligned} \Phi : \mathrm{Map}\,(G,V) \otimes_K W &\longrightarrow \mathrm{Map}\,(G, V \otimes_K W) \\ f \otimes w &\longrightarrow [\,\Phi\ (f \otimes w)\ :\ g \rightarrow f(g) \otimes gw] \end{aligned}$$

is a KG–isomorphism. Let $\{v_i \mid i \in I\}$ be a K–basis of V. Since $V = \bigoplus_{i \in I} Kv_i$ (direct sum) and $\mathrm{Map}(G,Kv_i) \subset \mathrm{Map}(G,V)$,

$$\mathrm{Map}(G,V) = \mathrm{Map}(G, \bigoplus_{i \in I} Kv_i) = \bigoplus_{i \in I} \mathrm{Map}(G,Kv_i)\ .$$

Hence we have

$$\mathrm{Map}(G,V) \otimes_K W = (\bigoplus_{i \in I} \mathrm{Map}(G,Kv_i)) \otimes_K W = \bigoplus_{i \in I} (\pi_i \otimes 1_W)(\mathrm{Map}(G,V) \otimes_K W)\ ,$$

where $\pi_i : \mathrm{Map}(G,V) \rightarrow \mathrm{Map}(G,Kv_i)$ is the projection. Notice that

$$\begin{aligned} \mathrm{Map}(G,Kv_i) \otimes'_K W &\cong (\pi_i \otimes 1_W)(\mathrm{Map}(G,V) \otimes_K W) \\ &= \mathrm{Map}(G,Kv_i) \otimes_K W \\ f \otimes' w &\longrightarrow f \otimes w \end{aligned}$$

as K–spaces.

Similarly since $V \otimes_K W = \bigoplus_{i \in I} Kv_i \otimes_K W$, we have

$$\mathrm{Map}(G,V \otimes_K W) = \bigoplus_{i \in I} \mathrm{Map}(G,Kv_i \otimes_K W)\ .$$

Thus it is reasonable to check that

$$\Phi|_{\mathrm{Map}(G,Kv_i)\otimes_K W} : \mathrm{Map}(G,Kv_i)\otimes_K W \longrightarrow \mathrm{Map}(G, Kv_i\otimes W)$$
$$f\otimes w \longrightarrow [\,\Phi(f\otimes w) : g \to f(g)\otimes gw]$$

is a K–isomorphism. Let

$$\varphi : \mathrm{Map}(G,K)\otimes_K W \longrightarrow \mathrm{Map}(G, W)$$
$$f\otimes w \longrightarrow [\,\varphi(f\otimes w) : g \to f(g)gw]\ .$$

Since $\mathrm{Map}(G,K) = \mathscr{O}_G(G)$ and

$$\varphi_1 : \mathscr{O}_G(G)\otimes_K W \longrightarrow \mathrm{Map}(G, W)$$
$$f\otimes w \longrightarrow [\,\varphi_1(f\otimes w) : g \to f(g)w]$$

is a KG–isomorphism (see Proposition 18.9.iii) and the map

$$\varphi_2 : \mathrm{Map}(G,W) \longrightarrow \mathrm{Map}(G,W)$$
$$\tau \longrightarrow [\,\varphi_2(\tau) : g \to g\tau(g)]$$

is a K–isomorphism with inverse $\varphi_2^{-1} : \tau \longrightarrow [g \to g^{-1}\tau(g)]$, $\varphi = \varphi_2\circ\varphi_1$ is a K– isomorphism. Hence Φ is a K–isomorphism. Let $f\otimes w \in \mathrm{Map}(G,V)\otimes_K W$, then

$$\Phi(g(f\otimes w))(x) = \Phi(g*f\otimes gw)(x) = (g*f)(x)\otimes xgw = f(xg)\otimes xgw = (g*\Phi(f\otimes w))(x)$$

for any $g, x \in G$. Thus Φ is a KG–isomorphism.

Next we show that $\Phi(V_H^G\otimes_K W) = (V\otimes_K W)_H^G$. Assume that $f\otimes w \in V_H^G\otimes_K W$, then

$$\Phi(f\otimes w)(hg) = f(hg)\otimes hgw = hf(g)\otimes hgw = h\Phi(f\otimes w)(g)$$

for any $h \in H$ and $g \in G$, which implies $\Phi(V_H^G\otimes_K W) \subset (V\otimes_K W)_H^G$.

Let $\tau \in (V\otimes_K W)_H^G$ and $\{w_j \mid j \in J\}$ be a K–basis of W , then we have

$$\tau(g) = \sum_{j\in J} \tau_j(g)\otimes gw_j\ ,$$

where $\tau_j(g) \in V$ $(j \in J)$. Suppose that there exists

$$\sum_{j\in J} f_j\otimes w_j \in \mathrm{Map}\,(G,V)\otimes_K W \ \text{ such that } \ \Phi\,(\sum_{j\in J} f_j\otimes w_j) = \tau\ ,$$

then
$$\tau(g) = \sum_{j\in J} \tau_j(g)\otimes gw_j = \sum_{j\in J} f_j(g)\otimes gw_j\ .$$

Therefore $f_j(g) = \tau_j(g)$ for any $j \in J$ and $g \in G$.
Thus it is enough to prove that each $f_j \in V_H^G$. Since

$$\tau(hg) = \sum_{j\in J} \tau_j(hg)\otimes hgw_j = h\tau(g) = \sum_{j\in J} h\tau_j(g)\otimes hgw_j\ ,$$

we have $f_j(hg) = hf_j(g)$ for any $h \in H$ and $g \in G$. Hence $f_j \in V_H^G$ and

$$\rho = \Phi\big|_{V_H^G\otimes_K W}$$

is a KG–isomorphism. Q.E.D.

CHAPTER IV

COALGEBRAS AND LIE ALGEBRAS OF LINEAR ALGEBRAIC GROUPS

In this chapter we shall introduce the idea of Lie algebra $\mathfrak{g}$ of a given linear algebraic group G relating it to the coalgebra of G and define the adjoint representation of G into $GL(\mathfrak{g})$. As references to coalgebras and the related topics and Lie algebras we give:

J. A. Green, Locally finite representations. Journal of Algebra, Vol.41, 137–171 (1976) and Humphreys [1].

19. Coalgebras and Lie algebras of linear algebraic groups

In this section k denotes an arbitrary field.

(19.1) Definition. A triple (R,μ,e) , where R is a vector space over k , and $\mu:R \to R \otimes_k R$, $e:R \to k$ are k–linear maps, is called a k–coalgebra if μ and e satisfy the following two conditions.

(i) $(1\otimes\mu)\circ\mu = (\mu\otimes 1)\circ\mu$ and

(ii) $(e\otimes 1)\circ\mu = 1 = (1\otimes e)\circ\mu$, where 1 is the identity map on R .

$$
\begin{tikzcd}
R \arrow[r, "\mu"] \arrow[d, "\mu"'] & R\otimes_k R \arrow[d, "1\otimes\mu"] \\
R\otimes_k R \arrow[r, "\mu\otimes 1"'] & R\otimes_k R\otimes_k R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\mu"] \arrow[d, "\mu"'] \arrow[dr, "1"] & R\otimes_k R \arrow[d, "e\otimes 1"] \\
R\otimes_k R \arrow[r, "1\otimes e"'] & R
\end{tikzcd}
$$

(commutative diagrams)

(19.2) Example. Let $(G,K[G])$ be a linear algebraic group over K . Put $R = K[G]$, $\mu = m^*$ (the comorphism of the multiplication $m:G\times G\to G$) and $e = p^*$ (the comorphism of $p:G \to G$, $x \to 1$), then the triple (R,μ,e) is a K–coalgebra, because

$$
\begin{tikzcd}
G & G\times G \arrow[l, "m"'] & \ni & (x,yz) \\
G\times G \arrow[u, "m"] & G\times G\times G \arrow[l, "m\times 1"'] \arrow[u, "1\times m"'] & \ni & (x,y,z) \arrow[u] \\
(xy,z) & (x,y,z) \arrow[l, mapsto] & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
G & G\times G \arrow[l, "m"'] & \ni (p(x),x) \\
G\times G \arrow[u, "m"] \arrow[ur, phantom] & G \arrow[l, "(1,p)"'] \arrow[u, "(p,1)"'] & \ni \; x \arrow[u] \\
(x,p(x)) & x \arrow[l, mapsto] &
\end{tikzcd}
$$

are commutative.

Exercise 56. Verify the above example, Example 19.2.

(19.3) Proposition. Let (R,μ,e) be a k–coalgebra, then the dual space $A = \mathrm{Hom}_k(R,k)$ of R becomes a k–algebra with the product $\alpha\beta$ of elements $\alpha,\beta \in A$ defined by

$$\alpha\beta = (\alpha\otimes\beta) \circ \mu : R \xrightarrow{\mu} R\otimes_k R \xrightarrow{\alpha\otimes\beta} k .$$

This multiplication is associative with the unity element $e \in A$.

'Exercise 57. Prove Proposition 19.3.

(19.4) Definition. A vector space $\mathfrak{g}$ over a field k , with a bilinear bracket product

$$[\ ,\ ] : \begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ (x,y) & \longrightarrow & [x,y] \end{array}$$

is said to be a Lie algebra over k if the following axioms are satisfied.

(i) $[x,x] = 0$ for all $x \in \mathfrak{g}$.

(ii) $[x, [y,z]] + [y, [z,x]] + [z, [x,y]] = 0$ (Jacobi identity) for all $x,y,z \in \mathfrak{g}$.

From (i) we have

(i)' $[x,y] = -[y,x]$ for all $x,y \in \mathfrak{g}$. When the characteristic of k is not 2, (i) and (i)' are equivalent.

We review the following definitions:

(i) A k–linear map φ of a Lie algebra $\mathfrak{g}$ over k into another Lie algebra $\mathfrak{g}'$ over k is called a homomorphism if

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \text{ for all } x,y \in \mathfrak{g}.$$

(ii) A k–subspace $\mathcal{J}$ of a Lie algebra $\mathfrak{g}$ over k is called an ideal of $\mathfrak{g}$ if $[x,y] \in \mathcal{J}$ for any $x \in \mathfrak{g}$ and $y \in \mathcal{J}$.

(iii) A k–subspace $\mathscr{C}$ of a Lie algebra $\mathfrak{g}$ over k is called a Lie subalgebra of $\mathfrak{g}$ if

$$[x,y] \in \mathscr{C} \text{ whenever } x,y \in \mathscr{C}.$$

(19.5) Example. Let A be an associative algebra over k . Define a bracket product as follows

$$[\ ,\ ] : \begin{array}{ccc} A\times A & \longrightarrow & A \\ (a,b) & \rightarrow & ab - ba \end{array} ,$$

then with this bracket product A becomes a Lie algebra over k .

(19.6) Definition. Let $\mathfrak{a}$ be a vector space over k with a bilinear product

$$\begin{array}{ccc} \mathfrak{a}\times\mathfrak{a} & \longrightarrow & \mathfrak{a} \\ (x,y) & \longrightarrow & xy \end{array}$$

(e.g. $\mathfrak{a}$ is an associative algebra or a Lie algebra over k). A k–linear map D of $\mathfrak{a}$ into itself is said to be a derivation of $\mathfrak{a}$ if it satisfies

$$D(xy) = xD(y) + D(x)y \text{ for all } x,y \in \mathfrak{a}.$$

<u>(19.7) Lemma</u>. (i) Let $\mathscr{a}$ be a vector space over k with a bilinear product $\begin{array}{l}\mathscr{a} \times \mathscr{a} \to \mathscr{a} \\ (x,y) \to xy\end{array}$ and let $\mathrm{End}_k(\mathscr{a}) = \{f \mid f$ is a k–linear endomorphism of $\mathscr{a}$ into itself$\}$.

Then the set of derivations,

$$\mathrm{Der}_k(\mathscr{a}) = \{D \in \mathrm{End}_k(\mathscr{a}) \mid D(xy) = xD(y) + D(x)y \text{ for all } x,y \in \mathscr{a}\},$$

forms a Lie algebra over k with the bracket product

$$[\ ,\ ] : \begin{array}{ccc}\mathrm{Der}_k(\mathscr{a}) \times \mathrm{Der}_k(\mathscr{a}) & \longrightarrow & \mathrm{Der}_k(\mathscr{a}) \\ (D_1, D_2) & \longrightarrow & D_1 D_2 - D_2 D_1\end{array},$$

i.e., $\mathrm{Der}_k(\mathscr{a})$ is a Lie subalgebra of $\mathrm{End}_k(\mathscr{a})$ (see Example 19.5).

(ii) Let $\mathscr{g}$ be a Lie algebra over k. Let ad x be a map of $\mathscr{g}$ into itself such that ad x takes each $y \in \mathscr{g}$ to $[x,y] \in \mathscr{g}$, where $x \in \mathscr{g}$. Then ad x is a derivation of $\mathscr{g}$ for each $x \in \mathscr{g}$, and the map

$$\mathrm{ad} : \begin{array}{ccc}\mathscr{g} & \longrightarrow & \mathrm{Der}_k(\mathscr{g}) \\ x & \longrightarrow & \mathrm{ad}\, x\end{array}$$

is a Lie algebra homomorphism.

<u>Exercise 58</u>. Prove the above lemma.

Now we define the Lie algebra of a given linear algebraic group.

<u>(19.8) Definition</u>. Let $(G, K[G])$ be a linear algebraic group over K. Let

$$\mathscr{L}(G) = \{D \in \mathrm{Der}_K(K[G]) \mid D \circ L_g = L_g \circ D \text{ for any } g \in G\},$$

then $\mathscr{L}(G)$ is a Lie subalgebra of $\mathrm{Der}_K(K[G])$ and we call $\mathscr{L}(G)$ the <u>Lie algebra of G</u>.

Now let $X \in \mathrm{Hom}_K(K[G], K)$, where $(G, K[G])$ is a linear algebraic group over K as usual, then we define the <u>right convolution</u> $*X$ by X as a K–linear map of $K[G]$ into itself such that

<u>(19.9)</u> $$(f * X)(g) = X(L_g(f)) \text{ for any } f \in K[G] \text{ and } g \in G.$$

One can easily check that (19.9) is well–defined, because for any $f \in K[G]$ we have

$$f * X = \sum_{i=1}^{1} X(f'_i)\, f_i,$$

where $$m^*(f) = \sum_{i=1}^{1} f_i \otimes f'_i \in K[G] \otimes K[G].$$

<u>(19.10) Lemma</u>. Let $(G, K[G])$ be a linear algebraic group over K. Let $\gamma \in T(G)_1$ $(\subset \mathrm{Hom}_K(K[G], K))$, then

$$*\gamma \in \mathscr{L}(G) .$$

<u>Proof</u>. Since $\{(f_1 f_2) * \gamma\}(g) = \gamma(L_g(f_1 f_2)) = \gamma(L_g(f_1) L_g(f_2)) = L_g(f_1)(1)\cdot\gamma(L_g(f_2)) + \gamma(L_g(f_1))\cdot L_g(f_2)(1)$ for any $f_1, f_2 \in K[G]$ and $g \in G$, where $\gamma \in T(G)_1$, we have

$$(f_1 f_2) * \gamma = f_1 (f_2 * \gamma) + (f_1 * \gamma) f_2 ,$$

which implies $*\gamma \in \mathrm{Der}_K(K[G])$ for any $\gamma \in T(G)_1$.

Now let $g,h \in G$, $\gamma \in T(G)_1$ and $f \in K[G]$, then

$$\{L_h \circ (*\gamma)\}(f)(g) = \{L_h (f * \gamma)\}(g) = (f * \gamma)(hg) = \gamma(L_{hg}(f))$$

and

$$\{(*\gamma) \circ L_h\}(f)(g) = \{(*\gamma)(L_h(f))\}(g) = (L_h(f) * \gamma)(g) = \gamma(L_g(L_h(f))) = \gamma(L_{hg}(f)) .$$

Thus we have $*\gamma \in \mathscr{L}(G)$ for all $\gamma \in T(G)_1$. Q.E.D.

The next theorem gives us a reason why $\mathscr{g} = T(G)_1$ can be considered as the Lie algebra of G.

<u>(19.11) Theorem</u>. Let $(G, K[G])$ be a linear algebraic group over K. Put $\mathscr{g} = T(G)_1$. Let θ be a map of $\mathrm{End}_K(K[G])$ into $\mathrm{Hom}_K(K[G], K)$ which takes each

$$D \in \mathrm{End}_K(K[G]) \text{ to } \gamma_D \in \mathrm{Hom}_K(K[G], K)$$

such that

$$\gamma_D(f) = \{D(f)\}(1) \text{ for all } f \in K[G] .$$

Then θ is a well–defined K–linear map, and

(1) $\theta(*X) = X$ for any $X \in \mathrm{Hom}_K(K[G], K)$;

(2) $\theta((*X)(*Y)) = XY$ for any $X,Y \in \mathrm{Hom}_K(K[G], K)$, where

$$XY = (X \otimes Y) \circ m^*$$

(see Example 19.2 and Proposition 19.3);

(3) let η be a K–linear map of $\mathrm{Hom}_K(K[G], K)$ to $\mathrm{End}_K(K[G])$ such that

$$\eta : \mathrm{Hom}_K(K[G], K) \longrightarrow \mathrm{End}_K(K[G]), \quad X \longmapsto *X$$

then $\eta(\mathscr{g}) \subset \mathscr{L}(G)$, $\theta(\mathscr{L}(G)) \subset \mathscr{g}$ and

$$(\theta \mid_{\mathscr{L}(G)}) \circ (\eta \mid_{\mathscr{g}}) = 1_{\mathscr{g}} \text{ and}$$

$$(\eta \mid_{\mathscr{g}}) \circ (\Theta \mid_{\mathscr{L}(G)}) = 1_{\mathscr{L}(G)} ;$$

(4) from (2) and (3) we have

$$\mathscr{L}(G) = \{*\gamma \mid \gamma \in \mathscr{g}\} \overset{\theta}{\cong} \mathscr{g} \quad \text{and}$$

$$\Theta([*\gamma_1, *\gamma_2]) = \gamma_1 \gamma_2 - \gamma_2 \gamma_1 \quad \text{for any} \quad \gamma_1, \gamma_2 \in \mathscr{g};$$

(5) from (4) $\mathscr{g}$ is a Lie subalgebra of $\mathrm{Hom}_K (K[G], K)$, the bracket product of which is

$$[\ ,\ ] : \mathrm{Hom}_K (K[G], K) \times \mathrm{Hom}_K (K[G], K) \longrightarrow \mathrm{Hom}_K (K[G], K)$$
$$(\gamma_1, \gamma_2) \longrightarrow \gamma_1 \gamma_2 - \gamma_2 \gamma_1$$

(see Example 19.2 and Proposition 19.3), and $\Theta : \mathscr{L}(G) \cong \mathscr{g}$ is a Lie algebra isomorphism, i.e., a bijective Lie algebra homomorphism.

<u>Proof</u>. (1) Let $X \in \mathrm{Hom}_K (K[G], K)$, then

$$\{\Theta(*X)\} (f) = (f*X) (1) = X(f) \quad \text{for any} \quad f \in K[G] .$$

Thus $\Theta(*X) = X$ for any $X \in \mathrm{Hom}_K (K[G], K)$.

(2) Let $X,Y \in \mathrm{Hom}_K (K[G], K)$, then

$$\{\Theta((*X)(*Y))\} (f) = \{(*X)(*Y)\} (f) (1)$$

$$= (*X) \left[\sum_{i=1}^{l} Y(f'_i) f_i \right] (1) = \left[\sum_{i=1}^{l} Y(f'_i) f_i*X \right] (1) = \sum_{i=1}^{l} Y(f'_i) X(f_i) ,$$

for any $f \in K[G]$, where

$$m^*(f) = \sum_{i=1}^{l} f_i \otimes f'_i \in K[G] \otimes K[G] .$$

Since

$$(XY)(f) = \{(X \otimes Y) \circ m^*\} (f) = (X \otimes Y) \left[\sum_{i=1}^{l} f_i \otimes f'_i \right] = \sum_{i=1}^{l} X(f_i) Y(f'_i) ,$$

we have proved (2).

(3) It is clear that η is a well–defined K–linear map and also $\eta(\mathscr{g}) \subset \mathscr{L}(G)$ (see Lemma 19.10). Now let $f_1, f_2 \in K[G]$, then

$$\begin{aligned} \gamma_D (f_1 f_2) &= \{D (f_1 f_2)\} (1) = \{f_1 D (f_2) + D (f_1) f_2\} (1) \\ &= f_1 (1) \cdot D (f_2) (1) + D(f_1) (1) \cdot f_2 (1) \\ &= f_1 (1) \gamma_D (f_2) + \gamma_D (f_1) f_2 (1) \end{aligned}$$

for any $D \in \mathscr{L}(G)$, which implies

$$\Theta(\mathscr{L}(G)) \subset \mathscr{g}.$$

From (1) we have

$$(\Theta \mid_{\mathscr{L}(G)}) \circ (\eta \mid_{\mathscr{g}}) = 1_{\mathscr{g}}.$$

Conversely, let $D \in \mathscr{L}(G)$, then

$$\{(\eta\circ\Theta)\,(D)\}\,(f)\,(g) = \{\eta\,(\gamma_D)\}\,(f)\,(g) = (f*\gamma_D)\,(g) = \gamma_D\,(L_g\,(f))$$
$$= D\,(L_g\,(f))\,(1) = L_g\,(D\,(f))\,(1) = D\,(f)\,(g)$$

for all $f \in K[G]$ and $g \in G$. Hence

$$(\eta\circ\Theta)\,(D) = D\ ,\ \text{i.e.,}\ \ (\eta \mid_{\mathscr{g}}) \circ (\Theta \mid_{\mathscr{L}(G)}) = 1_{\mathscr{L}(G)}\ .$$

(4) and (5) – straightforward. Q.E.D.

<u>(19.12) Corollary</u>. Let G, $\mathscr{g}$ and $\mathscr{L}(G)$ be as in the theorem, then

$$\dim_K \mathscr{L}(G) = \dim_K \mathscr{g} = \dim G$$

(see Exercise 50 on p.171).

<u>(19.13) Example</u>. The Lie algebra of $(G, K[G]) = (GL\,(n,K), K[f_{ij}, \delta \mid 1 \le i,j \le n])$.

We follow the same notation as in Example 14.2. We write $\mathscr{g}$ for $T\,(GL\,(n,K))_1$. Let φ be a map of $\mathscr{g}$ into $\mathscr{gl}(n,K)$, the set of all $n\times n$ matrices over K , such that

$$\varphi\colon \mathscr{g} \longrightarrow \mathscr{gl}(n,K)\ ,\quad \gamma \longmapsto (\gamma\,(f_{ij}))$$

then φ is a well–defined Lie algebra isomorphism, where $\mathscr{gl}(n,K)$ is considered as a Lie algebra with the bracket product as in Example 19.5.

<u>Proof</u>. It is easy to check that φ is a well–defined K–linear map. Since $\gamma(\Delta\delta) = \gamma(1) = 0 = \Delta(1)\,\gamma(\delta) + \gamma(\Delta)\,\delta(1)$, $\gamma(\delta)$ is determined by $\{f_{ij} \mid 1 \le i,j \le n\}$ for any $\gamma \in \mathscr{g}$. Thus φ is injective. Since $\dim_K \mathscr{g} = n^2$, φ is bijective.

Now let $\gamma, \gamma' \in \mathscr{g}$, then

$$\begin{aligned}\gamma\,\gamma'(f_{ij}) &= \{(\gamma\otimes\gamma') \circ m^*\}\,(f_{ij})\\ &= (\gamma\otimes\gamma')\left[\sum_{k=1}^{n} f_{ik} \otimes f_{kj}\right]\end{aligned}$$

$$= \sum_{k=1}^{n} \gamma(f_{ik})\ \gamma'(f_{kj}) .$$

Hence $\varphi([\gamma, \gamma'])$

$$\begin{aligned} &= \varphi(\gamma\gamma' - \gamma'\gamma) = ((\gamma\gamma' - \gamma'\gamma)\ (f_{ij})) \\ &= (\gamma\gamma'\ (f_{ij})) - (\gamma'\gamma\ (f_{ij})) \\ &= \Big[\sum_{k=1}^{n} \gamma(f_{ik})\ \gamma'(f_{kj}) \Big] - \Big[\sum_{k=1}^{n} \gamma'(f_{ik})\ \gamma(f_{kj}) \Big] \\ &= \varphi(\gamma)\ \varphi(\gamma') - \varphi(\gamma')\ \varphi(\gamma) \\ &= [\varphi(\gamma), \varphi(\gamma')] \end{aligned}$$

for any $\gamma, \gamma' \in \mathscr{g}$. Q.E.D.

The next proposition with Theorem 18.7 will show us that the Lie algebra of a given linear algebraic group is essentially a Lie subalgebra of $\mathscr{gl}(n,K)$ (see Example 19.13).

<u>(19.14) Proposition</u>. Let $\varphi:G \to H$ be a morphism of linear algebraic groups of G into H over K. Let $\mathscr{g} = T(G)_1$ and $\mathscr{H} = T(H)_1$ be Lie algebras of G and H respectively. Then

(i) $\widehat{d\varphi} : \mathrm{Hom}_K (K[G], K) \longrightarrow \mathrm{Hom}_K (K[H], K)$, $X \longmapsto X \circ \varphi^*$ is a K–algebra map, and

(ii) $d\varphi : \mathscr{g} \longrightarrow \mathscr{H}$, $\gamma \longmapsto \gamma \circ \varphi^*$ is a homomorphism of Lie algebras, where φ^* is the comorphism of φ.

<u>Proof</u>. (i) It is clear that $\widehat{d\varphi}$ is a well–defined K–linear map. Let e be the comorphism of $p:G \longrightarrow G$, $x \longmapsto 1$, then we have

$$e \circ \varphi^* : K[H] \xrightarrow{\varphi^*} K[G] \xrightarrow{e} K \cdot 1 \ (\subset K[G])\ .$$
$$f \longmapsto f \circ \varphi \longmapsto (f \circ \varphi)(1) \cdot 1$$

Hence $e \circ \varphi^*$ is the comorphism of p': $H \longrightarrow H$, $x \longmapsto 1$, because $\varphi(1) = 1$. Thus $\widehat{d\varphi}\,(e) = (p')^*$.

Now let $X, Y \in \mathrm{Hom}_K (K[G], K)$, then

$$\widehat{d\varphi}\,(XY) = (XY) \circ \varphi^* = (X \otimes Y) \circ m^* \circ \varphi^* ,$$

where $m:G \times G \to G$ is the multiplication of G. On the other hand

$$
\begin{aligned}
\widehat{d\varphi}\,(X)\,\widehat{d\varphi}\,(Y) &= (X\circ\varphi^*)\,(Y\circ\varphi^*) \\
&= \{(X\circ\varphi^*)\otimes(Y\circ\varphi^*)\}\circ m'^* \\
&= (X\otimes Y)\circ(\varphi^*\otimes\varphi^*)\circ m'^* ,
\end{aligned}
$$

where $m':H\times H \to H$ is the multiplication of H. Since the following diagram

$$
\begin{array}{ccccc}
(x,\ y) & \in & G\times G & \xrightarrow{m} & G \\
\downarrow & & \downarrow{\scriptstyle \varphi\times\varphi} & & \downarrow{\scriptstyle \varphi} \\
(\varphi(x),\ \varphi(y)) & \in & H\times H & \xrightarrow[m']{} & H
\end{array}
$$

is commutative, we have $m^*\circ\varphi^* = (\varphi^*\otimes\varphi^*)\circ m'^*$, which implies

$$\widehat{d\varphi}\,(XY) = \widehat{d\varphi}\,(X)\,\widehat{d\varphi}\,(Y)$$

for any $X, Y \in \mathrm{Hom}_K\,(K[G], K)$.

(ii) is clear from (i). Q.E.D.

<u>(19.15) Proposition</u>. Let G and H be linear algebraic groups over K. Let $\varphi:G \to H$ be a morphism of linear algebraic groups such that φ is an isomorphism of affine varieties of G onto $\varphi(G)$ (see Theorem 15.4). Then

(i) $d\varphi : \mathscr{g} \to \mathscr{H}$ is injective, where $\mathscr{g}$ and $\mathscr{H}$ are Lie algebras of G and H respectively.

(ii) $d\varphi(\mathscr{g}) = \{\gamma \in T(H)_1 \mid \gamma\,(\mathscr{I}(\varphi(G)) = 0\}$.

<u>Proof</u>. (i) Since φ is an isomorphism of affine varieties of G onto $\varphi(G)$,

$$\varphi^* : K[H] \to K[G]$$

is surjective. Hence

$$
\begin{aligned}
d\varphi : \mathscr{g} &\longrightarrow \mathscr{H} \\
\gamma &\longrightarrow \gamma\circ\varphi^*
\end{aligned}
$$

is injective.

(ii) Since $\mathrm{Ker}\,\varphi^* = \mathscr{I}(\varphi(G))$, we have

$$d\varphi\,(\mathscr{g}) \subset \{\gamma \in T(H)_1 \mid \gamma\,(\mathscr{I}(\varphi(G)) = 0\}.$$

Let $X \in T(H)_1$ such that $X\,(\mathrm{Ker}\,\varphi^*) = 0$, then we can define a map

$$
\begin{aligned}
\gamma_X :\ K[G] &\longrightarrow K[H]/\mathrm{Ker}\ \varphi^* \longrightarrow K \quad , \\
\varphi^*(f) &\longrightarrow f + \mathrm{Ker}\ \varphi^* \longrightarrow X(f)
\end{aligned}
$$

where $f \in K[H]$, which is well-defined and belongs to $\mathscr{g}$. Since $d\varphi\,(\gamma_X) = \gamma_X\circ\varphi^* = X$, we have $X \in d\varphi\,(\mathscr{g})$. Q.E.D.

20. Adjoint representations of linear algebraic groups

Let G be a linear algebraic group over K. Let

$$\begin{aligned} I_x : G &\longrightarrow G \\ g &\longmapsto xgx^{-1} \end{aligned}$$

be an inner automorphism of G defined by $x \in G$. Then I_x is a morphism of linear algebraic groups (see Lemma 14.6). Write $\mathrm{Ad}\, x$ for $d(I_x): \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Then $\mathrm{Ad}\, x \in GL(\mathfrak{g})$ for any $x \in G$, and we call the group homomorphism

$$\begin{aligned} \mathrm{Ad}: G &\longrightarrow GL(\mathfrak{g}) \\ x &\longmapsto \mathrm{Ad}\, x \end{aligned}$$

the <u>adjoint representation</u> of G.

<u>(20.1) Example</u>. (We follow the same notation as in Example 14.2 and 19.13). Let $G = GL(n,K)$ and $\mathfrak{gl}(n,K)$ be the Lie algebra of G under the identification

$$\begin{aligned} T(G)_1 &\cong \mathfrak{gl}(n,K) \\ \gamma &\longmapsto (\gamma(f_{ij})) \end{aligned}$$

(see Example 19.13). Then $\mathrm{Ad}\, x(\gamma) = x\gamma x^{-1}$ for any $\gamma \in \mathfrak{gl}(n,K)$ and $x \in G$.

<u>Proof</u>. Let $x = (x_{ij}) \in G$ and $x^{-1} = (z_{ij}) \in G$. Then

$$\{(I_x)^* (f_{ij})\} (y) = f_{ij} (xyx^{-1}) = \sum_l \Big(\sum_k x_{ik}\, y_{kl} \Big) z_{lj}$$

$$= \sum_{k,l} x_{ik}\, y_{kl}\, z_{lj} = \sum_{k,l} x_{ik}\, z_{lj}\, f_{kl}(y)\,,$$

where $y = (y_{ij}) \in G$. Hence

$$(I_x)^* (f_{ij}) = \sum_{k,l} x_{ik}\, z_{lj}\, f_{kl}\,,$$

and

$$\{\mathrm{Ad}\, x(\gamma)\} (f_{ij}) = \gamma \circ (I_x)^* (f_{ij})$$

$$= \gamma \Big(\sum_{k,l} x_{ik}\, z_{lj}\, f_{kl} \Big) = \sum_{k,l} x_{ik}\, \gamma(f_{kl})\, z_{lj}$$

for any $\gamma \in T(G)_1$. Thus we have shown that

$$\{\mathrm{Ad}\, x(\gamma)\} (f_{ij}) = x(\gamma(f_{ij}))\, x^{-1}$$

for any $x \in G$ and $\gamma \in T(G)_1$. Q.E.D.

Remark to Example 20.1. Following the same notation as in Example 19.13. Let $G = GL(n,K)$, $\mathfrak{g} = T(GL(n,K))_1$ and $x \in G$. Then we have got the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{Ad } x} & \mathfrak{g} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathfrak{gl}(n,K) & \longrightarrow & \mathfrak{gl}(n,K) \end{array}$$

Lie algebra homomorphism: $(m_{ij}) \longrightarrow x(m_{ij})x^{-1}$

(20.2) Proposition. (i) Let (U,A) and $(V,B) \in \mathscr{A}(K)$ and $(u,v) \in U{\times}V$. Let $\iota_1 : U \longrightarrow U{\times}V$, $x \longrightarrow (x, v)$ and $\iota_2 : V \longrightarrow U{\times}V$, $y \longrightarrow (u, y)$ be two injective morphisms as in the proof of Proposition 4.8 and

$$\varphi : T(U{\times}V)_{(u,v)} \cong T(U)_u \dot{+} T(V)_v$$
$$\eta \longrightarrow (\eta \circ \pi_1^*, \ \eta \circ \pi_2^*)$$

be also as in Proposition 4.8, where $\pi_1 : U{\times}V \to U$ and $\pi_2 : U{\times}V \to V$ are the projections. Then for any $(\eta_1, \eta_2) \in T(U)_u \dot{+} T(V)_v$ we have

$$\varphi(\eta_1 \circ \iota_1^* + \eta_2 \circ \iota_2^*) = (\eta_1, \eta_2)$$

and
$$(\eta_1 \circ \iota_1^* + \eta_2 \circ \iota_2^*) (f{\otimes}g) = \eta_1(f)\, g(v) + f(u)\, \eta_2(g) ,$$
where $f \in A$ and $g \in B$.

We often identify $(\eta_1, \eta_2) \in T(U)_u \dot{+} T(V)_v$ with $\eta_1 \circ \iota_1^* + \eta_2 \circ \iota_2^* \in T(U{\times}V)_{(u,v)}$.

(ii) Let G be a linear algebraic group over K with the operations $m: G{\times}G \longrightarrow G$, $(x, y) \to xy$ and $\tau: G \longrightarrow G$, $x \longrightarrow x^{-1}$ and the Lie algebra $\mathfrak{g} = T(G)_1$. Then

$$(dm)_{(1,1)} (\gamma_1, \gamma_2) = \gamma_1 + \gamma_2 \text{ for any } (\gamma_1, \gamma_2) \in \mathfrak{g} \dot{+} \mathfrak{g}$$

and
$$(d\tau)_1 (\gamma) = -\gamma \text{ for any } \gamma \in \mathfrak{g}.$$

Proof. (i) $(\eta_1{\circ}\iota_1^* + \eta_2{\circ}\iota_2^*)(f{\otimes}g) = \eta_1{\circ}\iota_1^* (f{\otimes}g) + \eta_2{\circ}\iota_2^*(f{\otimes}g) = \eta_1 (g(v){\cdot}f) + \eta_2 (f(u){\cdot}g)$ $= \eta_1(f)\, g(v) + f(u)\, \eta_2(g)$.

(ii) Let $f \in K[G]$ and assume that

$$m^*(f) = \sum_i f_i{\otimes}g_i \in K[G] \otimes K[G] .$$

Then for any $(\gamma_1, \gamma_2) \in \mathfrak{g} \dot{+} \mathfrak{g}$ we have

$$\{(dm)_{(1,1)}(\gamma_1,\gamma_2)\}(f) = (\gamma_1,\gamma_2)(m^*(f)) = (\gamma_1,\gamma_2)\left(\sum_i f_i \otimes g_i\right)$$
$$= \sum_i \gamma_1(f_i)\, g_i(1) + \sum_i f_i(1)\,\gamma_2(g_i) \quad \text{(see (i))}$$
$$= \gamma_1(R_1(f)) + (f * \gamma_2)(1) \quad \text{(see the proof of Lemma 18.4 and (19.9))}$$
$$= \gamma_1(f) + \gamma_2(f)\ .$$

Now let $(1_G,\tau)$ be a morphism of G into $G\times G$ which takes $x \in G$ to (x,x^{-1}), then $m \circ (1_G,\tau)$ is also a morphism of G into G the differential of which at 1 is zero. Thus we have $d(m \circ (1_G,\tau))_1 = (dm)_{(1,1)} \circ d(1_G,\tau)_1 = 0$. Let $\gamma \in \mathfrak{g}$, then for each $f\otimes g \in K[G] \otimes K[G]$

$$d(1_G,\tau)_1(\gamma)(f\otimes g) = \gamma \circ (1_G,\tau)^*(f\otimes g) = \gamma(1_G{}^*(f)\,\tau^*(g))$$
$$= 1_G{}^*(f)(1)\,\gamma(\tau^*(g)) + \gamma(1_G{}^*(f))\,\tau^*(g)(1)\ .$$

Since $(\gamma, (d\tau)_1(\gamma))(f\otimes g) = \gamma(f)\,g(1) + f(1)\,(d\tau)_1(\gamma)(g)$ from (i), we have

$$d(1_G,\tau)_1(\gamma) = (\gamma, (d\tau)_1(\gamma))\ .$$

Hence for all $\gamma \in \mathfrak{g}$ we have

$$d(m \circ (1_G,\tau))_1(\gamma) = (dm)_{(1,1)} \circ d(1_G,\tau)_1(\gamma)$$
$$= (dm)_{(1,1)}(\gamma, (d\tau)_1(\gamma)) = \gamma + (d\tau)_1(\gamma) = 0\ ,$$

which implies $(d\tau)_1(\gamma) = -\gamma$ for any $\gamma \in \mathfrak{g}$. Q.E.D.

<u>(20.3) Lemma.</u> (i) (see Exercise 11 on p.24). Let (U,A), (U',A'), (V,B) and $(V',B') \in \mathscr{A}(K)$ and $f: U \longrightarrow U'$, $u \longmapsto u'$, $g: V \longrightarrow V'$, $v \longmapsto v'$ be morphisms, then $f\times g: U\times V \longrightarrow U'\times V'$, $(u,v) \mapsto (u',v')$ is also a morphism with the comorphism

$$(f\times g)^* = f^*\otimes g^* : A'\otimes B' \longrightarrow A\otimes B,\quad a'\otimes b' \longmapsto f^*(a')\otimes g^*(b')$$

and the differential

$$d(f\times g)_{(u,v)} = (df_u, dg_v) : T(U)_u \dotplus T(V)_v \longrightarrow T(U')_{u'} \dotplus T(V')_{v'},\quad (\gamma,\delta) \longmapsto (df_u(\gamma), dg_v(\delta))$$

(ii) Let (U,A), (V,B) and $(W,C) \in \mathscr{A}(K)$ and $\xi: W \to U$, $\eta: W \to V$ be morphisms, then

$$\chi : W \longrightarrow U\times V,\quad w \longmapsto (\xi(w), \eta(w))$$

is also a morphism (see Proposition 3.3) with the comorphism

$$\chi^* : A\otimes B \longrightarrow C$$
$$a\otimes b \longmapsto \xi^*(a)\ \eta^*(b)$$

and the differential

$$(d\chi)_w : T(W)_w \longrightarrow T(U)_{\xi(w)} \dot{+} T(V)_{\eta(w)} .$$
$$\gamma \longmapsto ((d\xi)_w(\gamma),\ (d\eta)_w(\gamma))$$

Proof. (ii) Let $\gamma \in T(W)_w$, then

$$(d\chi)_w(\gamma)\ (a\otimes b) = \gamma \circ \chi^*\ (a\otimes b) = \gamma(\xi^*(a)\ \eta^*(b))$$

for all $a\otimes b \in A\otimes B$. Thus

$$(d\chi)_w\ (\gamma)\ (a\otimes b) = \{\xi^*(a)\ (w)\}\ \{\gamma(\eta^*(b))\} + \{\gamma(\xi^*(a))\}\ \eta^*(b)\ (w)$$
$$= \{(d\xi)_w\ (\gamma)\ (a)\}\ b(\eta(w)) + a(\xi(w))\ \{(d\eta)_w\ (\gamma)(b)\} = ((d\xi)_w\ (\gamma),\ (d\eta)_w\ (\gamma))\ (a\otimes b)$$

for all $a\otimes b \in A\otimes B$. Q.E.D.

(20.4) Proposition. We follow the same notation as in Example 20.1. Let $G = GL(n,K)$ and $\mathfrak{gl}(n,K)$ be the Lie algebra of G unter the identification

$$T(G)_1 \cong \mathfrak{gl}(n,K) .$$
$$\gamma \longmapsto (\gamma(f_{ij}))$$

Then

(i) $l:G \longrightarrow GL(\mathfrak{g})$, $x \longmapsto l_x$ is a morphism of affine varieties, where $\mathfrak{g} = \mathfrak{gl}(n,K)$ and $l_x:\mathfrak{g} \longrightarrow \mathfrak{g}$, $\gamma \longmapsto x\gamma$. The differential of l at 1 is

$$\hat{l}:\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}),\ \gamma \longmapsto \hat{l}_\gamma, \quad \text{where } \hat{l}_\gamma:\mathfrak{g} \longrightarrow \mathfrak{g},\ \delta \longmapsto \gamma\delta$$

and $\mathfrak{gl}(\mathfrak{g}) = \mathrm{End}_K(\mathfrak{g})$.

(ii) $r:G \longrightarrow GL(\mathfrak{g})$, $x \longmapsto r_x$ is a morphism of affine varieties, where $r_x:\mathfrak{g} \longrightarrow \mathfrak{g}$, $\gamma \longmapsto \gamma x$. The differential of r at 1 is $\hat{r}:\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, $\gamma \longmapsto \hat{r}_\gamma$, where $\hat{r}_\gamma:\mathfrak{g} \longrightarrow \mathfrak{g}$, $\delta \longmapsto \delta\gamma$.

(iii) $Ad:G \longrightarrow GL(\mathfrak{g})$, $x \longmapsto Ad\ x$ is a morphism of linear algebraic groups whose differential at 1 is ad, i.e., $d(Ad)_1 = ad$ where

$$(ad\ X)\ (Y) = XY - YX \quad \text{for any } X, Y \in \mathfrak{g}.$$

Proof. (i) It is clear that l is a well–defined map of G into $GL(\mathfrak{g})$. Let E_{st} be the element $\mathfrak{g}$ with 1 in the (s,t) position and zeros elsewhere. Let x be an $n\times n$

matrix over K, then

$$\text{the } (i,j)\text{th component of } x\,E_{st} = \begin{cases} 0 & \text{if } j \neq t \\ x_{is} & \text{if } j = t\,. \end{cases}$$

Thus the matrix of l_x, where $x \in G$, with respect to the basis $\{E_{st}\}$ of $\mathscr{g}$ is as follows.

$$\text{The } \{(i,j), (s,t)\}\text{th component of the matrix of } l_x = \begin{cases} 0 & \text{if } j \neq t \\ x_{is} & \text{if } j = t\,. \end{cases}$$

Hence the comorphism l^* of l is given by the following formula

$$l^* : K[GL(\mathscr{g})] \to K[G]$$

and

$$l^*\,(f_{ij,st}) = \begin{cases} 0 & \text{if } j \neq t \\ f_{is} & \text{if } j = t\,, \end{cases}$$

which implies l is a morphism.

Further we have

$$(dl)_1\,(\gamma)\,(f_{ij,st}) = \gamma\,(l^*\,(f_{ij,st})) = \begin{cases} 0 & \text{if } j \neq t \\ \gamma(f_{is}) & \text{if } j = t \end{cases} \text{ for any } \gamma \in \mathscr{g},$$

which implies $(dl)_1\,(\gamma) = \hat{l}_\gamma$.

(ii) (Exercise).

(iii) Since

$$\begin{array}{rcccccc} Ad : G & \xrightarrow{(1_G,\tau)} & G \times G & \xrightarrow{l\times r} & GL(\mathscr{g}) \times GL(\mathscr{g}) & \xrightarrow{m} & GL(\mathscr{g}) \\ x & \longrightarrow & (x,\, x^{-1}) & \longrightarrow & (l_x,\, r_{x^{-1}}) & \longrightarrow & l_x \circ r_{x^{-1}} \end{array},$$

we have

$$d(Ad)_1 = d(m \circ (l\times r) \circ (1_G,\, \tau))_1 = dm_{(1_{\mathscr{g}},\, 1_{\mathscr{g}})} \circ d(l\times r)_{(1,1)} \circ d(1_G,\, \tau)_1\,.$$

Hence from (i), (ii), Proposition 20.2 and Lemma 20.3 we have

$$\begin{aligned} d(Ad)_1\,(\gamma) &= dm_{(1_{\mathscr{g}},\, 1_{\mathscr{g}})} \circ d(l\times r)_{(1,1)}\,(\gamma, -\gamma) \\ &= dm_{(1_{\mathscr{g}},\, 1_{\mathscr{g}})}\,(\hat{l}_\gamma, -\hat{r}_\gamma) \\ &= \hat{l}_\gamma - \hat{r}_\gamma \quad \text{for any} \quad \gamma \in \mathscr{g}\,. \end{aligned}$$

Thus $\{d(Ad)_1\,(\gamma)\}\,(\gamma') = \gamma\gamma' - \gamma'\gamma$ for any $\gamma, \gamma' \in \mathscr{g}$, i.e.,

$$d(Ad)_1\,(\gamma) = ad\ \gamma \ \text{ for any } \ \gamma \in \mathscr{g}.$$

Q.E.D.

<u>Exercise 59</u>. Prove Proposition 20.4.ii.

<u>(20.5) Lemma.</u> Let H be a closed subgroup of a linear algebraic group G over K and φ be a rational representation of G into $GL(V)$, where V is a finite dimensional vector space over K. Let W be a $\varphi(H)$–invariant subspace of V, i.e., $\varphi(h)(W) \subset W$ for all $h \in H$. Then

(i) $\varphi_0 : H \to GL(W)$, defined by $\varphi_0(h) = \varphi(h)|_W$ $(h \in H)$, is a rational representation of H over K.

(ii) Let $\mathscr{H}$ and $\mathscr{g}$ be the Lie algebras of H and G respectively, then we can embed $\mathscr{H}$ into $\mathscr{g}$ by $d\iota$ where $\iota: H \subset G$ is the inclusion map of H into G, and

$$d\varphi(\gamma)(W) \subset W$$

and

$$d\varphi(\gamma)|_W = d\varphi_0(\gamma)$$

for any $\gamma \in \mathscr{H}$, where $d\varphi: \mathscr{g} \to \mathscr{gl}(V)$ and $d\varphi_0: \mathscr{H} \to \mathscr{gl}(W)$ be the differentials of $\mathscr{g}$ and $\mathscr{H}$ into the Lie algebras of linear transformations of V and W, i.e., $\mathscr{gl}(V) = \mathrm{End}_K(V)$ and $\mathscr{gl}(W) = \mathrm{End}_K(W)$ respectively.

<u>Proof.</u> Let $\{v_1, \ldots, v_s, v_{s+1}, \ldots, v_t\}$ be a K–basis of V such that $\{v_1, \ldots, v_s\}$ forms a K–basis of W. We identify $GL(V)$ with $GL(t,K)$ and $GL(W)$ with $GL(s,K)$ with respect to these basis. Let $(\varphi(h)_{ij})$ be a $t \times t$ matrix such that

$$\begin{array}{l} \varphi: G \longrightarrow GL(V) \\ \quad\; \cup \\ \quad\; H \\ \quad\; h \longrightarrow [\varphi(h): v_j \longrightarrow \sum_{i=1}^{t} \varphi(h)_{ij} v_i] \end{array}$$

and $(\varphi_0(h)_{ij})$ be an $s \times s$ matrix such that

$$\begin{array}{l} \varphi_0 : H \longrightarrow GL(W) \\ \qquad\; h \longrightarrow [\varphi(h)|_W : v_j \to \sum_{i=1}^{s} \varphi_0(h)_{ij} v_i], \end{array}$$

then we have

$$(\varphi(h)_{ij}) = \begin{array}{cc} & \overbrace{\qquad\qquad}^{s} \\ s\Big\{ & \left[\begin{array}{c|c} \varphi_0(h)_{ij} & * \\ \hline 0 & * \end{array}\right] \end{array}.$$

(i) It is clear that φ_0 is a group homomorphism. Since φ is a rational representation, all the maps

$$\begin{array}{l} \varphi_{ij}: G \longrightarrow K \\ \qquad\; g \longrightarrow \varphi(g)_{ij} \end{array}$$

where

$$\varphi(g): v_j \longrightarrow \sum_{i=1}^{t} \varphi(g)_{ij} v_i$$

belong to $K[G]$ $(1 \leq i,j \leq t)$ from Proposition 18.2. Hence we have $(\varphi_0)_{ij} \in K[H]$ for any $1 \leq i,j \leq s$, which shows that φ_0 is a rational representation.

(ii) Since the inclusion map $\iota: H \subset G$ has the surjective comorphism $\iota^*: K[G] \to K[H]$,

$$\begin{aligned} d\iota: \mathscr{H} &\subset \mathfrak{g} \\ \gamma &\longrightarrow [d\iota(\gamma): f \to \gamma(f|_H)] \end{aligned}$$

is an injective Lie algebra homomorphism, where $f \in K[G]$. Let $\gamma \in \mathscr{H}$, then we have

$$\Big(d\varphi\,(d\iota(\gamma))\,(f_{ij})\Big) = \Big(\gamma \circ (\varphi\circ\iota)^*\,(f_{ij})\Big)$$

$$= \begin{array}{c} \overbrace{\hphantom{d\varphi_0(\gamma)\,(f'_{ij})}}^{s} \\ s\left\{\left[\begin{array}{c|c} d\varphi_0(\gamma)\ (f'_{ij}) & * \\ \hline 0 & * \end{array}\right]\right. \end{array},$$

because $(\varphi\circ\iota)^*\,(f_{ij}) = \varphi_0^*\,(f'_{ij})$ for $1 \leq i,j \leq s$, where $K[GL(V)] = K[f_{ij}, \delta \mid 1 \leq i,j \leq t]$ and $K[GL(W)] = K[f'_{ij}, \delta' \mid 1 \leq i,j \leq s]$ according to the notation in Example 14.2. Hence

$$d\varphi\,(d\iota(\gamma))\,(W) \subset W$$

and

$$d\varphi\,(d\iota(\gamma))\,|_W = d\varphi_0(\gamma)$$

for any $\gamma \in \mathscr{H}$. Q.E.D.

<u>(20.6) Theorem</u>. Let G be a linear algebraic group over K with the Lie algebra $\mathfrak{g}$. Let $\mathrm{Ad}\,x = d(I_x)$ for $x \in G$. Then

(i) $\begin{aligned} \mathrm{Ad}: G &\longrightarrow GL(\mathfrak{g}) \\ x &\longrightarrow \mathrm{Ad}\,x \end{aligned}$ is a morphism of linear algebraic groups, and

(ii) $d(\mathrm{Ad})_1 = \begin{aligned} \mathrm{ad}: \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ X &\longrightarrow [\mathrm{ad}\,X : Y \to [X,Y]] \end{aligned}$, where $X,Y \in \mathfrak{g}$ and $\mathfrak{gl}(\mathfrak{g}) = \mathrm{End}_K(\mathfrak{g})$.

<u>Proof</u>. (i) Let φ be an embedding of G into $GL(n,K)$ for some n as in Theorem 18.7. Since $\varphi: G \cong \varphi(G)$ is an isomorphism of affine varieties, $\varphi^*: K[GL(n,K)] \to K[G]$ is surjective and $\begin{aligned} d\varphi: \mathfrak{g} &\to \mathfrak{gl}(n,K) \\ \gamma &\to \gamma\circ\varphi^* \end{aligned}$ is injective.

From Proposition 20.4 we have

$$d(\mathrm{Ad}^{GL})_1 = \mathrm{ad}^{GL},$$

where

$$\begin{aligned} \mathrm{Ad}^{GL}: GL(n,K) &\longrightarrow GL(\mathfrak{gl}(n,K)). \\ x &\longrightarrow \mathrm{Ad}_x^{GL} \end{aligned}$$

In case of $G = GL(n,K)$ we write Ad^{GL} and ad^{GL} for Ad and ad , respectively.

Now let $g \in G$ and $\hat{I}_{\varphi(g)}: \begin{array}{l} GL(n,K) \longrightarrow GL(n,K) \\ x \longmapsto \varphi(g)x\varphi(g)^{-1} \end{array}$, then $d(\hat{I}_{\varphi(g)} \circ \varphi) = d(\hat{I}_{\varphi(g)}) \circ d\varphi = d(I_{\varphi(g)}) \circ d\varphi$ where $I_{\varphi(g)} : \begin{array}{l} \varphi(G) \longrightarrow \varphi(G) \\ x \longmapsto \varphi(g)x\varphi(g)^{-1} \end{array}$. Thus for each $g \in G$ we have

$$Ad^{GL}_{\varphi(g)}(d\varphi(\mathfrak{g})) = d(\hat{I}_{\varphi(g)}) \circ d\varphi(\mathfrak{g}) = d(I_{\varphi(g)})(d\varphi(\mathfrak{g})) \subset d\varphi(\mathfrak{g}) .$$

Hence $d\varphi(\mathfrak{g})$ is an $Ad^{GL}(\varphi(G))$–invariant space of $\mathfrak{gl}(n,K)$. From Lemma 20.5

$$\begin{aligned} Ad : \varphi(G) &\longrightarrow GL(d\varphi(\mathfrak{g})) \\ \varphi(g) &\longmapsto Ad^{GL}\ \varphi(g)\,|_{d\varphi(\mathfrak{g})} \end{aligned}$$

is a morphism of linear algebraic groups, because

$$Ad^{GL}\ \varphi(g)\,|_{d\varphi(\mathfrak{g})} = d(I_{\varphi(g)}) = Ad_{\varphi(g)} .$$

(ii) Since $d(Ad^{GL})(d\varphi(\gamma))\,|_{d\varphi(\mathfrak{g})} = d(Ad)(d\varphi(\gamma))$ for each $\gamma \in \mathfrak{g}$ from Lemma 20.5, we have

$$d(Ad)(d\varphi(\gamma)) = ad^{GL}(d\varphi(\gamma))\,|_{d\varphi(\mathfrak{g})} = ad(d\varphi(\gamma))$$

for any $\gamma \in \mathfrak{g}$. Q.E.D.

<u>(20.7) Proposition</u>. Let $(G,K[G])$ be a linear algebraic group over K with Lie algebra $\mathfrak{g}$, and V be a finite dimensional left KG–submodule of $K[G]$ (see Proposition 18.5). Let φ be the rational representation of G into $GL(V)$ afforded by V and its K–basis $\{v_1, v_2,\dots,v_n\}$, i.e.,

$$g*(v_1,\dots,v_n) = (v_1,\dots,v_n)\begin{bmatrix} \varphi_{11}(g), \varphi_{12}(g),\dots, & \varphi_{1n}(g) \\ \varphi_{21}(g),\dots\dots\dots, & \varphi_{2n}(g) \\ \vdots & \vdots \\ \varphi_{n1}(g),\dots\dots\dots, & \varphi_{nn}(g) \end{bmatrix},$$

where $\varphi_{ij} \in K[G]$ for any $1 \le i,j \le n$. Then

$$d\varphi(\gamma) = *\gamma \text{ for any } \gamma \in \mathfrak{g},$$

where $(v*\gamma)(g) = \gamma(L_g(v))$ $(v \in V$ and $g \in G)$ (see (19.9)).

<u>Proof</u>. Let $m : \begin{array}{l} G\times G \longrightarrow G \\ (x,y) \longmapsto xy \end{array}$, then from the Remark to Proposition 18.5 we have

$$m^*(v_i) = \sum_{j=1}^{n} v_j \otimes \varphi_{ji}$$

for any $1 \leq i \leq n$. Hence we have

$$v_i * \gamma = \sum_{j=1}^{n} \gamma(\varphi_{ji})\, v_j \ .$$

Since $$d\varphi(\gamma) = \gamma \circ \varphi^* : f_{ij} \to \gamma(\varphi_{ij})$$

for any $1 \leq i,j \leq n$ where $K[GL(V)] = K[f_{ij}, \delta \mid 1 \leq i,j \leq n]$ as in Example 14.2, we have

$$d\varphi(\gamma) = *\gamma \text{ for any } \gamma \in \mathfrak{g}.$$

Q.E.D.

<u>(20.8) Proposition</u>. Let $(G_i, K[G_i])$ $(i = 1,2)$ be linear algebraic groups over K with Lie algebras $\mathfrak{g}_i$. Let

$$\varphi_i : G_i \longrightarrow GL(V_i) \quad (i = 1,2)$$

be finite dimensional rational representations of G_i . Then

(i) The Lie algebra of $G_1 \times G_2$ is $\mathfrak{g}_1 \dot{+} \mathfrak{g}_2$ where

$$\mathfrak{g}_1 \dot{+} \mathfrak{g}_2 = \{(\gamma_1, \gamma_2) \mid \gamma_i \in \mathfrak{g}_i \ , \ i = 1,2\}$$

and $$[(\gamma_1, \gamma_2), (\gamma_1', \gamma_2')] = ([\gamma_1, \gamma_1'], [\gamma_2, \gamma_2']) \ .$$

(ii) $\varphi_1 \dot{+} \varphi_2 : \begin{array}{l} G_1 \times G_2 \longrightarrow GL(V_1 \dot{+} V_2) \\ (g_1, g_2) \longrightarrow [\varphi_1 + \varphi_2 \ (g_1, g_2) : v_1 + v_2 \to \varphi_1(g_1)v_1 + \varphi_2(g_2)v_2] \end{array}$ is a rational representation of $G_1 \times G_2$ and

$$d(\varphi_1 \dot{+} \varphi_2)\,(\gamma_1, \gamma_2)\,(v_1 + v_2) = (d\varphi_1)\,(\gamma_1)\, v_1 + (d\varphi_2)\,(\gamma_2)\, v_2 \ ,$$

where $v_i \in V_i$ $(i = 1,2)$.

(iii) $\varphi_1 \otimes \varphi_2 : \begin{array}{l} G_1 \times G_2 \longrightarrow GL(V_1 \otimes V_2) \\ (g_1, g_2) \longrightarrow \varphi_1(g_1) \otimes \varphi_2(g_2) \end{array}$ is a rational representation of $G_1 \times G_2$ and

$$d(\varphi_1 \otimes \varphi_2)\,(\gamma_1, \gamma_2)\,(v_1 \otimes v_2) = \{(d\varphi_1)\,(\gamma_1)\, v_1\} \otimes v_2 + v_1 \otimes \{(d\varphi_2)\,(\gamma_2)\, v_2\} \ .$$

<u>Proof</u>. (i) Let $\pi_i : G_1 \times G_2 \longrightarrow G_i$ be the projection of $G_1 \times G_2$ onto G_i $(i = 1,2)$, then we have the K–isomorphism φ of $T(G_1 \times G_2)_{(1,1)}$ onto $\mathfrak{g}_1 \dot{+} \mathfrak{g}_2$ which takes each $\gamma \in T(G_1 \times G_2)_{(1,1)}$ to $(\gamma \circ \pi_1^*, \gamma \circ \pi_2^*)$ (see Proposition 20.2). Since

$$\begin{aligned} \varphi([\gamma, \gamma']) &= ([\gamma, \gamma'] \circ \pi_1^*, [\gamma, \gamma'] \circ \pi_2^*) \\ &= ([\gamma \circ \pi_1^*, \gamma' \circ \pi_1^*], [\gamma \circ \pi_2^*, \gamma' \circ \pi_2^*]) \\ &= [(\gamma \circ \pi_1^*, \gamma \circ \pi_2^*), (\gamma' \circ \pi_1^*, \gamma' \circ \pi_2^*)] \\ &= [\varphi(\gamma), \varphi(\gamma')] \end{aligned}$$

for any $\gamma,\gamma' \in T(G_1 \times G_2)_{(1,1)}$, φ is a Lie algebra isomorphism.

(ii) Since $\varphi_1 \dot{+} \varphi_2 : G_1 \times G_2 \longrightarrow GL(V_1 \dot{+} V_2)$ is the direct sum of rational representations

$$\varphi_i \circ \pi_i : G_1 \times G_2 \longrightarrow G_i \longrightarrow GL(V_i) \quad (i = 1,2) ,$$

$\varphi_1 \dot{+} \varphi_2$ is also a rational representation of $G_1 \times G_2$. Since

$$\begin{array}{l} \varphi_1 \dot{+} \varphi_2 : G_1 \times G_2 \longrightarrow GL(V_1) \times GL(V_2) \overset{\iota}{\subset} GL(V_1 \dot{+} V_2) , \\ \qquad\qquad (g_1, g_2) \longrightarrow (\varphi_1(g_1), \varphi_2(g_2)) \end{array}$$

we have

$$\begin{array}{l} d(\varphi_1 \dot{+} \varphi_2) : \mathfrak{g}_1 \dot{+} \mathfrak{g}_2 \longrightarrow \mathfrak{gl}(V_1) \dot{+} \mathfrak{gl}(V_2) \overset{d\iota}{\subset} \mathfrak{gl}(V_1 \dot{+} V_2) . \\ \qquad\qquad (\gamma_1, \gamma_2) \longrightarrow ((d\varphi_1)(\gamma_1), (d\varphi_2)(\gamma_2)) \end{array}$$

From Lemma 20.5 we have $d(\varphi_1 \dot{+} \varphi_2)(\gamma_1, \gamma_2)(v_1 + v_2) = (d\varphi_1)(\gamma_1)v_1 + (d\varphi_2)(\gamma_2)v_2$.

(iii) Since $\varphi_1 \otimes \varphi_2 : G_1 \times G_2 \longrightarrow GL(V_1 \otimes V_2)$ is the tensor product of rational representations

$$\varphi_i \circ \pi_i : G_1 \times G_2 \xrightarrow{\pi_i} G_i \xrightarrow{\varphi_i} GL(V_i) \quad (i = 1,2) ,$$

$\varphi_1 \otimes \varphi_2$ is a rational representation of $G_1 \times G_2$. Notice that

$$d(\varphi_1 \otimes \varphi_2)(\gamma_1, \gamma_2) = d(\varphi_1 \otimes \varphi_2)(\gamma_1, 0) + d(\varphi_1 \otimes \varphi_2)(0, \gamma_2)$$

and

$$d(\varphi_1 \otimes \varphi_2)(\gamma_1, 0) = d(\varphi_1 \otimes \varphi_2) \circ d\iota_1(\gamma_1) ,$$

where $\begin{array}{l}\iota_1 : G_1 \longrightarrow G_1 \times G_2 \\ \quad g \longrightarrow (g, 1)\end{array}$. Hence it is enough to show that

$$d((\varphi_1 \otimes \varphi_2) \circ \iota_1)(\gamma_1)(v_1 \otimes v_2) = \{d\varphi_1(\gamma_1) v_1\} \otimes v_2 .$$

Since

$$\begin{array}{l} (\varphi_1 \otimes \varphi_2) \circ \iota_1 : G_1 \longrightarrow GL(V_1 \otimes V_2) \\ \qquad\qquad g \longrightarrow [\varphi_1(g) \otimes 1_{V_2} : v_1 \otimes v_2 \to gv_1 \otimes v_2] \end{array}$$

where 1_{V_2} is the identity map of V_2, $V_1 \otimes V_2 = \sum_{j=1}^{m} V_1 \otimes Kw_j$ is a direct sum of G_1–subspaces $V_1 \otimes Kw_j$ where $\{w_1, w_2, \dots, w_n\}$ is a K–basis of V_2 . Since the map $\begin{array}{l}V_1 \otimes Kw_j \cong V_1 \\ v \otimes w_j \longrightarrow v\end{array}$ is a KG_1–isomorphism for any $1 \le j \le m$, we have

$$\begin{array}{l} (\varphi_1 \otimes \varphi_2) \circ \iota_1 : G_1 \longrightarrow GL(V_1) \times \dots \times GL(V_1) \subset GL\left(\sum_{j=1}^{m} V_1 \otimes Kw_j\right) . \\ \qquad\qquad g \longrightarrow (\varphi_1(g), \varphi_1(g), \dots, \varphi_1(g)) \end{array}$$

Hence $d((\varphi_1 \otimes \varphi_2) \circ \iota_1)(\gamma)(v_1 \otimes v_2) = d\varphi_1(\gamma) v_1 \otimes v_2$. Q.E.D.

CHAPTER V

HOMOGENEOUS SPACES

We construct the quotient G/H where $(H,K[H])$ is a closed subgroup of a linear algebraic group $(G,K[G])$. We first explain separable morphisms and Zariski's Main Theorem and finally show the conjugacy of Borel subgroups.

Let $(G,\mathcal{O}_G)$ be an algebraic group over K and $(H,\mathcal{O}_H)$ be a closed subgroup of G. Let G/H be the set of all left cosets H in G. Let

$$\begin{aligned}\nu: G &\longrightarrow G/H\\ g &\longrightarrow gH\end{aligned}$$

be the natural map. We define a topology on G/H as follows: a subset U of G/H is open if and only if $\nu^{-1}(U)$ is open in G.

It is clear that this topology is well–defined and ν is an open map. Now let U be an open subset of G/H. We define $\mathcal{O}(U)$ to be the set of all maps f of U into K such that

$$f\circ\nu\,|_{\nu^{-1}(U)} \in \mathcal{O}_G(\nu^{-1}(U)),$$

i.e.,
$$\mathcal{O}(U) = \{f:U \to K \mid f\circ\nu\,|_{\nu^{-1}(U)} \in \mathcal{O}_G(\nu^{-1}(U))\},$$

then <u>$\mathcal{O}$ is a sheaf of functions over K</u> and ν is a morphism of ringed spaces $(G,\mathcal{O}_G)$ onto $(G/H,\mathcal{O})$.

21. Separable morphisms

Let $(X, \mathscr{O}_X)$ be a prevariety over K and x be any fixed point of X. Let $\mathscr{M}_x$ be the unique maximal ideal of the local ring $\mathscr{O}_x$ (see Proposition 13.3). In §13 we defined the tangent space $T(X)_x$ of X at x to be

$$\mathrm{Hom}_K(\mathscr{M}_x/(\mathscr{M}_x)^2, K)$$

and have shown that

$$T(U)_x \cong T(X)_x$$

as K–linear spaces for any affine open set U of X containing x.

Now let $\varphi: X \to Y$ be a morphism of prevarieties $(X, \mathscr{O}_X)$ into $(Y, \mathscr{O}_Y)$ over K. Let $y = \varphi(x)$, then there exist affine open sets U $(\subset X)$ and V $(\subset Y)$ such that

$$\varphi(U) \subset V, \quad U \ni x \text{ and } V \ni y .$$

Let $\begin{array}{l}\varphi_0: U \longrightarrow V \\ \quad\;\; u \longrightarrow \varphi(u)\end{array}$, then φ_0 is a morphism of affine varieties and we have

$$(\varphi_0{}^*)^{-1}(\mathscr{I}_U(x)) = \mathscr{I}_V(y) .$$

Thus we have got a K–algebra homomorphism ψ of $\mathscr{O}_y$ into $\mathscr{O}_x$ such that

$$\begin{array}{rl}\psi : K[V]_{\mathscr{I}_V(y)} \longrightarrow & K[U]_{\mathscr{I}_U(x)} \quad , \\ a/s \longrightarrow & \varphi_0{}^*(a)/\varphi_0{}^*(s)\end{array}$$

where $a,s \in K[V]$ and $s \notin \mathscr{I}_V(y)$.

Let $\mathscr{M}_y = \{a/s \mid a \in \mathscr{I}_V(y) \text{ and } s \notin \mathscr{I}_V(y)\}$ and

$\mathscr{M}_x = \{a/s \mid a \in \mathscr{I}_U(x) \text{ and } s \notin \mathscr{I}_U(x)\}$

be the maximal ideals of $\mathscr{O}_y$ and $\mathscr{O}_x$ respectively, then we have

$$\psi^{-1}(\mathscr{M}_x) = \mathscr{M}_y .$$

Since $\psi(\mathscr{M}_y) \subset \mathscr{M}_x$ and $\psi(\mathscr{M}_y{}^2) \subset \mathscr{M}_x{}^2$, we can define a K–linear map $\tilde{\psi}$ of $\mathscr{M}_y/\mathscr{M}_y{}^2$ into $\mathscr{M}_x/\mathscr{M}_x{}^2$ as follows.

$$\begin{array}{rl}\tilde{\psi} : \mathscr{M}_y/\mathscr{M}_y{}^2 \longrightarrow & \mathscr{M}_x/\mathscr{M}_x{}^2 \\ m + \mathscr{M}_y{}^2 \longrightarrow & \psi(m) + \mathscr{M}_x{}^2\end{array}$$

Thus we can define a K–linear map $d\varphi_x$ of $T(X)_x$ into $T(Y)_y$ induced by $\tilde{\psi}$ as follows.

$$\begin{array}{rl}d\varphi_x : \mathrm{Hom}_K(\mathscr{M}_x/\mathscr{M}_x{}^2, K) \longrightarrow & \mathrm{Hom}_K(\mathscr{M}_y/\mathscr{M}_y{}^2, K) \\ \gamma \longrightarrow & \gamma \circ \tilde{\psi}\end{array}$$

(21.1) Definition. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be prevarieties over K and $\varphi: X \to Y$ be a morphism of prevarieties, then we call the map

$$d\varphi_x : T(X)_x \longrightarrow T(Y)_y$$
$$\gamma \longrightarrow \gamma \circ \tilde{\psi}$$

the differential of φ at x where $x \in X$ and $\tilde{\psi}$ is as before.

(21.2) Remark. Let $\varphi: X \to Y$ be a morphism of prevarieties $(X, \mathscr{O}_X)$ into $(Y, \mathscr{O}_Y)$ over K. Let $x \in X$ and $y = \varphi(x)$ and U and V be affine open sets of X and Y respectively such that $U \ni x$, $V \ni y$ and $\varphi(U) \subset V$. Let $\begin{array}{l}\varphi_0: U \longrightarrow V \\ \quad u \longrightarrow \varphi(u)\end{array}$. Then we have the following commutative diagram.

$$\begin{array}{llll}
T(U)_x \overset{\rho_1}{\cong} \mathrm{Hom}_K(\mathscr{I}_U(x)/\mathscr{I}_U(x)^2, K) & \overset{\rho_2}{\cong} \mathrm{Hom}_K(\mathscr{M}_x/\mathscr{M}_x^2, K) = T(X)_x \\
\gamma \longrightarrow [\rho_1(\gamma): a + \mathscr{I}_U(x)^2 \to \gamma(a)] & \to [\rho_2 \circ \rho_1(\gamma): a/1 + \mathscr{M}_x^2 \to \gamma(a)] \\
\Big\downarrow (d\varphi_0)_x & \qquad \Big\downarrow d\varphi_x \\
\gamma \circ \varphi_0^* & [\rho_2 \circ \rho_1(\gamma) \circ \tilde{\psi}: b/1 + \mathscr{M}_y^2 \to \gamma(b \circ \varphi_0)] \\
T(V)_y \overset{\rho_1'}{\cong} \mathrm{Hom}_K(\mathscr{I}_V(x)/\mathscr{I}_V(x)^2, K) & \overset{\rho_2'}{\cong} \mathrm{Hom}_K(\mathscr{M}_y/\mathscr{M}_y^2, K) = T(Y)_y \\
\gamma \circ \varphi_0^* \to [\rho_1'(\gamma \circ \varphi_0^*): b + \mathscr{I}_V(x)^2 \to \gamma(b \circ \varphi_0)] & \to [\rho_2' \circ \rho_1'(\gamma \circ \varphi_0^*): b/1 + \mathscr{M}_y^2 \to \gamma(b \circ \varphi_0)]
\end{array}$$

From the above diagram we have the following proposition.

(21.3) Proposition. Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ and $(Z, \mathscr{O}_Z)$ be prevarieties cver K and $\varphi: X \to Y$ and $\psi: Y \to Z$ be morphisms. then we have the following commutative diagram:

$$\begin{tikzcd}
T(X)_x \arrow[rr, "d\varphi_x"] \arrow[dr, "d(\psi\circ\varphi)_x"'] & & T(Y)_{\varphi(x)} \arrow[dl, "d\psi_{\varphi(x)}"] \\
& T(Z)_{\psi\circ\varphi(x)} &
\end{tikzcd}$$

for any $x \in X$.

Proof. See Proposition 4.7. Q.E.D.

(21.4) Example. Let V be an $n+1$ – dimensional vector space over K and $\{v_0, v_1,\dots, v_n\}$ be a K–basis of V. Let X_i be a map of V into K which takes

$$c_0 v_0 + c_1 v_1 + \dots + c_n v_n \text{ to } c_i, \text{ for } 0 \le i \le n,$$

then the pair $(V, K[X_0, X_1,\dots, X_n])$ is an affine variety over K (see Example 1.2).

Now let γ_i be a map of $K[V]$ into K such that

$$\gamma_i(f) = \frac{\delta f}{\delta x_i}(v_0) \text{ for any } f \in K[V],$$

then the set $\{\gamma_0, \gamma_1,\dots,\gamma_n\}$ forms a K–basis of $T(V)_{v_0}$ and $\gamma_i(X_j) = \delta_{ij}$ for any $0 \le i,j \le n$ (see Example 4.5). Let $P(V)$ be the projective space defined by V (see §11) and $P_j(V) = \{K(c_0v_0 + \dots + c_nv_n) \mid c_j \neq 0\}$.

Let
$$\pi : V - \{0\} \to P(V)$$
be a map of $V - \{0\}$ into $P(V)$ which takes each $v \in V - \{0\}$ to $Kv \in P(V)$. Then π is a morphism of the open subvariety $V - \{0\}$ of V onto $P(V)$, because for each $0 \le j \le n$

$$\pi|_{V_{X_j}} : V_{X_j} \longrightarrow P_j(V)$$

is a morphism of affine varieties where V_{X_j} is the principal open subset of V defined by $X_j \in K[V]$. Further we have

$$\operatorname{Ker}(d\pi)_{v_0} = K\gamma_0 .$$

Proof of the last statement: $\operatorname{Ker}(d\pi)_{v_0} = K\gamma_0$.

Since V_{X_0} is an affine open subset of V and $\pi(V_{X_0}) \subset P_0(V)$,

$$(d\pi)_{v_0} = (d\pi_0)_{v_0} : T(V_{X_0})_{v_0} \longrightarrow T(P_0(V))_{Kv_0},$$

where $\pi_0 = \pi|_{V_{X_0}} : V_{X_0} \to P_0(V)$. Let $\iota : V_{X_0} \subset V$ be the inclusion map, then

$$\begin{aligned}(d\iota)_{v_0} : T(V_{X_0})_{v_0} &\longrightarrow T(V)_{v_0}\\ \gamma &\longrightarrow \gamma \circ \iota^*\end{aligned}$$

is bijective where ι^* is the comorphism of ι, i.e.,

$$\begin{aligned}\iota^* : K[V] &\subset K[V]_{X_0} .\\ \alpha &\longrightarrow \frac{\alpha}{1}\end{aligned}$$

We shall write γ_i' for the element in $T(V_{X_0})_{v_0}$ such that $\gamma_i = \gamma_i' \circ \iota^*$ where $0 \le i \le n$. Thus

$$T(V_{X_0})_{v_0} = K\gamma_0' \oplus K\gamma_1' \oplus \dots \oplus K\gamma_n' .$$

Since
$$\pi_0 : V_{X_0} \longrightarrow P_0(V)$$
$$v \longmapsto Kv$$

and
$$\pi_0^* : K\left[\frac{X_1}{X_0}, \frac{X_2}{X_0}, \ldots, \frac{X_n}{X_0}\right] \longrightarrow K[X_0,\ldots,X_n]_{X_0},$$
$$f \longmapsto f \circ \pi_0$$

we have
$$\pi_0^*\left(\frac{X_i}{X_0}\right) = \frac{X_i}{X_0} \text{ for any } 1 \leq i \leq n .$$

Thus
$$(d\pi_0)_{v_0}(c_0\gamma_0' + c_1\gamma_1' + \ldots + c_n\gamma_n') = (c_0\gamma_0' + c_1\gamma_1' + \ldots + c_n\gamma_n') \circ \pi_0^* = 0$$

if and only if
$$(c_0\gamma_0' + c_1\gamma_1' + \ldots + c_n\gamma_n')\left(\frac{X_i}{X_0}\right) = c_i = 0 \text{ for any } 1 \leq i \leq n .$$

Hence $\operatorname{Ker}(d\pi)_{v_0} = K\gamma_0$. Q.E.D.

<u>(21.5) Definition</u>. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be irreducible varieties over K . We call a dominant morphism $\varphi : X \to Y$ <u>separable</u> if $K(X)$ is separable over $K(Y)$ (see Definition 6.17).

<u>(21.6) Remark</u>. Let $\varphi : X \to Y$ be a dominant morphism of irreducible varieties X into Y .

(i) Since $K(X)$ is finitely generated over $K(Y)$, φ is spearable if and only if $K(X)$ is separably generated over $K(Y)$ (see Corollary 6.19).

(ii) If $\dim X = \dim Y$, then $[K(X):K(Y)] < \infty$. If φ is separable and $\dim X = \dim Y$, then $K(X)$ is separably algebraic over $K(Y)$ (see Exercise 22 on p.49).

<u>(21.7) Definition</u>. Let $E \supset L \supset k$ be a sequence of fields. We define a <u>k–linear derivation</u> D of L into E to be a k–linear map of L into E such that
$$D(xy) = xD(y) + yD(x) \quad \text{for any } x,y \in L$$
(c.f. Proposition 6.21). We write $\operatorname{Der}_k(L,E)$ for the set of all k–linear derivations of L into E . $\operatorname{Der}_k(L,E)$ becomes a vector space over E by the following operations
$$(D_1 + D_2)(x) = D_1(x) + D_2(x) \qquad (D_1, D_2 \in \operatorname{Der}_k(L,E) \text{ and } x \in L)$$
$$(aD)(x) = aD(x) \qquad (D \in \operatorname{Der}_k(L,E)\ , \ a \in E \text{ and } x \in L) .$$

<u>(21.8) Lemma</u>. Let X be an irreducible variety over K, then for any extension field E of $K(X)$, we have

$$\dim X = \dim_E \operatorname{Der}_K (K(X), E) .$$

<u>Proof</u>. Since K is perfect, $K(X)$ is separable over K (see Corollary 6.20) Since $K(X)$ is finitely generated over K, $K(X)$ is separably generated over K (see Corollary 6.19). Hence there exists a transcendence base $\{t_1, t_2,\dots,t_r\}$ of $K(X)$ over K such that $K(X)$ is separably algebraic over $K(t_1, t_2,\dots, t_r)$.

$$\begin{tikzcd}
E \\
\cup \\
K(X) \arrow[r, "D_i"] & E \\
\cup \\
K(t_1, \dots ,t_r) \arrow[uur, "d_i"']
\end{tikzcd}$$

Let d_i be a map of $K(t_1,\dots,t_r)$ into E such that

$$d_i \left(\frac{g(t_1,\dots,t_r)}{f(t_1,\dots,t_r)} \right) = \frac{\delta}{\delta t_i} \left(\frac{g(t_1,\dots,t_r)}{f(t_1,\dots,t_r)} \right) ,$$

where $g(t_1,\dots,t_r)$, $f(t_1,\dots,t_r) \in K[t_1,\dots,t_r]$ and $f(t_1,\dots,t_r) \neq 0$, then d_i is a K–linear derivation of $K(t_1,\dots,t_r)$ into E and $d_i(t_j) = \delta_{ij}$ for any $1 \leq i,j \leq r$. From Proposition 6.21 we can extend d_i to a derivation D_i of $K(X)$ into E . It is clear that $\{D_i \mid 1 \leq i \leq r\}$ are linearly independent over E .

Now let D be any fixed element of $\operatorname{Der}_K (K(X), E)$. Let $e_i = D(t_i)$ and $D_0 = D - \sum_{i=1}^{n} e_i D_i$, then $D_0 \in \operatorname{Der}_K (K(X), E)$ and $D_0 (K(t_1,\dots,t_r)) = 0$. Since $K(X)$ is separably algebraic and finitely generated over $K(t_1,\dots,t_r)$, $K(X) = \{K(t_1,\dots,t_r)\}[\alpha]$ for some $\alpha \in K(X)$. Let $F(T)$ be a monic minimal polynomial of α over $K(t_1,\dots,t_r)$, then we have

$$D_0(F(\alpha)) = \{D_0(\alpha)\} F'(\alpha) = 0 ,$$

where $F'(T)$ is the derivative of $F(T)$. Since α is separably algebraic over $K(t_1,t_2,\dots,t_r)$, we have $D_0(\alpha) = 0$. Thus

$$D_0 = 0 \text{ , i.e., } D = \sum_{i=1}^{n} e_i D_i .$$

Hence $\dim_E \operatorname{Der}_K (K(X), E) = r = \dim X$. Q.E.D.

<u>(21.9) Theorem</u>. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be irreducible varieties over K and $\varphi : X \to Y$ be a dominant morphism. Then there exists a simple point $x \in X$ such

that $\varphi(x)$ is also simple in Y. For such a pair of simple points x and $\varphi(x)$ if

$$(d\varphi)_x : T(X)_x \longrightarrow T(Y)_{\varphi(x)}$$

is surjective, then φ is separable.

<u>Proof</u> (see Humphreys [2, Theorem 5.5]). Since the set of all simple points of a given irreducible variety forms a non–empty open set from Theorem 7.18, there exists a pair of simple points $x \in X$ and $\varphi(x) \in Y$ from the assumption that φ is dominant. Similarly if $x \in X$ and $\varphi(x) \in Y$ are simple, there exist smooth affine open subvarieties U and V in X and Y respectively such that $x \in U$, $\varphi(x) \in V$ and $\varphi(U) \subset V$. Since the closure of $\varphi(U)$ in V is the intersection of the closure of $\varphi(U)$ in Y with V,

$$\varphi|_U : U \to V$$

is also dominant. Thus it is enough to prove the theorem in case X and Y are affine and smooth.

Let $\varphi^* : K[Y] \to K[X]$ be the comorphism of φ, then φ^* is injective (see Lemma 8.3). Hence we can consider $K(Y)$ as a subfield of $K(X)$ by φ^*. We shall assume that the characteristic of K is a positive prime, say p. Let L be a subfield of $K(X)$ containing K and

$$D : L \to K(X)$$

be a derivation, i.e.,

$$D(x+y) = D(x) + D(y) \text{ and } D(xy) = xD(y) + yD(x)$$

for any $x,y \in L$. Since $K^p = K$ and

$$D(a^p) = a^p \cdot D(1) + pD(a)a^{p-1} = 0$$

for any $a \in K$, D is always K–linear.

Let $n = \dim X$ and $d = \dim Y$. Let δ be the restriction map of $\mathrm{Der}_K(K(X), K(X))$ into $\mathrm{Der}_K(K(Y), K(X))$, i.e.,

$$\delta : \mathrm{Der}_K(K(X), K(X)) \longrightarrow \mathrm{Der}_K(K(Y), K(X)) .$$
$$D \longmapsto D|_{K(Y)}$$

From Proposition 6.23 $K(X)$ is separable over $K(Y)$ if δ is surjective.

$$\begin{tikzcd} K(X) \arrow[r, "D"] & K(X) \\ K(Y) \arrow[u, phantom, "\cup"] \arrow[ur, "\delta(D)"'] & \end{tikzcd}$$

Since δ is K(X)–linear and $\dim_{K(X)} \mathrm{Der}_K(K(X),K(X)) = n$ and $\dim_{K(X)} \mathrm{Der}_K(K(Y), K(X)) = d$ from Lemma 21.8, it is enough to show that $\dim_{K(X)} \mathrm{Ker}\,\delta = n-d$ for the surjectivity of δ. Since $\dim_{K(X)} \mathrm{Ker}\ \delta \geq n-d$ and $\mathrm{Ker}\ \delta = \mathrm{Der}_{K(Y)}(K(X), K(X))$, we shall show that any $n-d+1$ K(Y)–linear derivations $\{D_k\}$ of K(X) into itself are linearly dependend over K(X) .

Now let $\mathscr{O}_x$ be the local ring of x in K(X) . Let $a,s \in K[X]$ and $s \neq 0$. Since

$$D_k\,(s\cdot\tfrac{a}{s}) = D_k(a) = D_k(s)\cdot\tfrac{a}{s} + sD_k\,(\tfrac{a}{s})\ ,$$

we have $s^2\, D_k(\frac{a}{s}) = D_k(a)\cdot s - aD_k(s)$, which implies

$$D_k(\tfrac{a}{s}) = \frac{D_k(a)\cdot s\ - aD_k(s)}{s^2}\ .$$

Since K[X] is finitely generated as K–algebra, multiplying non–zero element of K[X] to D_k we can assume that

$$D_k\,(K[X]) \subset K[X]\ .$$

Thus we can assume that

$$D_k(\mathscr{O}_x) \subset \mathscr{O}_x \ \text{ for any }\ 1 \leq k \leq n-d+1\ .$$

From the definition of derivation, we have

$$D_k(\mathscr{M}_x^{\,r}) \subset \mathscr{M}_x^{\,r-1} \ \text{ for any }\ r \geq 2\ ,$$

where $\mathscr{M}_x$ is the maximal ideal of $\mathscr{O}_x$. Since

$$D_k(\mathscr{M}_x^{\,r} + z) \subset \mathscr{M}_x^{\,r-1} + D_k(z) \quad (z \in \mathscr{O}_x)\ ,$$

D_k is continuous on $\mathscr{O}_x$ by the $\mathscr{M}_x$–adic topology (see Definition 6.31). Let

$$\begin{aligned}\tilde{\psi} : \mathscr{M}_y/\mathscr{M}_y^{\,2} &\longrightarrow \mathscr{M}_x/\mathscr{M}_x^{\,2}\ , \text{ where } y = \varphi(x)\\ m+\mathscr{M}_y^{\,2} &\longrightarrow \psi(m) + \mathscr{M}_x^{\,2}\end{aligned}$$

and

$$\begin{aligned}\psi : \mathscr{O}_y &\longrightarrow \mathscr{O}_x \qquad\qquad (a,s \in K[Y] \text{ and } s \notin \mathscr{I}_Y(y))\ ,\\ a/s &\longrightarrow \varphi^*(a)/\varphi^*(s)\end{aligned}$$

then $\tilde{\psi}$ is injective, because

$$\begin{aligned}(d\varphi)_x : \mathrm{Hom}_K\,(\mathscr{M}_x/\mathscr{M}_x^{\,2}, K) &\longrightarrow \mathrm{Hom}_K\,(\mathscr{M}_y/\mathscr{M}_y^{\,2}, K)\\ \gamma &\longrightarrow \gamma\circ\tilde{\psi}\end{aligned}$$

is surjective.

Now let $f_1,\dots,f_t \in \mathscr{M}_y$ be a minimal set of generators of $\mathscr{O}_y$–module $\mathscr{M}_y$, then from Lemma 7.15 the set $\{f_1 + \mathscr{M}_y^{\,2},\dots, f_t + \mathscr{M}_y^{\,2}\}$ forms a K–basis for $\mathscr{M}_y/\mathscr{M}_y^{\,2}$. Since y is simple, we have $t = d$. Since $\tilde{\psi}$ is injective, we can extend $\{\psi(f_1) + \mathscr{M}_x^{\,2},\dots, \psi(f_d) + \mathscr{M}_x^{\,2}\}$ to a K–basis $\{\psi(f_1) + \mathscr{M}_x^{\,2},\dots, \psi(f_d) + \mathscr{M}_x^{\,2}\ ,\ f_{d+1} + \mathscr{M}_x^{\,2},\dots, f_n + \mathscr{M}_x^{\,2}\}$ of $\mathscr{M}_x/\mathscr{M}_x^{\,2}$. We shall identify f_i with $\psi(f_i)$ $(1 \leq i \leq d)$. Let R be a K–subalgebra of $\mathscr{O}_x$ generated by $\{f_1,\dots,f_n\}$. By induction on $l \geq 1$ we can show that for any l we have

$$\mathscr{O}_x = R + \mathscr{M}_x^{\,l}\ .$$

Hence R is dense in the $\mathscr{M}_x$–adic topology on $\mathscr{O}_x$.

Since $\dim_{K(X)} \mathrm{Der}_K (K(f_{d+1},\dots, f_n), K(X)) \le n-d$ from Lemma 21.8, we have

$$\left(\sum_{k=1}^{n-d+1} g_k D_k\right) (K[f_{d+1},\dots,f_n]) = 0 ,$$

where $K[f_{d+1},\dots,f_n]$ is the K–subalgebra of $\mathscr{O}_x$ generated by $\{f_{d+1},\dots,f_n\}$ and $K(f_{d+1},\dots,f_n)$ is the quotient field of $K[f_{d+1},\dots,f_n]$ and $\{g_k\} \subset K(X)$ and $\{g_k\} \neq \{0\}$. We can assume that $\{g_k\} \subset \mathscr{O}_x$. Since $D_{k,s}$ are $K(Y)$–linear, we have

$$(\Sigma\, g_k D_k)\ (R) = 0 .$$

Since R is dense in $\mathscr{O}_x$ and $\mathscr{O}_x$ is Hausdorff, we have

$$(\Sigma\, g_k D_k)\ (\overline{R}) = (\Sigma\, g_k D_k)\ (\mathscr{O}_x) \subset \overline{\{0\}} = \{0\} ,$$

i.e., $(\Sigma\, g_k D_k)\ (\mathscr{O}_x) = 0$. Hence $\Sigma\, g_k D_k = 0$ on $K(X)$ and $\{D_k\}$ are linearly dependent over $K(X)$. Q.E.D.

Exercise 60. Let $\{f_1,\dots,f_n\}$ be as in the proof of Theorem 21.9. Show that $\{f_1,\dots,f_n\}$ are algebraically independent over K .

22. Zariski's Main Theorem

Zariski's Main Theorem. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be irreducible varieties over K. Assume that Y is smooth. Let $\varphi: X \to Y$ be a surjective morphism of X onto Y such that the comorphism of φ induces an isomorphism of $K(Y)$ onto $K(X)$ (see p.143) and $|\varphi^{-1}(y)| < \infty$ for any $y \in Y$, then φ is an isomorphism of varieties.

For the proof of this theorem we need the following propositions.

(22.1) Definition. An element a of a commutative ring R is said to be irreducible if a is not a unit and is not a product of any two non–units of R. We call an integral domain R is a unique factorization domain if

(i) every non–unit of R is a product of finite irreducible elements, and
(ii) if the factorization in (i) is unique up to order and unit elements.

(22.2) Lemma. Let R be a Noetherian integral domain, then R satisfies the condition (i) in Definition 22.1.

Proof. Let $a \neq 0$ be a non–unit element of R. Assume that $a = a_1 a_2$ for some non–unit elements $a_1, a_2 \in R$. Since $Ra \subsetneqq Ra_1$ and R satisfies the ascending chain condition (see p.25), a has an irreducible divisor. Thus we have a sequence $\{b_n\}$ of elements in R such that

$$b_0 = a \quad \text{and} \quad b_{n-1} = b_n p_n ,$$

where p_n is irreducible. Since

$$Rb_0 \subset Rb_1 \subset Rb_2 \subset \ldots \subset Rb_n = Rb_{n+1} = \ldots$$

for some n, we have

$$a = p_1 p_2 \ldots p_n b_n$$

a product of irreducible elements. Q.E.D.

(22.3) Proposition. A Noetherian integral domain R is a unique factorization domain if and only if every prime ideal $\mathfrak{p}$ of height 1 in R is principal.

Proof (see Nagata [1, Theorem 13.1]). Assume that R is a unique factorization domain. Let $\mathfrak{p}$ be a prime ideal of height 1, that is, $\mathfrak{p}$ contains no prime ideals except $\{0\}$ and itself. Let a be an irreducible element of R contained in $\mathfrak{p}$. Since R is a unique factorization domain, Ra is a prime ideal. Hence $Ra = \mathfrak{p}$.

Conversely assume that every prime ideal of height 1 is principal. Let $a_1 \dots a_m = b_1 \dots b_n$ be factorizations of an element $c \in R$ as products of irreducible elements $\{a_i\}$ and $\{b_j\}$ (see Lemma 22.2). We show the uniqueness by induction on n. When $n = 1$, c is irreducible and the assertion holds. From Lemma 7.14 and the assumption an irreducible element a in R generates a prime ideal Ra, because any prime ideal of R which is minimal among prime ideals of R containing Ra is principal. Since

$$a_1 \dots a_m \in Rb_1 \text{ and } Rb_1 \text{ is prime,}$$

$a_i \in Rb_1$ for some i $(1 \leq i \leq m)$. We may assume that

$$a_1 \in Rb_1 \text{, i.e., } a_1 = ub_1 \text{ for some unit } u \in R .$$

Thus we have $ua_2 \dots a_m = b_2 \dots b_n$. Hence by induction the uniqueness holds.

Q.E.D.

Let $(X, \mathscr{O}_X)$ be an irreducible variety over K and x be a simple point of X. Let $\mathscr{O}_x$ be the local ring at x with unique maximal ideal $\mathscr{M}_x$. Let $r = \dim X$, then any minimal set of generators of $\mathscr{O}_x$–module $\mathscr{M}_x$ consists of r–elements from Lemma 7.15. Hence

$$\text{the height of } \mathscr{M}_x = r$$

from Proposition 7.10 and Theorem 7.17.

From Proposition 22.3 we have the following theorem. For the proof of this theorem see Nagata [1, Theorem 28.7].

(22.4) Theorem. Let R be a Noetherian local ring with unique maximal ideal $\mathscr{M}$ and r be the height of $\mathscr{M}$. Assume that R is an integral domain and any minimal set of generators of R–module $\mathscr{M}$ consists of r–element s. Then R is a unique factorization domain.

Now we shall prove the Zariski's Main Theorem.

Proof of the Zariski's Main Theorem. Let y be any fixed point of Y and V be an affine open neighbourhood of y in Y. Let $U \subset \varphi^{-1}(V)$ be an affine open subset of Y such that

$$U \cap \varphi^{-1}(y) \neq \emptyset .$$

Let $K[V] = \mathscr{O}_Y(V)$ and $K[U] = \mathscr{O}_X(U)$, then we have an embedding

$$\varphi^*:K[V] \subset K[U]$$
$$a \longrightarrow a \circ \varphi$$

and an isomorphism
$$\varphi^*:K(V) \longrightarrow K(U)$$
$$a/b \longrightarrow \varphi^*(a)/\varphi^*(b)$$

(see p.143). Let $\{f_1,f_2,\dots,f_r\}$ be a set of generators of $K[U]$ over $\varphi^*(K[V])$, i.e., $K[U] = \varphi^*(K[V])\,[f_1,\dots,f_r]$.

"Assume that for any $f \in K[U]$ there exists $g \in K[V]$ such that $g(y) \neq 0$ and $f \in \varphi^*(K[V]_g)$ (for the definition of $K[V]_g$ see Lemma 2.7)."

Let $g_i \in K[V]$ such that $g_i(y) \neq 0$ and $f_i \in \varphi^*(K[V]_{g_i})$ where $1 \leq i \leq r$. Let $g = g_1 g_2 \dots g_r$, then we have $f_i \in \varphi^*(K[V]_g)$ and

$$K[U]_{\varphi^*(g)} = \varphi^*(K[V]_g)\,[f_1,\dots,f_r]$$
$$= \varphi^*(K[V]_g) \, .$$

Let V' be the principal open set in V defined by g , i.e.,

$$V' = V_g \, ,$$

then $\varphi:U_{\varphi^*(g)} \to V'$ is an isomorphism of varieties from Lemma 2.4 and there exists a morphism

$$\psi:V' \longrightarrow U$$

such that $\varphi \circ \psi = 1_{V'}$, i.e., $\varphi\circ\psi(y') = y'$ for all $y' \in V'$. Since $\varphi(\psi(V')) = V'$,

$$\varphi^{-1}(V') \supset \psi(V') \, .$$

Thus we have the following morphisms between $\varphi^{-1}(V')$ and V' :

$$\varphi:\varphi^{-1}(V') \longrightarrow V' \quad \text{and} \quad \psi:V' \longrightarrow \varphi^{-1}(V') \, .$$

Since $\varphi \circ (\psi\circ\varphi) = (\varphi\circ\psi) \circ \varphi = \varphi$ on $\varphi^{-1}(V')$ and φ is injective on a dense open subset of X from Theorem 13.14, $\psi \circ \varphi = 1_{\varphi^{-1}(V')}$ on a dense open subset of $\varphi^{-1}(V')$. From Remark 10.10 we have $\psi \circ \varphi = 1_{\varphi^{-1}(V')}$ on $\varphi^{-1}(V')$. Hence for any $y \in Y$ there exists an affine open neighbourhood V' of y in Y such that φ is an isomorphism of varieties on $\varphi^{-1}(V')$.

Thus it is enough to show that for any $f \in K[U]$ there exists $g \in K[V]$ such that

$$g(y) \neq 0 \quad \text{and} \quad f \in \varphi^*(K[V]_g) \, .$$

Let $f \in K[U]$, then $f = \frac{\varphi^*(a)}{\varphi^*(b)}$ for some $a,b \in K[V]$ $(b \neq 0)$. Since $\mathscr{O}_y$ is a unique factorization domain from Theorem 22.4, we may assume that a and b are relatively prime in $\mathscr{O}_y$. We have to show that

$$b(y) \neq 0$$

(then take $g = b$). Therefore we assume that $b(y) = 0$, and lead to a contradiction. Let $\beta \in \mathscr{O}_y$ be an irreducible factor of b such that $b = \beta \cdot b'$. Let W be an affine open subset of Y such that

$$y \in W \subset V \text{ and } \beta, b' \in \mathscr{O}_Y(W).$$

Let E be an irreducible component of

$$\{x \in \varphi^{-1}(W) \mid \varphi^*(\beta)(x) = 0\}$$

such that $E \cap \varphi^{-1}(y) \neq \emptyset$. Since φ is surjective, E is non–empty and $\dim E = \dim X - 1$ (see Exercise 14 on p.32 and Theorem 7.8). Since

$$\varphi^*(a) = f \cdot \varphi^*(b) = f \cdot \varphi^*(b') \cdot \varphi^*(\beta),$$

we have $$\varphi(E) \subset \{w \in W \mid a(w) = 0 \text{ and } \beta(w) = 0\}.$$

Let Z be an irreducible component of $\{w \in W \mid a(w) = \beta(w) = 0\}$ containing $\varphi(E)$. Then $\varphi|_E$ is a morphism of E into Z.

Let $\mathfrak{p} = \beta \mathscr{O}_y \cap K[W]$, then $\mathfrak{p}$ is a prime ideal in K[W] and $a \notin \mathfrak{p}$. We shall show that

$$\mathscr{I}_{K[W]}(Z) \supset \mathfrak{p}.$$

Let $\mathfrak{q}$ be a prime ideal of K[W] which is contained in $\mathscr{I}_{K[W]}(Z)$ and is minimal among prime ideals containing $\beta K[W]$ (see Lemma 7.13). Let $S = K[W] - \mathscr{I}_{K[W]}(y)$' then we have $S^{-1}\beta K[W] \subset S^{-1}\mathfrak{q}$. Since

$$\beta K[W] \subset \mathfrak{p} = (S^{-1}\beta K[W]) \cap K[W] \subset S^{-1}\mathfrak{q} \cap K[W] = \mathfrak{q},$$

we have $\mathfrak{p} = \mathfrak{q} \subset \mathscr{I}_{K[W]}(Z)$.

Thus we have got a strictly descending sequence of prime ideals

$$\mathscr{I}_{K[W]}(Z) \supsetneqq \mathfrak{p} \supsetneqq \{0\}.$$

Hence we have a strictly increasing sequence of closed irreducible sets

$$Z \subsetneqq \mathscr{V}_W(\mathfrak{p}) \subsetneqq W,$$

which implies $$\dim Z \leqq \dim Y - 2.$$

From Corollary 13.8 we have

$$\dim (\varphi|_E)^{-1}(w) \geqq 1 \text{ for any } w \in \varphi(E),$$

contradicting to the fact that $|\varphi^{-1}(w)| < \infty$ for any $w \in Y$. Q.E.D.

23. Quotient spaces of linear algebraic groups

We shall construct the quotient space G/H in case $(G, K[G])$ is a linear algebraic group over K and $(H, K[H])$ is its closed subgroup.

(23.1) Lemma. Let $(G, \mathscr{A}_G)$ be an algebraic group over K and V be a finite dimensional rational left KG–module, i.e.,

$$\begin{array}{rl} \varphi : G \longrightarrow & GL(V) \qquad (v \in V) \\ g \longrightarrow & [\,\varphi(g): v \to gv] \end{array}$$

is a rational representation.

Let $d \leq \dim_K V$, then

(i) $\Lambda^d V$ is also a finite dimensional rational left KG–module under the following operation:

$$\begin{array}{c} G \times \Lambda^d\, V \longrightarrow \Lambda^d\, V \\ (g, v_1\wedge \ldots \wedge v_d) \to gv_1 \wedge \ldots \wedge gv_d \,. \end{array}$$

We write $\Lambda^d\, \varphi$ for this representation of G into $GL(\Lambda^d\, V)$;

(ii) If G is a linear algebraic group with Lie algebra $\mathfrak{g}$, then

$$d(\Lambda^d\, \varphi)\,(\gamma)\,(v_1 \wedge \ldots \wedge v_d) = \sum_{i=1}^{d} v_1 \wedge \ldots \wedge (d\varphi)\,(\gamma)\, v_i \wedge \ldots \wedge v_d \,,$$

where $\gamma \in \mathfrak{g}$ and $v_1 \wedge \ldots \wedge v_d \in \Lambda^d\, V$.

Proof. (i) Since the map $\begin{array}{rl} g_L : V \times \ldots \times V \longrightarrow & \Lambda^d\, V \\ (v_1, \ldots, v_d) \to & gv_1 \wedge \ldots \wedge gv_d \end{array}$ is K–multilinear and alternating, $g(v_1 \wedge \ldots \wedge v_d) = gv_1 \wedge \ldots \wedge gv_d$ is a well–defined left operation of G on $\Lambda^d V$.

$$\begin{tikzcd}
 & v_1 \wedge \ldots \wedge v_d & \\
 & & \Lambda^d\, V \arrow[dd] \\
(v_1, \ldots, v_d) \in V \times \ldots \times V \arrow[uur] \arrow[ur] \arrow[dr, "g_L"'] \arrow[ddr] & & \\
 & & \Lambda^d\, V \\
 & gv_1 \wedge \ldots \wedge gv_d &
\end{tikzcd}$$

(commutative diagram)

Let $\{m_1,\ldots,m_n\}$ be a K–basis of V, then

$$\{m_{i_1} \wedge\ldots\wedge m_{i_d} \mid i_1 < i_2 <\ldots< i_d\}$$

forms a K–basis of $\wedge^d V$ (see Proposition 11.9). Let

$$g(m_1,\ldots,m_n) = (m_1,\ldots,m_n) \begin{bmatrix} \varphi_{11}(g), \varphi_{12}(g),\ldots, \varphi_{1n}(g) \\ \varphi_{21}(g),\ldots\ldots\ldots, \varphi_{2n}(g) \\ \vdots \\ \varphi_{n1}(g),\ldots\ldots\ldots, \varphi_{nn}(g) \end{bmatrix},$$

where $\varphi_{ij} \in \mathscr{O}_G(G)$ for any $1 \le i,j \le n$. Since

$$g(m_1 \wedge\ldots\wedge m_d) = gm_1 \wedge\ldots\wedge gm_d = \left(\sum_{i=1}^{n} \varphi_{i_1}(g)\, m_i\right) \wedge\ldots\wedge \left(\sum_{i=1}^{n} \varphi_{i_d}(g)\, m_i\right)$$

and only products and sums of $\{\varphi_{ij} \mid 1 \le i \le n,\ 1 \le j \le d\}$ appear as coefficients of the linear combination of $\{m_{i_1} \wedge\ldots\wedge m_{i_d} \mid i_1 < i_2 <\ldots< i_d\}$ which expresses $g(m_1 \wedge\ldots\wedge m_d)$, $\wedge^d V$ is also a rational KG–module.

(ii) Let $\mathfrak{a}_d$ be the K–subspace of $T^d(V) = \underbrace{V \otimes\ldots\otimes V}_{d}$ generated by all elements of the form

$$x_1 \otimes\ldots\otimes x_d \in V \otimes\ldots\otimes V$$

where $x_i = x_j$ for some $i \ne j$. Clearly $\mathfrak{a}_d$ is a KG–submodule of $T^d(V)$ (see Proposition 18.2). Hence

$$\begin{aligned} \wedge^d \varphi : G &\longrightarrow GL(T^d(V)/\mathfrak{a}_d) \\ g &\longrightarrow [v_1 \otimes\ldots\otimes v_d + \mathfrak{a}_d \to \otimes^d \varphi(g)(v_1 \otimes\ldots\otimes v_d) + \mathfrak{a}_d] \end{aligned}$$

where

$$\begin{aligned} \otimes^d \varphi : G &\longrightarrow G \times\ldots\times G \longrightarrow GL(T^d(V)) \\ g &\longrightarrow (\underbrace{g,\ldots,g}_{d}) \longrightarrow \underbrace{\varphi(g) \otimes\ldots\otimes \varphi(g)}_{d}. \end{aligned}$$

Since $d(\wedge^d \varphi) = (d(\otimes^d \varphi))^\sim$, where

$$\begin{aligned} (d(\otimes^d \varphi))^\sim : \mathfrak{g} &\longrightarrow \mathfrak{gl}(T^d(V)/\mathfrak{a}_d) \\ \gamma &\longrightarrow [v + \mathfrak{a}_d \to d(\otimes^d \varphi)(\gamma)v + \mathfrak{a}_d], \end{aligned}$$

from Exercise 61, we have

$$d(\wedge^d \varphi)(\gamma)(v_1 \wedge\ldots\wedge v_d) = \sum_{i=1}^{d} v_1 \wedge\ldots\wedge (d\varphi)(\gamma)v_i \wedge\ldots\wedge v_d$$

$(\gamma \in \mathfrak{g},\ v_1 \wedge\ldots\wedge v_d \in \wedge^d V)$ from Lemma 20.3 and Proposition 20.8. Q.E.D.

<u>Exercise 61</u>. Let $(G, K[G])$ be a linear algebraic group over K and $(H, K[H])$ be its closed subgroup. Let $\varphi: G \to GL(V)$ be a finite dimensional rational representation of G over K and let W be a $\varphi(H)$–invariant subspace of V. Let

$$\tilde{\varphi} : H \longrightarrow GL(V/W)$$
$$h \longrightarrow \tilde{\varphi}(h) : v + W \to hv + W$$

be a rational representation of H defined by KH–module V/W (see Proposition 18.2). Then show that

$$d\tilde{\varphi} = (d\varphi)^{\sim}$$

where
$$(d\varphi)^{\sim} : \mathscr{H} \longrightarrow \mathfrak{gl}(V/W)$$
$$\gamma \longrightarrow (d\varphi)^{\sim}(\gamma) : v + W \to (d\varphi)(\gamma)(v) + W$$

and $\mathscr{H}$ is a Lie algebra of H .

(23.2) Proposition. Let (G, K[G]) be a linear algebraic group over K with Lie algebra $\mathfrak{g}$ and (H, K[H]) be a closed subgroup of G with Lie algebra $\mathscr{H}$. Then there exist a finite dimensional left KG–submodule V of K[G] (see Lemma 18.4 and Proposition 18.5) together with a subspace W of V such that

$$H = \{g \in G \mid g*W = W\}$$

and
$$\mathscr{H} = \{\gamma \in \mathfrak{g} \mid (d\varphi)(\gamma)W \subset W\}$$

where
$$\varphi : G \longrightarrow GL(V)$$
$$g \longrightarrow [\varphi(g) : v \to g*v] \quad (v \in V) .$$

Further let $d = \dim_K W$, then for any $g \in G$ we have $g*W = W$ if and only if $g*(\Lambda^d W) = \Lambda^d W$, and for any $\gamma \in \mathfrak{g}$ we have $W*\gamma \subset W$ i.e. $(d\varphi)(\gamma)W \subset W$ if and only if $(d\Lambda^d \varphi)(\gamma)\Lambda^d W \subset \Lambda^d W$ (see Proposition 20.7).

Proof. Let $\mathscr{I}(H)$ be the ideal of functions of K[G] vanishing on H . Since K[G] is Noetherian, $\mathscr{I}(H)$ has a finite set of generators $f_1, f_2, \ldots, f_r$ as ideal. Let

$$V = KG*f_1 + \ldots + KG*f_r ,$$

then from Corollary 18.6 V is finite dimensional. Let

$$W = V \cap \mathscr{I}(H) ,$$

then we have $h*W = W$ for any $h \in H$.

Conversely assume that $g*W = W$ for some $g \in G$, then

$$g*f_i \in \mathscr{I}(H) \text{ for any } 1 \le i \le r .$$

Hence $g*\mathscr{I}(H) \subset \mathscr{I}(H)$, which implies $g \in H$ from Proposition 1.7. Therefore, we have

$$H = \{g \in G \mid g*W = W\} .$$

Now let $\gamma \in \mathscr{H}$, then from Lemma 20.5 we have

$$\gamma \in \{\gamma \in \mathfrak{g} \mid d\varphi(\gamma)\, W \subset W\} .$$

Assume that

$$d\varphi(\gamma)W \subset W ,$$

i.e., $W*\gamma \subset W$ for some $\gamma \in \mathfrak{g}$. Since

$$\mathscr{I}(H)*\gamma = (WK[G])*\gamma \subset (W*\gamma)K[G] + W(K[G]*\gamma)$$

(see Lemma 19.10), we have

$$\mathscr{I}(H)*\gamma \subset \mathscr{I}(H) .$$

Hence $f*\gamma(1) = \gamma(f) = 0$ for any $f \in \mathscr{I}(H)$. Therefore, we have $\gamma \in \mathscr{H}$ from Proposition 19.15. Hence we have shown that

$$\mathscr{H} = \{\gamma \in \mathfrak{g} \mid d\varphi(\gamma)W \subset W\} .$$

By definition $g*(\wedge^d W) = \wedge^d W$ if $g*W = W$ $(g \in G)$ (see Lemma 23.1). Assume that $g*(\wedge^d W) = \wedge^d W$ for some $g \in G$. Let $\{v_1,\ldots,v_d,\ldots,v_{d+t-1}\}$ be a K–basis of $W + g*W$ such that $\{v_1,\ldots,v_d\}$ forms a K–basis of W and $\{v_t,\ldots,v_d\}$ forms a K–basis of $W \cap g*W$ and $\{v_t, v_{t+1},\ldots,v_d,\ldots,v_{d+t-1}\}$ forms a K–basis of $g*W$. Since $g*(v_1 \wedge\ldots\wedge v_d) = cv_t \wedge\ldots\wedge v_{d+t-1}$ for some $c \in K - \{0\}$ and $cv_t \wedge\ldots\wedge v_{d+t-1} \in Kv_1 \wedge\ldots\wedge v_d$ and $\{v_1 \wedge\ldots\wedge v_d, v_t \wedge\ldots\wedge v_{d+t-1}\}$ form a part of a K–basis of $\wedge^d V$, we have $t = 1$. Hence $g*W = W$.

Since

$$d(\wedge^d \varphi)(\gamma)(w_1 \wedge\ldots\wedge w_d) = \sum_{i=1}^{d} w_1 \wedge\ldots\wedge (d\varphi)(\gamma)w_i \wedge\ldots\wedge w_d$$

from Lemma 23.1 where $w_1,\ldots,w_d \in V$, we have

$$d(\wedge^d \varphi)(\gamma) \wedge^d W \subset \wedge^d W$$

if $(d\varphi)(\gamma) W \subset W$. Assume that

$$d(\wedge^d \varphi)(\gamma) \wedge^d W \subset \wedge^d W$$

for some $\gamma \in \mathfrak{g}$. Let $\{v_1,\ldots,v_d,\ldots,v_n\}$ be a K–basis of V and

$$(d\varphi)(\gamma)v_i = \sum_{k=1}^{n} c_{ki} v_k .$$

Then
$$d(\wedge^d \varphi)(\gamma)(v_1 \wedge\ldots\wedge v_d) = \sum_{i=1}^{d} v_1 \wedge\ldots\wedge (d\varphi)(\gamma) v_i \wedge\ldots\wedge v_d$$

$$= \sum_{i=1}^{d}\sum_{k=1}^{n} c_{ki} v_1 \wedge\ldots\wedge v_{i-1} \wedge v_k \wedge v_{i+1} \wedge\ldots\wedge v_d \in Kv_1 \wedge\ldots\wedge v_d .$$

Hence $c_{ki} = 0$ if $k > d$. Therefore, $(d\varphi)(\gamma)W \subset W$. Q.E.D.

<u>(23.3) Corollary</u> (C. Chevalley). Let $(G, K[G])$ be a linear algebraic group over K and $(H, K[H])$ be a closed subgroup of G. Let $\mathfrak{g}$ and $\mathscr{H}$ be the Lie algebras of G and H respectively. Then there exists a finite dimensional rational left KG–module V with one dimensional subspace $L \subset V$ such that

$$H = \{g \in G \mid \varphi(g)L = L\}$$

and
$$\mathscr{H} = \{\gamma \in \mathfrak{g} \mid (d\varphi)(\gamma)L \subset L\}$$

where
$$\begin{aligned} \varphi : G &\longrightarrow GL(V) \\ g &\longrightarrow [\varphi(g):v \to gv] \quad (v \in V) . \end{aligned}$$

Now let G, H, V and L be as in Corollary 23.3. Let $P(V)$ be the projective space defined by V (see §11). Let

$$\pi : V - \{0\} \to P(V)$$

be a map of $V - \{0\}$ into $P(V)$ which takes each $v \in V - \{0\}$ to $Kv \in P(V)$, then π is a morphism of the open subvariety $V - \{0\}$ into $P(V)$ (see Example 21.4). Let

$$\begin{aligned} G \times P(V) &\longrightarrow P(V) \\ (g, \pi(v)) &\longrightarrow \pi(gv) , \end{aligned}$$

then G operates on $P(V)$ morphically by the following Lemma.

<u>(23.4) Lemma</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and $\varphi : G \to GL(V)$ be a finite dimensional rational representation. Let $\begin{aligned} \pi : V - \{0\} &\to P(V) \\ v &\longrightarrow Kv \end{aligned}$ be the morphism as above, then by the map

$$\begin{aligned} \psi : G \times P(V) &\longrightarrow P(V) \\ (g, \pi(v)) &\longrightarrow \pi(gv) \end{aligned}$$

$P(V)$ becomes a G–variety (see Definition 17.1), where $gv = \varphi(g)v$.

<u>Proof</u>. Let $V' = V - \{0\}$ and $\begin{aligned} \varphi' : G \times V' &\longrightarrow V' \\ (g,v) &\longrightarrow gv \end{aligned}$, then φ' is also a morphism of varieties and we have the following commutative diagram.

$$\begin{array}{ccc} G \times V' & \xrightarrow{\varphi'} & V' \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times P(V) & \xrightarrow[\psi]{} & P(V) \end{array}$$

Let $\{v_0, v_1, \dots, v_n\}$ be a K–basis of V . Since

$$\begin{aligned} \rho_j : \; G \times P_j(V) &\longrightarrow G \times V' \\ (g, K(c_0v_0 + \dots + c_nv_n)) &\to (g, \frac{c_0}{c_j} v_0 + \dots + v_j + \dots + \frac{c_n}{c_j} v_n) \end{aligned}$$

is a morphism of varieties and $\pi \circ \varphi' \circ \rho_j = \psi|_{G\times P_j(V)}$ where $0 \leq j \leq n$, ψ is a morphism of varieties. Q.E.D.

Let $X = G{\cdot}x$ where $x = L \in P(V)$, then from Proposition 17.8 X is open in $\overline{X}$. Hence X is a quasi–projective variety over K (see Definition 11.5). Thus we have got a quasi–projective homogeneous space X of G (see Definition 17.9) with certain point $x \in X$ such that

$$H = \{g \in G \mid gx = x\} .$$

Further we have

(23.5) Proposition. Let G, H, V and L be as in Corollary 23.3. Let $P(V)$ be the projective space defined by V and $x = L \in P(V)$. Let X be the quasi–projective homogeneous space of G as above, then

(i) $(d\varphi_x)_1 : \mathscr{g} \longrightarrow T(X)_x$ is surjective where $\begin{array}{l} \varphi_x : G \longrightarrow X \\ \qquad\; g \longrightarrow g\cdot x \end{array}$;

(ii) the map $\begin{array}{l} \psi : G^0 \longrightarrow G^0\cdot x \\ \qquad\; g \longrightarrow g\cdot x \end{array}$ is a separable morphism where G^0 is the irreducible component of G containing 1.

Proof. Since $\begin{array}{l} \varphi_x : G \longrightarrow X \\ \qquad\; g \longrightarrow g\cdot x \end{array}$ is a morphism of varieties and G^0 is closed in G, the map

$$\begin{array}{ccccc} G^0 & \subset & G & \xrightarrow{\varphi_x} & X \\ g & \longrightarrow & g & \longrightarrow & g\cdot x \end{array}$$

is also a morphism. Hence from Exercise 40 on p.113 the map $\begin{array}{l} G^0 \longrightarrow \overline{G^0\cdot x} \\ g \;\longrightarrow g\cdot x \end{array}$ is a morphism. From the proof of Lemma 17.12 $G^0{\cdot}x$ is open and closed in X . Hence the map

$$\begin{array}{l} \psi : G^0 \longrightarrow G^0\cdot x \\ \qquad\; g \longrightarrow g\cdot x \end{array}$$

is a morphism.

Since G^0 is open in G and $\varphi_x|_{G^0} = \psi$,

$$T(G)_1 = T(G^0)_1 \; , \;\; T(X)_x = T(G^0{\cdot}x)_x \;\text{ and }\; (d\varphi_x)_1 = (d\psi)_1 \, .$$

From Theorem 13.14 there exists a point $g'{\cdot}x$ in $G^0{\cdot}x$ such that

$$\dim \psi^{-1}(g'{\cdot}x) = \dim G^0 - \dim G^0{\cdot}x \; ,$$

where $g' \in G^0$.

Since $\psi^{-1}(g'{\cdot}x) = \{g \in G^0 \mid g{\cdot}x = g'{\cdot}x\} = g'(G^0 \cap H)$ and $H \supset H \cap G^0 \supset H^0$ where H^0 is the irreducible component of H containing 1, we have

$$\dim \psi^{-1}(g'{\cdot}x) = \dim G^0 \cap H = \dim H \, .$$

Thus

$$\dim_K T(G)_1 = \dim_K T(H)_1 + \dim_K T(X)_x \, .$$

(i) From the above argument it is enough to show that $\mathrm{Ker}(d\varphi_x)_1 = \mathscr{H}$. Let $L = Kv$ for some $v \in L - \{0\}$ and $\{v = v_0, v_1, \ldots, v_n\}$ be a K–basis of V. Then the map

$$\begin{aligned} \rho : GL(V) &\longrightarrow V - \{0\} \\ f &\longmapsto f(v) \end{aligned}$$

is a morphism of varieties and

$$\begin{aligned} (d\rho)_1 : \mathscr{gl}(V) &\longrightarrow T(V)_v \\ \gamma &\longmapsto \gamma(v) \, . \end{aligned}$$

Since

$$\begin{array}{lllllll} \varphi_x : G & \xrightarrow{\varphi} & GL(V) & \xrightarrow{\rho} & V\text{-}\{0\} & \xrightarrow{\pi} & X \\ \quad\;\; g & \longmapsto & \varphi(g) & \longmapsto & gv & \longmapsto & \pi(gv) = g\cdot x \end{array}$$

and

$$(d\varphi_x)_1 : \mathscr{g} \xrightarrow{(d\varphi)_1} \mathscr{gl}(V) \xrightarrow{(d\rho)_1} T(V)_v \xrightarrow{(d\pi)_v} T(X)_x \, ,$$

we have

$$\begin{aligned} \mathrm{Ker}\,(d\varphi_x)_1 &= (d(\rho\circ\varphi)_1)^{-1}\,(\mathrm{Ker}\,(d\pi)_v) \\ &= (d(\rho\circ\varphi)_1)^{-1}\,(L) && \text{(see Example 21.4)} \\ &= \mathscr{H} && \text{(see Corollary 23.3).} \end{aligned}$$

(ii) is clear from Theorem 21.9. Q.E.D.

<u>Exercise 62</u>. Verify the statement $\begin{aligned} (d\rho)_1 : \mathscr{gl}(V) &\longrightarrow T(V)_v \\ \gamma &\longmapsto \gamma(v) \end{aligned}$ in the above proof.

Now we define the quotient space G/H.

<u>(23.6) Definition</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and $(H, \mathscr{O}_H)$ be a closed subgroup of G. We call a pair $(G/H, a)$ of a homogeneous space G/H of G and a point a in G/H whose isotropy group is H <u>a quotient of G by H</u> if for any pair (Y, b) of a homogeneous space Y of G and a point $b \in Y$ whose isotropy group contains H there exists a unique G–morphism φ of G/H into Y such that $\varphi(a) = b$.

$$\begin{tikzcd} & g\cdot a \in G/H \arrow[d, "\exists_1 \varphi"] \\ g \in G \arrow[ur] \arrow[dr] & \\ & g\cdot b \in Y \end{tikzcd}$$

<u>(23.7) Theorem</u>. Let $(G, K[G])$ be a linear algebraic group over K and $(H, K[H])$ be a closed subgroup of G. Let V be a finite dimensional rational left KG–module with one dimensional subspace $L \subset V$ such that

$$H = \{g \in G \mid gL = L\} .$$

Let $\pi : \begin{array}{l} V - \{0\} \longrightarrow P(V) \\ v \longrightarrow Kv \end{array}$ and $x = L$. Let $X = G{\cdot}x$ where G operates on $P(V)$ as follows

$$\begin{array}{l} G \times P(V) \longrightarrow P(V) \\ (g, \pi(v)) \longrightarrow \pi(gv) . \end{array}$$

Then the pair (X, x) is a quotient of G by H and is unique up to G–isomorphism of G–varieties.

Proof. From the definition of quotient the uniqueness is clear. Let $\nu : G \to G/H$ and $(G/H, \mathscr{O})$ be as in p.218 and let (Y, b) be as in Definition 23.6. Let

$$\varphi : \begin{array}{l} G/H \longrightarrow Y \\ gH \longrightarrow gb , \end{array}$$

then φ is a well defined G–map. Let

$$\varphi_b : \begin{array}{l} G \longrightarrow Y \\ g \longrightarrow g{\cdot}b \end{array} \quad \text{(see Remark 17.2),}$$

then we have the following commutative diagram

$$\begin{tikzcd} & G/H \arrow[dd, "\varphi"] \\ G \arrow[ur, "\nu"] \arrow[dr, "\varphi_b"'] & \\ & Y \end{tikzcd}$$

Let O be an open set in Y , then

$$\begin{aligned} \varphi^{-1}(O) &= \{gH \mid g \in G \text{ and } g{\cdot}b \in O\} \\ &= \nu(\{g \in G \mid g{\cdot}b \in O\}) \\ &= \nu(\varphi_b^{-1}(O)) . \end{aligned}$$

Hence $\varphi^{-1}(O)$ is open in G/H .

$$\begin{tikzcd} G/H \arrow[r, "\varphi"] & Y & \\ \cup & \cup & \\ \varphi^{-1}(O) \arrow[r] & O \arrow[r, "f"] & K \\ G \supset \varphi_b^{-1}(O) \arrow[u, "\nu"] \arrow[ur, "\varphi_b"'] & & \end{tikzcd}$$

Since $\nu^{-1}(\varphi^{-1}(O)) = \varphi_b^{-1}(O)$ and $f{\circ}(\varphi_b \mid_{\varphi_b^{-1}(O)}) \in \mathscr{O}_G(\varphi_b^{-1}(O))$ for any $f \in \mathscr{O}_Y(O)$, we have

$$f \circ (\varphi \mid_{\varphi^{-1}(O)}) \in \mathscr{O}(\varphi^{-1}(O)) .$$

Therefore φ is a morphism of ringed space.

Now we shall show that if (Y, b) were (X, x) then $\varphi : \begin{array}{l} G/H \longrightarrow X \\ gH \longrightarrow g\cdot x \end{array}$ would be an isomorphism of ringed spaces. Clearly φ is bijective. Let U be an open set in G/H. Since $\varphi(U) = \varphi_x(\nu^{-1}(U))$ and φ_x is an open map from Lemma 17.12, $\varphi(U)$ is open in X. Hence φ is a homeomorphism.

$$\begin{tikzcd} & gH \in G/H \arrow[dd, "\varphi"] \\ g \in G \arrow[ur, "\nu"] \arrow[dr, "\varphi_x"'] & \\ & g\cdot x \in X \end{tikzcd}$$

(commutative diagram)

Finally we shall show that φ^{-1} is also a morphism of ringed spaces. Let O be an open set in X. If the map

$$\Phi : \begin{array}{l} \mathscr{O}_X(O) \longrightarrow \mathscr{O}(\varphi^{-1}(O)) \\ \qquad f \longrightarrow f\circ\varphi|_{\varphi^{-1}(O)} \end{array}$$

is a K–algebra isomorphism, then $\varphi^{-1} : X \to G/H$ is a morphism of ringed spaces. Since the map Φ is clearly an injective K–algebra homomorphism, it is enough to show that Φ is surjective, that is, for any

$$\hat{f} \in \mathscr{O}(\varphi^{-1}(O))$$

there exists $F \in \mathscr{O}_X(O)$ such that

$$\hat{f} = F \circ \varphi|_{\varphi^{-1}(O)} .$$

Now let

$$\mathscr{O}_G(\varphi_x{}^{-1}(O))^H$$

$$= \{f \in \mathscr{O}_G(\varphi_x{}^{-1}(O)) \mid f(gh) = f(g) \text{ for any } g \in \varphi_x{}^{-1}(O) \text{ and } h \in H\} .$$

(Notice that $\varphi_x{}^{-1}(O) = \nu^{-1}(\varphi^{-1}(O))$ and $\varphi_x{}^{-1}(O)h = \varphi_x{}^{-1}(O)$ for any $h \in H$.) Then we have got the following bijective correspondence:

$$\begin{array}{ccc} \mathscr{O}(\varphi^{-1}(O)) = \{\hat{f} : \varphi^{-1}(O) \to K \mid \hat{f}\circ\nu|_{\varphi_x{}^{-1}(O)} \in \mathscr{O}_G(\varphi_x{}^{-1}(O))\} & \longrightarrow & \mathscr{O}_G(\varphi_x{}^{-1}(O))^H \\ \hat{f} & \longrightarrow & \hat{f}\circ\nu \end{array}$$

Thus we only have to show that for any $f \in \mathscr{O}_G(\varphi_x{}^{-1}(O))$ such that $f(gh) = f(g)$ ($g \in \varphi_x{}^{-1}(O)$ and $h \in H$) there exists $F \in \mathscr{O}_X(O)$ such that

$$f = F \circ \varphi_x \text{ on } \varphi_x{}^{-1}(O) .$$

Let G^0 be the connected component of G containing 1, then $G = \bigcup_{i=1}^{t} G^0 g_i$ (disjoint union of open subsets) and

$$X = G \cdot x = \bigcup_{i=1}^{t} G^0 g_i x = G^0 g_{i_1} x \cup \ldots \cup G^0 g_{i_s} x$$

(disjoint union of G^0–orbits in X, $s \leqq t$). Then $G^0 g_{i_1} x, \ldots, G^0 g_{i_s} x$ are the irreducible components of X and each $G^0 g_{i_j} x$ is open and closed in X.

Now let $U_i = \varphi_x^{-1}(O) \cap G^0 g_i$ $(1 \leq i \leq t)$, then

$$\varphi_x^{-1}(O) = \bigcup_{i=1}^{t} (\varphi_x^{-1}(O) \cap G^0 g_i) = \bigcup_{i=1}^{t} U_i .$$

Let $f_i = f\,|_{U_i}$, then $f \in \mathscr{O}_G(\varphi_x^{-1}(O))$ if and only if

$$f_i \in \mathscr{O}_G(U_i) \text{ for any } 1 \leq i \leq t .$$

Let φ_i be a map of G^0 into $G^0 g_i x$ which takes each $g \in G^0$ to $g_i gx$, i.e.,

$$\begin{aligned} \varphi_i : G^0 &\longrightarrow g_i G^0 g_i^{-1} \longrightarrow G^0 g_i x \\ g &\longrightarrow g_i\ g g_i^{-1} \longrightarrow g_i g g_i^{-1} g_i x . \end{aligned}$$

Since
$$\begin{aligned} g_i gx \in 0 \quad &\Leftrightarrow g_i g \in \varphi_x^{-1}(O) \\ &\Leftrightarrow g \in g_i^{-1} \varphi_x^{-1}(O) \\ &\Leftrightarrow g \in g_i^{-1} U_i \ (= g_i^{-1} \varphi_x^{-1}(O) \cap G^0) \end{aligned}$$
for any $g \in G^0$, we have

$$\varphi_i^{-1}(O) = g_i^{-1} U_i \quad (1 \leq i \leq t) .$$

Let $H^0 = H \cap G^0$ and

$$\begin{aligned} f_i^0 : \varphi_i^{-1}(O) &\longrightarrow K \\ g &\longrightarrow f_i(g_i g) \end{aligned}$$

where $1 \leq i \leq t$, then $f_i^0 \in \mathscr{O}_G(\varphi_i^{-1}(O)) = \mathscr{O}_{G^0}(\varphi_i^{-1}(O))$ (see Exercise 37 on p.111) and

$$f_i^0(gh) = f_i(g_i gh) = f(g_i gh) = f(g_i g) = f_i^0(g)$$

for any $g \in \varphi_i^{-1}(O)$ and $h \in H^0$. Thus if we showed that for each $f_i^0 \in \mathscr{O}_{G^0}(\varphi_i^{-1}(O))$ such that $f_i^0(gh) = f_i^0(g)$ $(g \in \varphi_i^{-1}(O)$ and $h \in H^0)$ there exists

$$F_i \in \mathscr{O}_{G^0 g_i x}(O \cap G^0 g_i x)$$

such that $f_i^0 = F_i \circ \varphi_i$ on $\varphi_i^{-1}(O)$, we should have

$$f_i = F_i \circ \varphi_x$$

on each $\varphi_x^{-1}(O) \cap G^0 g_i$ and the function $F \in \mathscr{O}_X(O)$ such that

$$F\,|_{O \cap G^0 g_i x} = F_i$$

for each $1 \leq i \leq t$ should be the desired one.

Hence from now on we assume that G is connected. Let

$$\Gamma = \{(g, f(g)) \mid g \in \varphi_x^{-1}(O)\} \subset \varphi_x^{-1}(O) \times K$$

and $$\Gamma' = (\varphi_x|_{\varphi_x^{-1}(O)} \times 1_K)(\Gamma)$$

where $$\varphi_x|_{\varphi_x^{-1}(O)} \times 1_K : \varphi_x^{-1}(O) \times K \longrightarrow O{\times}K$$
$$(g,k) \longrightarrow (g\cdot x, k)\,.$$

Then $\Gamma' \subset O{\times}K$ and
$$O{\times}K - \Gamma' = (\varphi_x|_{\varphi_x^{-1}(O)} \times 1_K)(\varphi_x^{-1}(O) \times K - \Gamma)\,.$$

Since the map $\begin{array}{c}\varphi_x^{-1}(O) \times K \longrightarrow K \times K\\ (g,k) \longrightarrow (f(g),k)\end{array}$ is a morphism of varieties and Γ is its inverse image of $\Delta(K) = \{(k,k) \mid k \in K\}$, Γ is closed in $\varphi_x^{-1}(O) \times K$. Let $G{\times}K$ be the product of algebraic groups $(G, K[G])$ and $(K, K[X])$ (see Examples 14.9), then $G{\times}K$ operates on $X{\times}K$ morphically as follows:
$$(G{\times}K) \times (X{\times}K) \longrightarrow X{\times}K$$
$$((g,k),(y,z)) \longrightarrow (g\cdot y, k+z)$$
Since $(G{\times}K)(x,0) = X{\times}K$, $X{\times}K$ is a homogeneous space of $G{\times}K$. Hence the map
$$\varphi_x \times 1_K : G{\times}K \longrightarrow X{\times}K$$
$$(g,k) \longrightarrow (g\cdot x, k)$$
is an open map from Lemma 17.12. Thus
$$\varphi_x|_{\varphi_x^{-1}(O)} \times 1_K : \varphi_x^{-1}(O) \times K \longrightarrow O{\times}K$$
is also an open map and Γ' is closed in $O{\times}K$.

Let $\lambda = \pi_1|_{\Gamma'} : \Gamma' \to O$ where $\pi_1 : O{\times}K \to O$ is the projection, then
$$\lambda : \Gamma' \subset O{\times}K \xrightarrow{\pi_1} O$$
is a morphism of varieties. Since $\Gamma' = \{(g{\cdot}x, f(g)) \mid g \in \varphi_x^{-1}(O)\}$, λ is bijective. Since Γ' is the image of the morphism
$$\varphi_x^{-1}(O) \longrightarrow O{\times}K$$
$$g \longrightarrow (g\cdot x, f(g))\,,$$
Γ' is irreducible. If
$$\lambda : \Gamma' = \{(g{\cdot}x, f(g)) \mid g \in \varphi_x^{-1}(O)\} \longrightarrow O$$
$$(g\cdot x, f(g)) \longrightarrow g{\cdot}x$$
is an isomorphism of varieties, then the map
$$F : O \xrightarrow{\lambda^{-1}} \Gamma' \xrightarrow{\pi_2|_{\Gamma'}} K \longrightarrow K$$
$$g\cdot x \longrightarrow (g{\cdot}x, f(g)) \longrightarrow f(g) \longrightarrow f(g)$$
belongs to $\mathscr{O}_X(O)$, where $\pi_2 : O{\times}K \to K$ is the projection, and satisfies the condition
$$f = F \circ \varphi_x \text{ on } \varphi_x^{-1}(O)\,.$$

Now we shall show that λ is an isomorphism of varieties. Since λ is a bijective morphism, λ is dominant and we can embed $K(O)$ into $K(\Gamma')$. Since Γ' is the image of the morphism

$$\begin{aligned} \varphi_x^{-1}(O) &\longrightarrow O \times K \\ g &\longrightarrow (g \cdot x, f(g)) \end{aligned}$$

and Γ' is closed in $O \times K$, the map

$$\begin{aligned} \varphi_x^{-1}(O) &\longrightarrow \Gamma' \\ g &\longrightarrow (g \cdot x, f(g)) \end{aligned}$$

is a surjective morphism (Exercise 40 on p.113). Hence we can also embed $K(\Gamma')$ into $K(\varphi_x^{-1}(O)) = K(G)$. Thus we have

$$K(O) = K(X) \subset K(\Gamma') \subset K(G) .$$

Since the map $\begin{aligned} \varphi_x : G &\longrightarrow X \\ g &\longrightarrow g \cdot x \end{aligned}$ is a separable morphism from Proposition 23.5, $K(G)$ is separable over $K(X)$. From Corollary 6.19 $K(\Gamma')$ is also separable over $K(O)$. Since λ is bijective, we have $r = \dim \Gamma' - \dim O = 0$ from Theorem 13.14.ii. Therefore, from Theorem 13.14.iii we have $[K(\Gamma') : K(O)] = 1$, because λ is bijective. Hence $K(\Gamma') = K(O)$. From the Zariski's Main Theorem λ becomes an isomorphism of varieties. Q.E.D.

Thus we have constructed the quotient $(G/H, \mathscr{O})$ or (X, x) of G by H. Notice that

(i) $\begin{aligned} \varphi : G/H &\longrightarrow X \\ gH &\longrightarrow g \cdot x \end{aligned}$ is an isomorphism of varieties which makes the following diagram commutative

$$\begin{tikzcd} & gH \in G/H \arrow[dd, "\varphi"] \\ g \in G \arrow[ur, "\nu"] \arrow[dr, "\varphi_x"'] & \\ & g \cdot x \in X \end{tikzcd}$$

(ii) for any open subset U of G/H we have $\varphi_x^{-1}(\varphi(U)) = \nu^{-1}(U)$, and the following correspondence is a bijective K–algebra homomorphism

$$\begin{aligned} \mathscr{O}_G(\nu^{-1}(U))^H &\longrightarrow \mathscr{O}(U) \\ f &\longrightarrow [\hat{f} : gH \to f(g)] \end{aligned}$$

where

$$\mathscr{O}_G(\nu^{-1}(U))^H = \{f \in \mathscr{O}_G(\nu^{-1}(U)) \mid f(gh) = f(g) \text{ for any } g \in \nu^{-1}(U) \text{ and } h \in H\}$$

and

$$\mathscr{O}(U) = \{f : U \to K \mid f \circ \nu|_{\nu^{-1}(U)} \in \mathscr{O}_G(\nu^{-1}(U))\} ;$$

(iii) $(G/H, \mathscr{O})$ is a quasi-projective variety of dimension $\dim G - \dim H$ (see Proposition 23.5 and its proof);

(iv) $\nu_0 : G^0 \longrightarrow \{gH \mid g \in G^0\}$ $(\subset G/H)$ is a separable morphism where G^0 is the
$\quad g \longrightarrow gH$
connected component of G containing 1 (see Proposition 23.5);

(v) $\nu : G \to G/H$ is a separable morphism if G is connected.

<u>(23.8) Lemma</u>. Let $(G, K[G])$ be a linear algebraic group over K and $(H, K[H])$ be a closed subgroup of G. Let X be a G-variety over K. Assume that $H = G_x$ and $X = G{\cdot}x$ for some $x \in X$ and the map

$$\varphi_x\,|_{G^0} : G^0 \longrightarrow G^0{\cdot}x, \quad g \longrightarrow g{\cdot}x$$

is separable where $\varphi_x : G \longrightarrow X$, $g \longrightarrow g{\cdot}x$ and G^0 is the connected component of G containing 1. Then the map

$$\varphi : G/H \longrightarrow X, \quad gH \longrightarrow g{\cdot}x$$

is an isomorphism of varieties.

<u>Proof</u>. From the definition of the quotient G/H φ is a bijective G-morphism. Let

$$G = \bigcup_{i=1}^{t} G^0\, g_i \quad \text{(disjoint union of open subsets)},$$

then $X = G{\cdot}x = \bigcup\limits_{i=1}^{t} G^0\, g_i\, x = G^0{\cdot}x \cup G^0\, g_{i_2}{\cdot}x \cup \ldots \cup G^0 g_{i_s}{\cdot}x$ (disjoint union of G^0-orbits in X, $s \le t$). $G^0\, g_{i_1}{\cdot}x$ $(= G^0{\cdot}x), \ldots, G^0\, g_{i_s}{\cdot}x$ are the irreducible components of X and each $G^0\, g_{i_j}$ is open and closed in X $(1 \le j \le s)$. Since φ is bijective, we also have

$$G/H = G^0\, H \cup G^0\, g_{i_2}\, H \cup \ldots \cup G^0\, g_{i_s}\, H \quad \text{(disjoint union)}.$$

From Proposition 10.7. it is enough to prove that

$$\varphi^0 : G^0\, gH \longrightarrow G^0\, g{\cdot}x \quad (t \in G^0), \quad tgH \longrightarrow t\, gx$$

is an isomorphism of varieties for any $g \in G$. Notice that

$$\{t \in G^0 \mid tgx = gx\} = G^0 \cap gHg^{-1},$$

We shall show that φ_1 and φ_2 in the following commutative diagrams

$$\begin{tikzcd} & G^0/G^0 \cap gHg^{-1} \arrow[d, "\exists_1 \varphi_1"] \\ t \in G^0 \arrow[ur, "\nu"] \arrow[r, "\varphi_{gH}"'] & tgH \in G^0 gH \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & G^0/G^0 \cap gHg^{-1} \arrow[d, "\exists_1 \varphi_2"] \\ t \in G^0 \arrow[ur, "\nu"] \arrow[r, "\varphi_{gx}"'] & tgx \in G^0 gx \end{tikzcd}$$

are isomorphisms of varieties.

Since φ_{gH} and φ_{gx} are separable, this is an application of the Lemma in case G is connected. Hence it is enough to prove the Lemma in case G is connected.

Now we assume that G is connected. Since $K(G)$ is separable over $K(X)$ and

$$K(X) \subset K(G/H) \subset K(G),$$

$K(G/H)$ is separable over $K(X)$. Since $K(G/H)$ is algebraic over $K(X)$ from Theorem 13.14.ii, $K(G/H)$ is separably algebraic over $K(X)$ and we have $K(G/H) = K(X)$ from Theorem 13.14.iii. Hence from Zariski's Main Theorem φ is an isomorphism of varieties. Q.E.D.

<u>(23.9) Proposition</u>. Let $(G_i, K[G_i])$ be a linear algebraic group over K and $(H_i, K[H_i])$ be a closed subgroup of G_i $(i = 1,2)$, then the map

$$\varphi : G_1{\times}G_2/H_1{\times}H_2 \longrightarrow G_1/H_1 \times G_2/H_2$$
$$(g_1, g_2)H_1{\times}H_2 \longrightarrow (g_1H_1,\ g_2H_2)$$

is an isomorphism of varieties (see Examples 14.9.iii).

<u>Proof</u>. From Exercise 43 on p.118 the map

$$(G_1{\times}G_2) \times (G_1/H_1 \times G_2/H_2) \to (G_1 \times G_1/H_1) \times (G_2 \times G_2/H_2) \to G_1/H_1 \times G_2/H_2$$
$$((g_1, g_2), (g_1'H_1,\ g_2'H_2)) \to ((g_1, g_1'H_1), (g_2, g_2'H_2)) \to (g_1g_1'H_1,\ g_2g_2'H_2)$$

Hence $G_1/H_1 \times G_2/H_2$ is a homogeneous $G_1{\times}G_2$–variety.

Let $x = (H_1, H_2) \in G_1/H_2 \times G_2/H_2$, then

$$(G_1{\times}G_2)_x = H_1{\times}H_2 \text{ and } G_1/H_1 \times G_2/H_2 = (G_1{\times}G_2)\cdot x .$$

Let $G_i{}^0$ be the connected component of G_i containing 1 $(i = 1,2)$, then $G_1{}^0 \times G_2{}^0$ is also the connected component of $G_1{\times}G_2$ containing $(1,1)$. Let

$$\varphi_x : G_1{}^0 \times G_2{}^0 \longrightarrow (G_1{}^0 \times G_2{}^0)x \quad (\subset G_1/H_1 \times G_2/H_2)$$
$$(g_1, g_2) \longrightarrow (g_1H_1, g_2H_2)$$

then $(d\varphi_x)_{(1,1)}$ is surjective from Lemma 20.3 and Proposition 23.5. From Theorem 21.9 φ_x is a separable morphism. Thus from Lemma 23.8 φ is an isomorphism of varieties. Q.E.D.

(23.10) Theorem. Let G be a linear algebraic group over K and H be a closed normal subgroup of G. Then there exists a rational representation

$$\varphi : G \longrightarrow GL(V)$$

such that $\operatorname{Ker}\varphi = H$ and $\operatorname{Ker} d\varphi = \mathscr{H}$ where $\mathscr{H}$ is the Lie algebra of H.

Proof (see Humphreys [2, Theorem 11.5]). Let $\varphi : G \to GL(V)$ be a finite dimensional rational representation of G and L be a one dimensional subspace of V as in Corollary 23.3, that is,

$$H = \{g \in G \mid \varphi(g)L = L\}$$

and

$$\mathscr{H} = \{\gamma \in \mathscr{g} \mid (d\varphi)(\gamma)L \subset L\}$$

where $\mathscr{g}$ is the Lie algebra of G. Let λ be a rational linear character of H, i.e., λ is a group homomorphism of H into $K^\times = K - \{0\}$ and $\lambda \in K[H]$, and Λ be the set of rational linear characters of H. Let

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \ \text{ for all } \ h \in H\},$$

then

$$gV_\lambda \subset V_{\lambda^g}$$

for any $g \in G$ where

$$\lambda^g : H \longrightarrow K^\times$$
$$h \longrightarrow \lambda(g^{-1}hg).$$

We assume that $V_{\lambda_0} \supset L$ for some $\lambda_0 \in \Lambda$. Let V' be the sum of all nonzero V_λ $(\lambda \in \Lambda)$, then V' is a KG–submodule of V and the direct sum of the nonzero V_λ's, because the set Λ is linearly independent over K (see e.g. Lang [1, Theorem 4.1 on p.319]). From Lemma 20.5 without loosing generality we can assume that $V = V'$.

Now let

$$\mathrm{Ad} : GL(V) \longrightarrow GL(\mathscr{gl}(V))$$
$$x \longrightarrow [\mathrm{Ad}_x : f \to xfx^{-1}]$$

be the adjoint representation of $GL(V)$ (see Proposition 20.4), then

$$\mathrm{Ad} \circ \varphi : G \xrightarrow{\varphi} GL(V) \xrightarrow{\mathrm{Ad}} GL(\mathscr{gl}(V))$$

is a rational representation of G into $GL(\mathscr{gl}(V))$. Let

$$W = \{f \in \mathscr{gl}(V) \mid f(hv) = hf(v) \ \text{ for any } \ h \in H \ \text{ and } \ v \in V\},$$

then $\mathrm{Ad} \circ \varphi(g)W \subset W$ for each $g \in G$, because H is normal in G. Thus we have got a rational representation ψ of G into $GL(W)$ defined by

$$\psi : G \longrightarrow GL(W)$$
$$g \longrightarrow \mathrm{Ad} \circ \varphi(g)|_W ,$$

We shall show that $\mathrm{Ker}\ \psi = H$. Since

$$\psi(h) = \mathrm{Ad} \circ \varphi(h)\ |_W : f \longrightarrow \varphi(h)\, f \varphi(h)^{-1} = f$$

$(h \in H,\ f \in W)$, we have $\mathrm{Ker}\ \psi \supset H$. Conversely let g be an element of G such that $\psi(g)f = f$ for all $f \in W$. Let f_1 be a map of V into V such that

$$f_1(v) = v \quad \text{if } v \in V_{\lambda_0}$$

and $$f_1(v) = 0 \quad \text{if } v \in V_\lambda \ (\lambda \in \Lambda - \{\lambda_0\}) .$$

Since $hV_\lambda \subset V_{\lambda h} = V_\lambda$ for any $h \in H$, where $\lambda \in \Lambda$, f_1 is contained in W . Thus we have

$$\psi(g)\, f_1 = f_1 ,$$

which implies $$g f_1\, (g^{-1} v) = v \ \text{ for any } v \in V_{\lambda_0} .$$

Therefore $f_1\, (g^{-1} v) = g^{-1} v \in V_{\lambda_0}$ and $g^{-1} V_{\lambda_0} = V_{\lambda_0}$, i.e.,

$$V_{\lambda_0} = gV_{\lambda_0} .$$

Let $\varphi_0 : V_{\lambda_0} \to V_{\lambda_0}$ be a K–linear map such that

$$\varphi_0 : V_{\lambda_0} \longrightarrow V_{\lambda_0}$$
$$v \longrightarrow gv \ .$$

Let α be any element of $\mathrm{End}_K\, (V_{\lambda_0})$, then we can define a K–linear map f_α of V into itself by

$$f_\alpha\, (v) = \alpha(v) \quad \text{if } v \in V_{\lambda_0}$$

and $$f_\alpha\, (v) = 0 \quad \text{if } v \in V_\lambda \text{ where } \lambda \in \Lambda - \{\lambda_0\} .$$

Since $f_\alpha\, (hv) = hf_\alpha\, (v)$ for any $h \in H$ and $v \in V$, we have $f_\alpha \in W$. Hence $\psi(g)\, f_\alpha = f_\alpha$ for any $\alpha \in \mathrm{End}_K\, (V_{\lambda_0})$, which implies $gf_\alpha(g^{-1} v) = g\alpha(g^{-1} v) = \alpha(v)$ for any $v \in V_{\lambda_0}$. Thus we have $\varphi_0 \circ \alpha = \alpha \circ \varphi_0$ for any $\alpha \in \mathrm{End}_K\, (V_{\lambda_0})$, i.e., φ_0 is an element of the centre of $\mathrm{End}_K\, (V_{\lambda_0})$. Hence

$$\varphi_0\, (L) = L ,$$

i.e., $$gL = L ,$$

because φ_0 is a scalar multiplication. Therefore, $g \in H$, i.e.,

$$\mathrm{Ker}\ \psi = H .$$

Finally we shall show that $\mathrm{Ker}\ d\psi = \mathscr{H}$. Notice that from Lemma 20.5 we have

$$d\psi(\gamma) = d(\mathrm{Ad} \circ \varphi)\gamma\,|_W$$

for any $\gamma \in \mathscr{g}$. Assume that $d\psi(\gamma) = 0$ for some $\gamma \in \mathscr{g}$, then

$$d(\mathrm{Ad} \circ \varphi)\gamma\,|_W = 0 .$$

Hence $\mathrm{ad}\,(d\varphi(\gamma))\,(f) = [d\varphi(\gamma), f] = d\varphi(\gamma) \circ f - f \circ d\varphi(\gamma) = 0$ for any $f \in W$. Thus

$$d\varphi(\gamma) \circ f_\alpha = f_\alpha \circ d\varphi(\gamma)$$

for any $\alpha \in \mathrm{End}_K\,(V_{\lambda_0})$, which implies

$$d\varphi(\gamma)\, V_{\lambda_0} \subset V_{\lambda_0}$$

and
$$d\varphi(\gamma)\,|_{V_{\lambda_0}} \circ \alpha = \alpha \circ d\varphi(\gamma)\,|_{V_{\lambda_0}}$$

for any $\alpha \in \mathrm{End}_K\,(V_{\lambda_0})$. Hence $d\varphi(\gamma) \mid V_{\lambda_0}$ is a scalar multiplication and we have

$$d\varphi(\gamma)L \subset L \text{ , i.e. } \gamma \in \mathscr{H} .$$

Therefore, $\mathrm{Ker}\, d\psi \subset \mathscr{H}$. Since $\dim G = \dim H + \dim \psi(G)$,

$$\dim_K \mathscr{g} = \dim_K \mathscr{H} + \dim_K T(\psi(G))_1$$

and
$$\dim_K \mathscr{g} = \dim_K \ker d\psi + \dim_K \mathrm{Im}\, d\psi ,$$

we have
$$\dim_K \mathscr{H} \leq \dim_K \mathrm{Ker}\, d\psi .$$

Hence $\mathrm{Ker}\, d\psi = \mathscr{H}$ and ψ is a desired rational representation of G.

Q.E.D.

(23.11) Theorem. Let G be a linear algebraic group over K and H be a closed normal subgroup of G. Then

(i) the quotient variety G/H is affine;
(ii) the factor group G/H is a linear algebraic group with respect to the variety structure of the quotient of G by H.

Proof. (i) Let φ and V be as in Theorem 23.10, i.e.,

$$\varphi : G \longrightarrow GL(V)$$

is a rational representation such that

$$\mathrm{Ker}\,\varphi = H \text{ and } \mathrm{Ker}\, d\varphi = \mathscr{H}$$

where $\mathscr{H}$ is the Lie algebra of H. Let $X = \varphi(G)$, then X is closed in $GL(V)$ and G operates on X morphically as follows

$$G{\times}X \longrightarrow X$$
$$(g, x) \longrightarrow \varphi(g)x .$$

Let $x = 1$ and $\begin{array}{l} \varphi_x : G \longrightarrow X \\ \qquad g \longrightarrow g\cdot x \end{array}$, then we have

$$\begin{array}{l} \varphi : G \xrightarrow{\varphi_x} X \overset{\iota}{\subset} GL(V) \\ \qquad g \longrightarrow \varphi(g) \longrightarrow \varphi(g) \ . \end{array}$$

Since $\dim X = \dim G - \dim H$ and $\operatorname{Ker} d\varphi = \operatorname{Ker} d\varphi_x = \mathscr{H}$,

$$d\varphi_x : \mathscr{g} \longrightarrow T(X)_x$$

is surjective where $\mathscr{g}$ is the Lie algebra of G . Hence from Theorem 21.9

$$\begin{array}{l} \varphi_x \mid G^0 : G^0 \longrightarrow G^0\cdot x \\ \qquad\qquad g \longrightarrow g\cdot x \end{array}$$

is separable, where G^0 is the connected component of G containing 1. Therefore G/H is isomorphic to X as variety from Lemma 23.8, and G/H has turned out to be an affine variety.

(ii) Since H is normal in G, G/H is also a right G–variety. Hence $\begin{array}{l} G \times G/H \to G/H \\ (g,g'H) \to g'Hg \end{array}$ is a morphism of varieties and G/H becomes a left G–variety by the following operation

$$\begin{array}{l} G \times G/H \longrightarrow G/H \\ (g,g'H) \longrightarrow g'g^{-1}H \ . \end{array}$$

Thus the map $\begin{array}{lll} G \times (G \times G/H) \longrightarrow & G \times G/H & \longrightarrow G/H \\ (g_1,(g_2,gH)) \longrightarrow & (g_1,gg_2^{-1}H) & \longrightarrow g_1gg_2^{-1}H \end{array}$

is a morphism of varieties and the group $G\times G$ operates on G/H morphically from the left. From Proposition 23.9 there exist a morphism of varieties φ:G/H×G/H → G/H which makes the following diagram commutative:

$$\begin{tikzcd}
& (g_1H,g_2H) \in G/H \times G/H \arrow[dd, "\varphi"] \\
(g_1,g_2) \in G \times G \arrow[ur, "\nu"] \arrow[dr, "\varphi_H"'] & \\
& g_1\, g_2^{-1}\, H \in G/H
\end{tikzcd}$$

Hence the group operations are morphic with respect to the variety structure of G/H .

Q.E.D.

24. Fixed Point Theorem and Borel subgroups

We shall prove the Fixed Point Theorem, i.e., a connected solvable linear algebraic group operating on a complete variety has a fixed point, and define Borel subgroups and parabolic subgroups.

We also show the Lie–Kolchin Theorem as a corollary to the Fixed Point Theorem.

(24.1) Definition. Let G be an abstract group and x, y be arbitrary elements of G. We shall write $[x,y]$ for the commutator of x and y, i.e.,

$$[x,y] = xy\, x^{-1} y^{-1} .$$

(24.2) Definition. Let H, K be subgroups of an abstract group G. We use $[H,K]$ to denote the subgroup of G generated by the set $\{[h,k] \mid h \in H \text{ and } k \in K\}$. We call $[G,G]$ the commutator subgroup of G and inductively we shall define

$$G' = [G,G],\ G'' = [G',G'],\dots,\ G^{(i)} = [G^{(i-1)}, G^{(i-1)}] \quad (i > 1) .$$

(24.3) Definition. An abstract group G is said to be solvable if $G^{(n)} = \{1\}$ for some $n > 0$.

From the definition of solvable groups it is clear that subgroups and homomorphic images of a solvable group are solvable. The following proposition is well–known.

(24.4) Proposition. Let G be an abstract group.

(i) If G has a normal solvable subgroup N such that G/N is solvable, then G is also solvable.

(ii) If A, B are solvable subgroups of G and A normalizes B, then AB is a solvable subgroup of G.

<u>(24.5) Proposition</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and A, B be closed subgroups of G .

(i) If A normalizes B , then AB is a closed subgroup of G .
(ii) If A is connected, then $[A,B]$ is a closed and connected subgroup of G .

<u>Proof</u>. (i) Since A normalizes B , AB is a subgroup of G . Since AB is the image of $A\times B$ under the following morphism

$$A\times B \subset G\times G \longrightarrow G$$
$$(x,y) \to (x,y) \to xy \ ,$$

AB is constructible from Theorem 8.6. Thus from Proposition 15.3 AB is closed.

(ii) Let $\varphi_y : A \to G$ be the map defined by $\varphi_y(x) = [x,y]$, where $y \in B$. Since A is connected and $\varphi_y(1) = 1$, from Proposition 15.5 $[A,B]$ is closed and connected.

Q.E.D.

<u>(24.6) Lemma</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and X, Y be homogeneous G–varieties. Assume that Y is complete and there exists a bijective G–morphism φ of X onto Y , then X is also complete.

<u>Proof</u>. We shall show that the projection

$$P_2 : X\times Z \longrightarrow Z$$

is a closed map for any affine variety Z . Since Y is complete, it is enough to prove that

$$\varphi\times 1 : X\times Z \longrightarrow Y\times Z$$
$$(x,z) \to (\varphi(x),z)$$

is a closed map.

$$\begin{tikzcd}
 & & Y\times Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
\varphi(x) \in & Y & & Z & \in z \\
 & & X\times Z \arrow[ul] \arrow[ur, "P_2"'] \arrow[uu, "\varphi\times 1"] & & \\
 & & (x,z) \arrow[uull] \arrow[uurr] & &
\end{tikzcd}$$

(commutative diagram)

From Theorem 2.3 we can embed Z into a certain affine n–space $(K^n, K[X_1, X_2, \ldots$ $\ldots, X_n])$ as a closed subset. Since $X\times Z$ and $Y\times Z$ are closed subvarieties of $X\times K^n$ and $Y\times K^n$ respectively (see Exercises 42 on p.118), it is sufficient to prove that $\varphi\times 1$ is a closed map only in case $Z = K^n$.

Notice that by componentwise addition K^n is an additive linear algebraic group. Thus $X \times K^n$ and $Y \times K^n$ are homogeneous $G \times K^n$–varieties. From Lemma 17.12 $\varphi \times 1$ is an open map. Since $\varphi \times 1$ is bijective, $\varphi \times 1$ is a homeomorphism. Hence $\varphi \times 1$ is a closed map and X is complete. Q.E.D.

<u>Fixed Point Theorem</u> (A. Borel). Let $(G, K[G])$ be a connected solvable linear algebraic group over K. Let X be a G–variety. If X is complete, then G has a fixed point in X.

<u>Proof</u>. We follow the induction on $\dim G$. Assume that $\dim G = 0$, then $G = \{1\}$ and the assertion holds.

If $\dim G > 0$, then $G' = [G,G]$ is a closed connected solvable subgroup of G of dimension less than $\dim G$ (see Proposition 24.5 and Exercise 48 on p.140). By induction the set Y of fixed points of G' in X is non–empty. Since Y is closed from Proposition 17.4, Y is complete from Proposition 12.2. Since G' is normal in G, we have

$$gY \subset Y \text{ for any } g \in G .$$

Hence it is enough to find a G–fixed point in Y. Since $G' \subset G_y$, G_y is normal in G for any $y \in Y$. Thus from Theorem 23.11 G/G_y is an affine variety. We can choose $y \in Y$ such that $G{\cdot}y$ is closed (see Corollary 17.7). Hence $G{\cdot}y$ is complete from Proposition 12.2. Thus we have got a canonical morphism of the affine variety G/G_y onto $G{\cdot}y$. From Lemma 24.6 G/G_y is complete. Therefore

$$G = G_y$$

from Proposition 12.2 and y is one of the fixed points which we want. Q.E.D.

<u>(24.7) Lemma</u>. Let $(G, \mathscr{O}_G)$ be an algebraic group over K and V be an n–dimensional rational left KG–module $(n > 0)$, i.e.,

$$\begin{aligned} \varphi : G &\longrightarrow GL(V) \\ g &\longrightarrow [\,\varphi(g): v \to gv] \quad (v \in V) \end{aligned}$$

is a rational representation. Then

(i) For any $0 < d \le n$, G operates on $P(\Lambda^d V)$ morphically as follows

$$\begin{aligned} G \times P(\Lambda^d V) &\longrightarrow P(\Lambda^d V) \\ (g, \pi(v_1 \wedge \ldots \wedge v_d)) &\longrightarrow \pi(gv_1 \wedge \ldots \wedge gv_d)\,, \end{aligned}$$

where $v_1 \wedge \ldots \wedge v_d \in \Lambda^d V - \{0\}$ and $\pi(v_1 \wedge \ldots \wedge v_d) = K(v_1 \wedge \ldots \wedge v_d) \in P(\Lambda^d V)$ (see Lemma 23.1 and Lemma 23.4).

(ii) Let $0 < d \leq n$ and $\mathcal{G}_d(V)$ be the Grassman variety of all d–dimensional subspaces of V, then the set

$$\{\Lambda^d D \mid D \in \mathcal{G}_d(V)\}$$

is closed in $P(\Lambda^d V)$ and $g\{\Lambda^d D \mid D \in \mathcal{G}_d(V)\} \subset \{\Lambda^d D \mid D \in \mathcal{G}_d(V)\}$ for any $g \in G$, i.e., $\mathcal{G}_d(V)$ is a projective variety on which G operates morphically as follows.

$$\begin{aligned} G \times \mathcal{G}_d(V) &\longrightarrow \mathcal{G}_d(V) \\ (g,D) &\longrightarrow g\cdot D \end{aligned}$$

(iii) Let $\mathcal{F}(V)$ be the flag variety defined by V, i.e., the set of all sequences of K–subspaces $\{0, V_1,\ldots,V_n\}$ of V such that

$$0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \ldots \subsetneqq V_n = V ,$$

then G operates on $\mathcal{F}(V)$ morphically as follows.

$$\begin{aligned} G \times \mathcal{F}(V) &\longrightarrow \mathcal{F}(V) \\ (g,\{0,V_1,V_2,\ldots,V_n\}) &\longrightarrow \{0,gV_1,gV_2,\ldots,gV_n\} \end{aligned}$$

Proof. (i) is from Lemma 23.1 and Lemma 23.4.
(ii) is from Proposition 11.10.
(iii) is from Proposition 11.13. Q.E.D.

As an application of the Fixed Point Theorem we can prove the Lie–Kolchin Theorem.

Lie–Kolchin Theorem. Let $(G, K[G])$ be a connected solvable linear algebraic group over K, then any non–zero finite dimensional rational KG–module V has a one–dimensional KG–submodule.

Proof. Let $\mathcal{F}(V)$ be the flag variety defined by V. Since G operates on $\mathcal{F}(V)$ morphically (see Lemma 24.7.iii) and $\mathcal{F}(V)$ is a projective variety from Proposition 11.13, by the Fixed Point Theorem G has a fixed point in $\mathcal{F}(V)$, i.e., there exists a sequence of K–subspaces $\{0, V_1,\ldots, V_n\}$ of V such that

$$0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \ldots \subsetneqq V_n = V$$

and

$$\{0, gV_1,\ldots, gV_n\} = \{0, V_1,\ldots, V_n\}$$

for any $g \in G$. Thus V_1 is the desired KG–submodule. Q.E.D.

Now we define Borel subgroups.

(24.8) Definition. Let $(G, \mathscr{O}_G)$ be an algebraic group over K. We call a maximal element in the set of all closed connected solvable subgroup of G a Borel subgroup.

Remark to Definition 24.8. (i) Let B be a closed connected solvable subgroup of G of the largest dimension, then B is a Borel subgroup.

(ii) Let G^0 be the connected component of G which contains 1, then Borel subgroups of G and G^0 coincide with each other.

(24.9) Theorem. Let $(G, K[G])$ be a connected linear algebraic group over K and S be a Borel subgroup of G of the largest possible dimension, then

(i) the quotient G/S is a projective variety, and
(ii) all other Borel subgroups are conjugate to S.

Proof. (i) Let V be a finite dimensional rational left KG–module with one dimensional subspace L $(\subset V)$ such that
$$S = \{g \in G \mid gL = L\}$$
(see Corollary 23.3). We shall write ρ for the rational representation of G defined by this G–module V. Applying the proof of Lie–Kolchin Theorem to the S–module V/L, there exists a sequence of K–subspaces
$$f = \{0 \subsetneqq L \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_n = V\}$$
of V such that
$$S = \{g \in G \mid gf = f\},$$
where $n = \dim_K V$.

Hence we have got a canonical bijective G–morphism of G/S onto the quasi–projective variety
$$G{\cdot}f \ \ (\subset \overline{G{\cdot}f} \subset \mathscr{F}(V)).$$
Since $\operatorname{Ker} \rho \subset S$ and S is solvable, the stabilizer of any fixed point of $\mathscr{F}(V)$ is closed and solvable. Hence the orbit $G{\cdot}f$ has the smallest possible dimension
$$\dim G{\cdot}f = \dim G - \dim S$$
from Theorem 13.14.ii.

Therefore $G \cdot f$ is closed from Corollary 17.7. Since $G \cdot f$ is complete from Proposition 12.2, G/S is complete from Lemma 24.6. Since G/S is a complete quasi-projective variety, G/S is a complete open subvariety of certain projective variety X. Hence G/S is closed in X from Proposition 12.2. Thus G/S is projective.

(ii) Let B be a Borel subgroup of G, then B is operating on G/S morphically as follows,

$$B \times G/S \longrightarrow G/S$$
$$(b, xS) \longrightarrow bxS .$$

Since G/S is complete, from the Fixed Point Theorem, B fixes xS for some $x \in G$. Since $BxS = xS$, we have

$$x^{-1} Bx \subset S .$$

From the definition of Borel subgroups we have $x^{-1} Bx = S$. Q.E.D.

(24.10) Definition. Let $(G, K[G])$ be a linear algebraic group over K. We call a closed subgroup P of G parabolic if G/P is projective.

(24.11) Proposition. Let $(G, K[G])$ be a linear algebraic group over K and G^0 is the connected component of G which contains 1. Then

(i) a closed subgroup P of G is parabolic if and only if G/P is complete;
(ii) a closed subgroup P of G is parabolic in G if and only if $G^0 \cap P$ is parabolic in G^0.

Proof. (i) Since projective varieties are complete (see Theorem 12.4), G/P is complete if P is parabolic. Assume that G/P is complete. Since G/P is a quasi-projective variety, G/P is open in certain projective variety X. Thus from Proposition 12.2.ii G/P is also closed in X, because G/P is the image of the embedding $G/P \subset X$. Hence G/P is projective, i.e., P is parabolic.

(ii) We first assume that P is parabolic, i.e., G/P is complete. Let

$$G = G^0 \cup g_2 G^0 \cup \ldots \cup g_t G^0$$

be a disjoint union of left cosets of G^0 in G, then we have the following G^0–orbits decomposition of G/P:

$$G/P = G^0 P/P \ \cup\ g_{j_1} G^0 P/P \ \cup \ldots \cup\ g_{j_s} G^0 P/P ,$$

where $\{g_{j_1}, \ldots, g_{j_s}\} \subset \{g_2, \ldots, t_t\}$. Since the canonical map

$$\begin{aligned} \nu : G &\longrightarrow G/P \\ g &\longrightarrow gP \end{aligned}$$

is an open map, $G^0\, P/P$ is open and closed in G/P. Hence $G^0\, P/P$ is complete from Proposition 12.2. Since the map

$$\begin{aligned} \varphi : G^0/G^0 \cap P &\longrightarrow G^0\, P/P \quad (g \in G^0) \\ gG^0 \cap P &\longrightarrow gP \end{aligned}$$

is a bijective G^0–morphism, $G^0/G^0 \cap P$ is complete from Lemma 24.6. Hence $G^0 \cap P$ is parabolic in G^0.

Conversely suppose that $G^0/G^0 \cap P$ is complete. Since φ is surjective, $G^0\, P/P$ is a complete closed subvariety of G/P. Thus

$$G/P = G^0\, P/P \;\cup\; g_{j_1} G^0\, P/P \;\cup \ldots \cup\; g_{j_s} G^0\, P/P$$

is a disjoint union of complete closed subvarieties. Hence G/P is also complete.

Q.E.D.

<u>(24.12) Proposition</u>. Let $(G, K[G])$ be a connected linear algebraic group over K, then

(i) a closed subgroup P of G is parabolic if and only if P contains a Borel subgroup of G;

(ii) a closed connected subgroup H of G is a Borel subgroup if and only if H is solvable and G/H is projective.

<u>Proof</u>. (i) Assume that G/P is projective. Let B be a Borel subgroup of G. Since B is connected and solvable, B fixed a point in G/P from the Fixed Point Theorem, i.e., there exists a left coset xP in G/P such that

$$BxP = xP \quad (x \in G)\,.$$

Hence P contains a Borel subgroup $x^{-1} Bx$.

Conversely let H be a closed subgroup of G which contains a Borel subgroup B of G. Since the morphism

$$\begin{aligned} G/B &\longrightarrow G/H \quad (g \in G) \\ gB &\longrightarrow gH \end{aligned}$$

is surjective and G/B is projective from Theorem 24.9, G/H is complete, i.e., H is parabolic from Proposition 12.2 and Proposition 24.11.

(ii) is clear from (i). Q.E.D.

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