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FOUNDATIONS OF LINEAR ALGEBRAIC GROUPS

Part I

by

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To Masahide, Eri and Emiko

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Preface

Algebraic group theory is one of the basic subjects of graduate level algebra. However, most of graduate programs of algebra do not teach this important theory. This is because graduate students are expected to have understood the theory before they entered the graduate programs. But it is often the case that they have not acquired the basic knowledge of the algebraic group theory. Furthermore, there are a few appropriate textbooks with which they can learn it by themselves.

The objective of these notes is to provide graduate students with completely self-contained lectures with which they can learn the basic theory of algebra. I explained most of the proofs of the theorems from commutative algebras to algebraic geometry (Chapters 1 and 2). These would help them understand the basic concepts of algebraic groups (Chapter 3) and construct homogeneous spaces of linear algebraic groups (Chapter 5). Also I attempted to relate a particular theory of this topics to other subjects of algebra with which graduate students may be familiar.

The original lectures started in 1980 when I was a Humboldt-fellow at the University of Essen and continued sporadically at Sophia University since then. The manuscript was completed in 1988, one year after the second visit to the University of Essen as a Humboldt and DFG-fellow.

I am very grateful to my colleagues who were involved in this project, especially Prof. Dr. Gerhard Michler, who gave me a chance of giving the lectures at the University of Essen and invited me again in 1987. Sections 21, 22 and 23 are the result of seminars with Dr. Klaus Timmerscheidt in 1987. Although only I am the person who is responsible to these notes, I should say that these sections are the joint work with him.

I am also grateful to Prof. Dr. Charles W. Curtis, who kindly gave me his informal lecture notes on linear algebraic groups which were very useful for preparing Chapter 1. Finally I should like to thank Sophia University for granting me the study leave twice and Frau Sabine Weber for her beautiful and careful typing.

Sophia University, November 1989 Hideki Sawada

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I. Affine algebraic varieties

In this chapter we explain basic ideas from affine algebraic varieties necessary for linear algebraic groups, which are defined by affine algebraic varieties and their morphisms.

Throughout these lectures unless otherwise stated, K always denotes an algebraically closed field, and an affine variety means an affine algebraic variety.

1. Definitions of affine algebraic varieties and morphisms

Let S be a non-empty set and $M(S,K)$ be the set of maps of S into K. Then $M(S,K)$ has a natural commutative K-algebra structure as follows:

For $f,g \in M(S,K)$, $\lambda \in K$ and $s \in S$,

$$(f+g)(s) = f(s) + g(s) ,$$
$$(fg)(s) = f(s)g(s) \quad \text{and}$$
$$(\lambda f)(s) = \lambda f(s) .$$

Assume that A is a K-subalgebra of $M(S,K)$, then we can define a natural map, called an evaluation map, of S into $\mathrm{Hom}_{K\text{-alg}}(A,K)$ which takes s to ϵ_s where

$$\epsilon_s(f) = f(s) \quad \text{for any} \quad f \in A .$$

ϵ_s is said to be an evaluation at s.

(1.1) Definition (see Steinberg [2]). Let (V,A) be a pair where V is a non-empty set and A is a K-subalgebra of $M(V,K)$. We call (V,A) an affine algebraic variety over K if

- A is finitely generated as K-algebra and the evaluation map of V into $\mathrm{Hom}_{K\text{-alg}}(A,K)$ is bijective.
- A is called the coordinate ring of V and we write $A = K[V]$.

(1.2) Example. Affine n-space. Let K^n be the n-times direct product of K where n is a positive integer and X_i be a map of K^n into K which takes $(x_1,x_2,\ldots,x_n) \in K^n$ to $x_i \in K$, where $1 \leq i \leq n$. Then the pair $(K^n, K[X_1,X_2,\ldots,X_n])$ is an affine algebraic variety over K. We call $(K^n, K[X_1,\ldots,X_n])$ affine n-space.

From Lang [1, Cor.4.6 on P.192] we can consider the coordinate ring of K^n as a polynomial ring in n-variables over K.

Exercise 1. Verify Example 1.2.

Next we define a morphism between two affine varieties. Let S_1 and S_2 be non-empty sets and $\varphi:S_1 \to S_2$ be a map. Then the map

$$\varphi^*:M(S_2,K) \to M(S_1,K)$$

$$(\varphi^*:f \longrightarrow f\circ\varphi)$$

is a K-algebra homomorphism. In case S_1 is a non-empty subset of S_2 and $\varphi:S_1 \longrightarrow S_2$ is an inclusion map we write π for φ^*.

(1.3) Definition. Let (U,A) and (V,B) be affine varieties over K. We call a map $\varphi:U \to V$ a morphism of affine varieties if $\varphi^*(B) \subset A$. The map

$$\varphi^*|_B : B \longrightarrow A$$

$$(\varphi^*|_B : f \longrightarrow f\circ\varphi)$$

is called a comorphism of φ and is usually denoted by φ^*. When φ is bijective and φ^{-1} is also a morphism, we call φ an isomorphism of affine varieties.

Exercise 2. Verify the following properties of morphisms.

(1) Let $\varphi:(U,A) \to (V,B)$ be a morphism of affine varieties. Assume that $\varphi(U) = V$, then φ^* is injective.

(2) Assume that φ and ψ are two morphisms of (U,A) into (V,B) and (V,B) into (W,C) respectively, then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$$

and $\psi \circ \varphi:(U,A) \to (W,C)$ is also a morphism.

(3) Let (U,A) be an affine variety. Then the identity map 1_U of U is a morphism of affine varieties and we have $(1_U)^* = 1_A$.

(4) If $\varphi:(U,A) \to (V,B)$ is an isomorphism of affine varieties, then $\varphi^*:B \to A$ is a K-algebra isomorphism.

Now we introduce the Zariski topology on affine variety (V,A), in which a morphism of affine varieties is continuous.

(1.4) Definition. Let W be a subset of V, where (V,A) is an affine variety over K. Then W is said to be closed (in V) if

$$W = \{v \in V \mid f(v) = 0 \text{ for all } f \in X\}$$

for some subset X of A. We sometimes write $\mathcal{V}(X)$ for W, i.e., the set of common zeros of X in V.

Let X and W etc. be as in Definition 1.4. Assume that I is the ideal generated by X, then $W = \mathcal{V}(I)$. Hence a subset W of V is closed if and only if there exists an ideal I of A such that $W = \mathcal{V}(I)$. If $J_1 \subset J_2$ are ideals of A, then we have

$$\mathcal{V}(J_2) \subset \mathcal{V}(J_1) .$$

(1.5) Proposition. Let (V,A) be an affine variety over K, then the closed sets in V define a topology, called the Zariski topology.

Proof. Since $V = \mathcal{V}(\{0\})$ and $\emptyset = \mathcal{V}(A)$ (notice that $1(v) = 1$ for all $v \in V$), V and the empty set are closed.

Assume that $\{V_\lambda\}_{\lambda\in\Lambda}$ are closed sets such that $V_\lambda = \mathcal{V}(I_\lambda)$ for some ideal I_λ where $\lambda \in \Lambda$, then we have $\bigcap_{\lambda\in\Lambda} V_\lambda = \mathcal{V}(I)$ where $I = \sum_{\lambda\in\Lambda} I_\lambda$ is the ideal generated by $\{I_\lambda\}_{\lambda\in\Lambda}$.

Now let $W_1 = \mathcal{V}(J_1)$ and $W_2 = \mathcal{V}(J_2)$ be closed sets defined by the ideals J_1 and J_2 respectively. Since

$$W_1 \cup W_2 = \mathcal{V}(J_1 \cap J_2) ,$$

$W_1 \cup W_2$ is closed in V. Q.E.D.

(1.6) Examples.

(1) The closed subsets of the affine 1-space $(K, K[X])$ are $K, \emptyset$ and all the finite sets, because $K[X]$ is a principal ideal domain.

(2) The circle $V = \{(x_1,x_2) \in K^2 \mid x_1^2 + x_2^2 = 1\}$ is a closed set of the affine 2-space $(K^2, K[X_1,X_2])$.

(3) $V = \{(x_1,x_2) \in K^2 \mid (x_2 - x_1^2)(x_2 - x_1) = 0\}$ is also a closed set of $(K^2, K[X_1,X_2])$.

(4) Let $x = (x_1,x_2,\ldots,x_n) \in K^n$, then $\{x\}$ is closed in the affine n-space, because $\{x\} = \mathcal{V}(\sum_{i=1}^{n} K[X_1,\ldots,X_n](X_i - x_i))$.

Let (V,A) be an affine variety over K and S be a subset of V . Let

$$\mathcal{I}(S) = \{f \in A \mid f(s) = 0 \quad \text{for any} \quad s \in S\}$$

be a set of elements of the coordinate ring of V which vanishes on S . Then $\mathcal{I}(S)$ is an ideal of A , and

$$\mathcal{I}(S_2) \subset \mathcal{I}(S_1)$$

if $S_1 \subset S_2$, and further we have

(1.7) Proposition. A subset S of (V,A) is closed if and only if $\mathcal{V}(\mathcal{I}(S)) = S$.

Exercise 3. Prove Proposition 1.7.

(1.8) Proposition. Let $\varphi:(U,A) \to (V,B)$ be a morphism of affine varieties (U,A) into (V,B) , then φ is continuous in the Zariski topology.

Proof. Let F be a closed subset of V , then there exists an ideal I of B such that $F = \mathcal{V}(I)$. Since $\varphi^*(f)(u) = f\circ\varphi(u) = 0$ for any $f \in I$ and $u \in \varphi^{-1}(F)$,

$$\varphi^{-1}(F) \subset \mathcal{V}(\varphi^*(I)) .$$

Let $u \in \mathcal{V}(\varphi^*(I))$, then $\varphi^*(f)(u) = f(\varphi(u)) = 0$ for any $f \in I$. Hence $\varphi(u) \in F$, i.e., $u \in \varphi^{-1}(F)$. Thus we have $\varphi^{-1}(F) = \mathcal{V}(\varphi^*(I))$, which implies that φ is continuous. Q.E.D.

Remark. A continuous map (with respect to the Zariski topology) between affine varieties needs not always to be a morphism. For example let $(K,K[X])$ be an affine 1-space and define a map $\varphi:K \to K$ which takes 0 to 0 and $\chi \in K-\{0\}$ to $\chi^{-1} \in K-\{0\}$, then φ is continuous in the Zariski topology but not a morphism of affine varieties.

Exercise 4. Verify the above Remark.

(1.9) Proposition. Let $\mathcal{A}(K)$ be a collection of all affine varieties over an algebraically closed field K, then $\mathcal{A}(K)$ forms a category of affine varieties with morphisms of affine varieties.

2. Subvarieties of affine algebraic varieties

In this section we introduce a notion of subvarieties of an affine variety and show that any affine variety over K is isomorphic to a subvariety of $(K^n, K[X_1,X_2,\ldots,X_n])$ for some n. For the rest of these lectures we always denote by $\mathcal{A}(K)$ the category of affine varieties over K.

<u>(2.1) Definition</u>. Let W be a non-empty closed subset of an affine variety (V,A) over K, then we call $(W,\pi(A))$ a <u>subvariety</u> of V, where π is the comorphism of the inclusion map $\varphi: W \to V$.

The following proposition justifies the above definition.

<u>(2.2) Proposition</u>. Let $(V,A) \in \mathcal{A}(K)$ and W be a non-empty subset of V. Then W is closed in V if and only if $(W,\pi(A))$ is an affine variety.

<u>Proof</u>. Let $\varphi: W \to V$ be an inclusion map, then π was defined to be the comorphism of φ as follows

$$\pi: M(V,K) \to M(W,K)$$
$$(\pi: f \longrightarrow \pi(f): w \to f(w))\ .$$

Let $\epsilon_v: A \to K$ be an evaluation at $v \in V$, and let $\epsilon'_w: \pi(A) \to K$ be an evaluation at $w \in W$. From the definition of π we have

$$\epsilon_w = \epsilon'_w \circ \pi \quad \text{for any} \quad w \in W\ .$$

We first assume that $(W,\pi(A))$ is an affine variety. It is clear that $\mathcal{V}(\mathcal{I}(W)) \supset W$. We shall apply Proposition 1.7. Let $z \in \mathcal{V}(\mathcal{I}(W))$, then $f(z) = 0$ for all $f \in \mathcal{I}(W)$. Since $\mathrm{Ker}(\pi|A) = \mathcal{I}(W)$, we can define a K-algebra homomorphism

$$\bar{\epsilon}_z : A/\mathrm{Ker}(\pi|A) \longrightarrow K$$
$$(\bar{\epsilon}_z: f + \mathrm{Ker}(\pi|A) \longrightarrow f(z))\ .$$

Since $\pi(A) \cong A/\mathrm{Ker}(\pi|A)$ and $(W,\pi(A))$ is an affine variety, there exists $w \in W$ such that $\bar{\epsilon}_z = \epsilon'_w$. Thus we have $z = w \in W$, because $\bar{\epsilon}_z \circ \pi = \epsilon_z = \epsilon'_w \circ \pi = \epsilon_w$.

Next we assume that W is closed in V . Since A is finitely generated as K-algebra, so is $\pi(A)$. Thus all we have to do is to show that the following evaluation map

$$E:W \to \mathrm{Hom}_{K\text{-alg}}(\pi(A),K)$$
$$(E:w \longrightarrow \epsilon'_w)$$

is bijective. Let $\theta \in \mathrm{Hom}_{K\text{-alg}}(\pi(A),K)$, then $\theta\circ\pi \in \mathrm{Hom}_{K\text{-alg}}(A,K)$. Therefore, there exists $z \in V$ such that $\theta\circ\pi = \epsilon_z$. Since $\pi(f) = 0$ for all $f \in \mathscr{I}(W)$, we have $\epsilon_z(f) = f(z) = 0$ for any $f \in \mathscr{I}(W)$. Hence $z \in \mathscr{V}(\mathscr{I}(W)) = W$ (see Proposition 1.7). Since $\theta(\pi(f)) = \epsilon_z(f) = f(z) = \pi(f)(z) = \epsilon'_z(\pi(f))$ for all $\pi(f) \in \pi(A)$, we have $\theta = \epsilon'_z$. Thus the evaluation map E is onto. Finally assume that $\epsilon'_{w_1} = \epsilon'_{w_2}$ for certain w_1 and $w_2 \in W$, then we have

$$\epsilon'_{w_1} \circ \pi = \epsilon_{w_1} = \epsilon'_{w_2} \circ \pi = \epsilon_{w_2} ,$$

which implies $w_1 = w_2$, because $(V,A) \in \mathscr{A}(K)$. Q.E.D.

<u>Exercise 5</u>. Let $(V,A) \in \mathscr{A}(K)$ and W be a non-empty closed subset of V . Let S be a subset of W . Show that S is closed in $(W,\pi(A))$, i.e., $S = \mathscr{V}_W(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset \pi(A)$ if and only if there exists an ideal I of A such that $S = \mathscr{V}_V(I) \cap W$.

<u>(2.3) Theorem</u>. Let (V,A) be any affine variety over K , then (V,A) is isomorphic to a subvariety of $(K^n, K[X_1,X_2,\dots,X_n])$ for some n . More precisely assume that A is generated by n-elements $\{a_1,a_2,\dots,a_n\}$ and let φ be a map which takes $v \in V$ to $(a_1(v),a_2(v),\dots,a_n(v)) \in K^n$, then $\varphi:V \to K^n$ is a morphism of affine varieties such that $\varphi(V) = \mathscr{V}_{K^n}(\mathrm{Ker}\ \varphi^*)$ and the map

$$\varphi_o:V \longrightarrow \varphi(V)$$
$$(\varphi_o:v \longrightarrow \varphi(v))$$

is an isomorphism of varieties.

For the proof of this theorem we need the following lemma and proposition.

<u>(2.4) Lemma</u>. Let (U,A), $(V,B) \in \mathcal{A}(K)$ and let $\theta : B \to A$ be a K-algebra homomorphism. Then there exists a unique morphism $\varphi : U \to V$ such that $\theta = \varphi^*$.

<u>Proof</u>. For any $u \in U$ we define $\varphi(u) \in V$ to be a unique element in V such that $\epsilon_u \circ \theta = \epsilon_{\varphi(u)}$. Notice that $\epsilon_u \circ \theta \in \mathrm{Hom}_{K\text{-alg}}(B,K)$ and $(V,B) \in \mathcal{A}(K)$. Since $\epsilon_u \circ \theta(b) = \theta(b)(u) = \epsilon_{\varphi(u)}(b) = b(\varphi(u)) = \varphi^*(b)(u)$ for $b \in B$ and $u \in U$, we have $\theta(b) = \varphi^*(b)$ for all $b \in B$, i.e., $\theta = \varphi^*$.

Next we shall prove the uniqueness. Assume that $\psi^* = \theta$ for some morphism $\psi : U \to V$, then for $b \in B$ and $u \in U$, $\varphi^*(b)(u) = \psi^*(b)(u)$, i.e., $b(\varphi(u)) = b(\psi(u))$. Thus we have $\epsilon_{\varphi(u)} = \epsilon_{\psi(u)}$, which implies $\varphi(u) = \psi(u)$. Hence φ is unique.

Q.E.D.

<u>(2.5) Proposition</u>. Let (U,A), $(V,B) \in \mathcal{A}(K)$ and $\varphi : U \to V$ be a morphism such that $\varphi^*(B) = A$. Then $\varphi(U) = \mathcal{V}(\mathrm{Ker}\,\varphi^*)$, i.e., $\varphi(U)$ is closed in V and the map

$$\varphi_o : U \longrightarrow \varphi(U)$$

$$(\varphi_o : u \longrightarrow \varphi(u))$$

defined from φ is an isomorphism of varieties.

<u>Proof</u>. First we show that $\varphi(U)$ is closed. Let $I = \mathrm{Ker}\,\varphi^*$ and $f \in I$, then $f(\varphi(u)) = \varphi^*(f)(u) = 0$ for any $u \in U$. Hence $\varphi(U) \subset \mathcal{V}(I)$. Next let $v \in \mathcal{V}(I)$. Since $\epsilon_v(I) = 0$, the map

$$\theta : A \longrightarrow K$$

$$(\theta : \varphi^*(f) \longrightarrow \epsilon_v(f))$$

is well-defined, where $f \in B$. Hence $\theta = \epsilon_u$ for some $u \in U$, because $(U,A) \in \mathcal{A}(K)$. Thus we have

$$\epsilon_u(\varphi^*(f)) = \varphi^*(f)(u) = f(\varphi(u)) = f(v)$$

for all $f \in B$. Hence $\epsilon_{\varphi(u)} = \epsilon_v$, which implies $v = \varphi(u)$. Therefore, $\mathcal{V}(I) \subset \varphi(U)$. Thus we have $\varphi(U) = \mathcal{V}(I)$.

Now let $\pi: M(V,K) \to M(\varphi(U),K)$ be the comorphism of the inclusion map $\varphi(U) \to V$. Then from Proposition 2.2 we have $(\varphi(U),\pi(B)) \in \mathcal{A}(K)$. Since

$$\varphi_o^*(\pi(f))(u) = \pi(f) \circ \varphi_o(u) = f(\varphi(u)) = \varphi^*(f)(u)$$

for any $f \in B$, $u \in U$, $\varphi_o^*(\pi(f)) = \varphi^*(f)$ for any $f \in B$. Since $\varphi^*(B) = A$ from the assumption, $\varphi_o^*: \pi(B) \to A$ is also surjective. However, since $\mathrm{Ker}(\pi|B) = \mathrm{Ker}\ \varphi^*$, φ_o^* is injective and φ_o^* is a K-algebra isomorphism. From Lemma 2.4 there exists a unique morphism

$$\psi: \varphi(U) \to U \quad \text{such that} \quad \psi^* = (\varphi_o^*)^{-1} .$$

Finally we only have to check that $\varphi_o \circ \psi = 1_{\varphi(U)}$ and $\psi \circ \varphi_o = 1_U$. Since

$$(\varphi_o \circ \psi)^* = \psi^* \circ \varphi_o^* = 1_{\pi(B)} = (1_{\varphi(U)})^* \quad \text{and}$$

$$(\psi \circ \varphi_o)^* = \varphi_o^* \circ \psi^* = 1_A = (1_U)^* ,$$

from Lemma 2.4 we have $\varphi_o \circ \psi = 1_{\varphi(U)}$ and $\psi \circ \varphi_o = 1_U$. Q.E.D.

<u>Proof of Theorem 2.3</u>. Since A is finitely generated, let $\{a_1, a_2, \ldots, a_n\}$ be a set of generators of A . Then we can define a K-algebra homomorphism $\theta: K[X_1, X_2, \ldots, X_n] \to A$ such that θ takes X_i to a_i . From Lemma 2.4 there exists a morphism $\varphi: V \to K^n$ such that $\theta = \varphi^*$. Since $\epsilon_v \circ \theta = \epsilon_{\varphi(v)}$ for all $v \in V$ from the proof of Lemma 2.4, we have

$$\varphi(v) = (a_1(v), a_2(v), \ldots, a_n(v))$$

for any $v \in V$. From Proposition 2.5 V is isomorphic to the subvariety $\varphi(V) = \mathcal{V}(\mathrm{Ker}\ \varphi^*)$ of K^n , because θ is surjective.

Q.E.D.

Exercise 6. Let $(V,A) \in \mathscr{A}(K)$ and x be any point in V. Show that $\{x\}$ is closed in V.

Finally we shall end this section introducing an idea of principal open sets which are open sets of a given affine variety but have another affine variety structures.

(2.6) Definition. Let $(V,A) \in \mathscr{A}(K)$, and f be an element of A. We call a subset of the form

$$V_f = \{v \in V \mid f(v) \neq 0\}$$

a principal open set of V.

(2.7) Lemma. Let $(V,A) \in \mathscr{A}(K)$. Let $f \in A - \{0\}$ and A_f be the ring of fractions of A by $\{f^n \mid n \in \mathbb{N}\}$ (for the definition of a ring of fractions see Lang [1, Chap.II,§3]), where $\mathbb{N}$ is the set of natural numbers including 0. Then the ring A_f is also a K-algebra (we define

$$c(\frac{a}{f^n}) = \frac{ca}{f^n} \quad \text{for any} \quad c \in K \quad \text{and} \quad \frac{a}{f^n} \in A_f \ ),$$

and we can define an injective K-algebra homomorphism

$$\rho : A_f \to M(V_f, K)$$

such that $\rho(\frac{a}{f^n})(v) = \frac{a(v)}{f(v)^n}$ for any $v \in V_f$, where $\frac{a}{f^n} \in A_f$.

Exercise 7. Verify Lemma 2.7.

Remark. Let $\iota_f : A \to A_f$ be the map such that $\iota_f(a) = \frac{a}{1}$, where A and f are as in Lemma 2.7, then ι_f is a K-algebra homomorphism.

(2.8) Proposition. Assume that $(V,A) \in \mathscr{A}(K)$ and $f \in A - \{0\}$. Then considering A_f as a K-subalgebra of $M(V_f, K)$ by Lemma 2.7, we have $(V_f, A_f) \in \mathscr{A}(K)$.

Proof. Since A_f is generated by $a_1, a_2, \ldots, a_n$, $\frac{1}{f}$ where $(a_1, a_2, \ldots, a_n)$ is a set of generators of A , we only have to show that the evaluation map of V_f into $\mathrm{Hom}_{K\text{-alg}}(A_f, K)$ is bijective.

Let $\theta \in \mathrm{Hom}_{K\text{-alg}}(A_f, K)$, then we have $\theta \circ \iota_f = \epsilon_v$ for some $v \in V$, because $\theta \circ \iota_f \in \mathrm{Hom}_{K\text{-alg}}(A, K)$. Since $\theta(1) = 1 = \theta(\frac{f}{1} \cdot \frac{1}{f}) = f(v)\theta(\frac{1}{f})$, we have $f(v) \neq 0$, i.e., $v \in V_f$, and $\theta(\frac{1}{f}) = \frac{1}{f(v)}$. Hence $\theta(\frac{a}{f^n}) = \theta(\frac{a}{1})(\frac{1}{f^n}) = a(v)(\frac{1}{f(v)})^n = \frac{a(v)}{f(v)^n}$ for any $\frac{a}{f^n} \in A_f$ and θ is an evaluation at $v \in V_f$.

Since $\epsilon_v = \epsilon_v'$ as elements of $\mathrm{Hom}_{K\text{-alg}}(A_f, K)$ means $\epsilon_v \circ \iota_f = \epsilon_{v'} \circ \iota_f$, where $v, v' \in V_f$, we have

$$\epsilon_v \circ \iota_f(a) = a(v) = a(v') = \epsilon_{v'} \circ \iota_f(a) \quad \text{for any } a \in A .$$

Hence $v = v'$ and the evaluation map of V_f into $\mathrm{Hom}_{K\text{-alg}}(A_f, K)$ is injective. Q.E.D.

(2.9) Remark. Let $(V, A) \in \mathcal{A}(K)$ and $O \neq \emptyset$ be an open set of V , then there exist $f_1, f_2, \ldots, f_l \in A$ such that

$$O = V_{f_1} \cup V_{f_2} \cup \cdots \cup V_{f_l} .$$

Exercise 8. Let $(V, A) \in \mathcal{A}(K)$ and $f \in A - \{0\}$.

(1) Verify that a subset O of V_f is open in (V_f, A_f) if and only if O is open in (V, A) .

(2) Show that $V_f \cap F$ is closed in (V_f, A_f) for any closed subset F in (V, A) .

3. Products of affine algebraic varieties

Let (U,A), $(V,B) \in \mathcal{A}(K)$. We shall show how to construct a product of affine varieties (U,A) and (V,B) in $\mathcal{A}(K)$ by making use of the tensor product of the two commutative algebras A and B .

Let R be a commutative ring and E_1 , E_2 , ... , E_n and F be left R-modules. We call a map

$$f : E_1 \times E_2 \times \ldots \times E_n \longrightarrow F$$

R-multilinear if

$$f(e_1,\ldots,e_i + e_i',\ldots,e_n) = f(e_1,\ldots,e_i,\ldots,e_n) + f(e_1,\ldots,e_i',\ldots,e_n)$$

for all $(e_1,\ldots,e_n) \in E_1 \times E_2 \times\ldots\times E_n$ and $e_i' \in E_i$, where $1 \leq i \leq n$, and

$$f(e_1,\ldots,re_i,\ldots,e_n) = rf(e_1,\ldots,e_i,\ldots,e_n)$$

for all $(e_1,\ldots,e_n) \in E_1 \times E_2 \times\ldots\times E_n$, $r \in R$ and $1 \leq i \leq n$.

A tensor product of E_1 , E_2 ,..., E_n is a pair (T,θ) where T is a left R-module and

$$\theta : E_1 \times E_2 \times\ldots\times E_n \longrightarrow T$$

is an R-multilinear map with the following universal property: if F is any left R-module and

$$f : E_1 \times E_2 \times\ldots\times E_n \longrightarrow F$$

is any multilinear map then there exists a unique R-homomorphism $\tilde{f} : T \longrightarrow F$ such that

$$\begin{array}{ccc} & T & \\ {}^{\theta}\nearrow & \circlearrowleft & \searrow^{\exists_1 \tilde{f}} \\ E_1 \times E_2 \times \ldots \times E_n & \xrightarrow[f]{} & F . \end{array}$$

We write $e_1 \otimes e_2 \otimes\ldots\otimes e_n$ for $\theta(e_1,e_2,\ldots,e_n)$ where $(e_1,e_2,\ldots,e_n) \in E_1 \times E_2 \times\ldots\times E_n$.

In order to introduce necessary notations we should like to review some results from tensor products.

Proposition. Let R be a commutative ring. Assume that A is a free R-module with R-basis $(a_i)_{i \in I}$ and B is also a free R-module with R-basis $(b_j)_{j \in J}$, then $A \otimes_R B$ is a free R-module with R-basis

$$(a_i \otimes b_j)_{(i,j) \in I \times J} .$$

(3.1) Proposition. Let A and B be commutative algebras over a field k. Then $A \otimes_k B$ is a commutative k-algebra with the following multiplication

$$(\sum_i a_i \otimes b_i)(\sum_j a_j \otimes b_j) = \sum_{i,j} a_i a_j \otimes b_i b_j ,$$

where $\sum_i a_i \otimes b_i$, $\sum_j a_j \otimes b_j \in A \otimes_k B$.

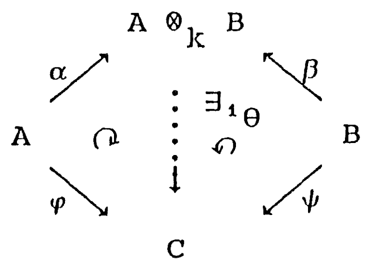
Proposition. Let A and B be commutative algebras over a field k. Let $\alpha : A \longrightarrow A \otimes_k B$ and $\beta : B \longrightarrow A \otimes_k B$ be two k-algebra homomorphisms such that

$$\alpha(a) = a \otimes 1 \quad \text{for any } a \in A \text{ and}$$
$$\beta(b) = 1 \otimes b \quad \text{for any } b \in B .$$

Then the triple $(A \otimes_k B, \alpha, \beta)$ satisfies the following property: for any commutative k-algebra C and k-algebra homomorphisms $\varphi : A \longrightarrow C$ and $\psi : B \longrightarrow C$ there exists a unique k-algebra homomorphism

$$\theta : A \otimes_k B \longrightarrow C$$

which makes the following diagram commutative,



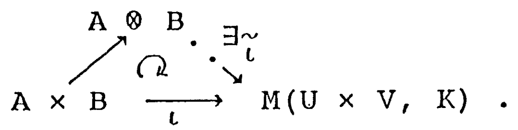
.

Remark. $\theta(a \otimes b) = \varphi(a)\psi(b)$ for any $a \otimes b \in A \otimes_k B$.

<u>(3.2) Lemma</u>. Let (U,A) , $(V,B) \in \mathscr{A}(K)$ and ι be a map of $A \times B$ into $M(U \times V, K)$ which takes $(a,b) \in A \times B$ to $\iota((a,b)) : U \times V \longrightarrow K$. Then there exists a K-algebra homo-

$(\iota((a,b)) : (u,v) \longrightarrow a(u)b(v))$

morphism $\tilde{\iota}$ of $A \otimes_K B$ into $M(U \times V, K)$ which makes the following diagram commutative.



Further the map $\tilde{\iota}$ is injective.

<u>Proof</u>. We only have to check that $\tilde{\iota}$ is injective. Take a K-basis $\{a_i\}_{i \in I}$ of A , then it is clear that every element of $A \otimes_K B$ is a sum of elements $\{a_i \otimes b_i \mid i \in I\}$ for some b_i's $\in B$. Assume $\tilde{\iota}(\sum_{i \in I} a_i \otimes b_i) = 0$, then we have

$\tilde{\iota}(\sum_{i \in I} a_i \otimes b_i)\ (u,v) = \sum_{i \in I} a_i(u)\ b_i(v) = 0$ for all $(u,v) \in U \times V$.

Hence we have $\sum_{i \in I} a_i\ b_i(v) = 0$ for any fixed $v \in V$. Since $\{a_i\}_{i \in I}$ is a K-basis of A , we have $b_i(v) = 0$ for any i . Thus $b_i = 0$. Hence $\sum_{i \in I} a_i \otimes b_i = 0$ and $\tilde{\iota}$ is injective. Q.E.D.

From the above lemma we shall regard $A \otimes_K B$ as a K-subalgebra of $M(U \times V, K)$ by

$$\tilde{\iota} : A \otimes_K B \longrightarrow M(U \times V, K) .$$

Notice that $\tilde{\iota}(a \otimes b)\ (u,v) = a(u) \cdot b(v)$ for any $a \otimes b \in A \otimes_K B$ and $(u,v) \in U \times V$.

<u>(3.3) Proposition</u>. Let (U,A) and (V,B) be affine varieties over K , then

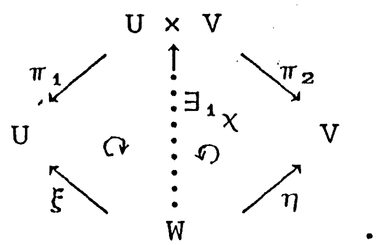
(1) $(U \times V, A \otimes_K B) \in \mathscr{A}(K)$,

(2) let $\pi_1 : U \times V \longrightarrow U$ and $\pi_2 : U \times V \longrightarrow V$

$(\pi_1 : (u,v) \longrightarrow u)$ $(\pi_2 : (u,v) \longrightarrow v)$

be projections, then π_1 and π_2 are morphisms and are open maps (i.e. they map any open set Y of $U \times V$ onto the open sets $\pi_1(Y)$ of U and $\pi_2(Y)$ of V respectively), and

(3) for any affine variety (W,C) over K and morphisms $\xi : W \longrightarrow U$ and $\eta : W \longrightarrow V$ there exists a unique morphism $\chi : W \longrightarrow U \times V$ which makes the following diagram commutative,



<u>Proof</u>. (1) Assume that A is generated by $\{a_1, a_2, \ldots a_m\}$ and B is generated by $\{b_1, b_2, \ldots, b_n\}$ as K-algebras, then $A \otimes_K B$ is generated by $\{a_i \otimes 1, 1 \otimes b_j\} \mid 1 \leq i \leq m$ and $1 \leq j \leq n\}$ as K-algebra. Thus we only have to show that the evaluation map : $U \times V \longrightarrow \mathrm{Hom}_{K\text{-alg}}(A \otimes_K B, K)$ is bijective.

Let θ be any element of $\mathrm{Hom}_{K\text{-alg}}(A \otimes_K B, K)$, then the maps

$$\theta_1 : A \longrightarrow A \otimes_K B \xrightarrow{\theta} K$$
$$(a \longrightarrow a \otimes 1)$$

and

$$\theta_2 : B \longrightarrow A \otimes_K B \xrightarrow{\theta} K$$
$$(b \longrightarrow 1 \otimes b)$$

belong to $\mathrm{Hom}_{K\text{-alg}}(A,K)$ and $\mathrm{Hom}_{K\text{-alg}}(B,K)$ respectively. Since (U,A) , $(V,B) \in \mathscr{A}(K)$, there exist $u \in U$ and $v \in V$ such that $\theta_1 = \epsilon_u$ and $\theta_2 = \epsilon_v$. It is easy to check that $\theta = \epsilon_{(u,v)}$. Thus the evaluation map is onto.

Now assume that $\epsilon_{(u,v)} = \epsilon_{(u',v')}$ for some (u,v) and $(u',v') \in U \times V$, then $\epsilon_{(u,v)}(a \otimes 1) = \epsilon_{(u',v')}(a \otimes 1)$ for any

$a \in A$. Hence we have $a(u) = a(u')$ for any $a \in A$, i.e., $\epsilon_u = \epsilon_{u'}$. Hence $u = u'$. Similarly we have $v = v'$. Thus the evaluation map is injective.

(2) First we shall prove that $\pi_1^*(A) \subset A \otimes_K B$. Let f be an element of A ,

$$\pi_1^*(f) : U \times V \xrightarrow{\pi_1} U \xrightarrow{f} K$$

$$(\pi_1^*(f) : (u,v) \longrightarrow u \longrightarrow f(u)) .$$

Since $f(u) = (f \otimes 1)(u,v)$ for any $(u,v) \in U \times V$, we have $\pi_1^*(f) = f \otimes 1 \in A \otimes_K B$. Thus π_1 is a morphism. Similarly π_2 is also a morphism.

Now let Y be an open set of $U \times V$ and v be a fixed element of V . Let $j_v : U \longrightarrow U \times V$ be a map which takes $u \in U$ to $(u,v) \in U \times V$. Then we have

$$j_v^*(\sum_i f_i \otimes g_i)(u) = \sum_i f_i(u)\ g_i(v) \quad \text{for any} \quad u \in U$$

and $\quad \sum_i f_i \otimes g_i \in A \otimes_K B$.

$$j_v^*(\sum_i f_i \otimes g_i) : U \xrightarrow{j_v} U \times V \xrightarrow{\sum_i f_i \otimes g_i} K .$$

$$(j_v^*(\sum_i f_i \otimes g_i) : u \longrightarrow (u,v) \longrightarrow \sum_i f_i(u) g_i(v))$$

Hence $j_v^*(\sum_i f_i \otimes g_i) = \sum_i g_i(v) f_i \in A$. Therefore j_v is a morphism.

Since $\pi_1(Y) = \bigcup_{v \in \pi_2(Y)} j_v^{-1}(Y)$, $\pi_1(Y)$ is open (see Proposition 1.8).

Similarly, $\pi_2(Y)$ is also open.

(3) Define $\chi : W \to U \times V$ to be the map which takes $w \in W$ to $(\xi(w), \eta(w)) \in U \times V$. Then we have

$$\chi^*(f) : W \xrightarrow{\chi} U \times V \xrightarrow{f} K$$

$$(\chi^*(f) : w \longrightarrow (\xi(w), \eta(w)) \longrightarrow \sum_i f_i \xi(w) \cdot g_i \eta(w))$$

for any $f = \sum_i f_i \otimes g_i \in A \otimes_K B$.

Since $\sum_i f_i\xi(w)\cdot g_i\eta(w) = \sum_i \xi^*(f_i)(w)\cdot\eta^*(g_i)(w) = (\sum_i \xi^*(f_i)\cdot\eta^*(g_i))(w)$ for any $w \in W$, we have $\chi^*(f) = \sum_i \xi^*(f_i)\cdot\eta^*(g_i) \in C$. Thus χ is a desired morphism. The uniqueness is clear from the definition of χ.

Q.E.D.

We call $(U \times V, A \otimes_K B)$ the <u>product</u> of affine varieties (U,A) and (V,B).

<u>(3.4) Proposition</u>. Let (U,A) and (V,B) be affine varieties over K. Then

(1) $F_1 \times F_2$ is closed in the affine variety $(U \times V, A \otimes B)$ for any closed sets F_1 in (U,A) and F_2 in (V,B).

(2) $O_1 \times O_2$ is open in the affine variety $(U \times V, A \otimes B)$ for any open sets O_1 in (U,A) and O_2 in (V,B).

<u>Exercise 9</u>. Prove Proposition 3.4.

4. Tangent spaces to affine algebraic varieties

We first introduce an algebra of dual numbers over an arbitrary field k .

(4.1) Definition. An algebra of dual numbers $k[\epsilon]$ over a field k is defined by $k[\epsilon] = k \times k$,

$$(x,y) + (x',y') = (x+x', y+y') \quad \text{and}$$
$$(x,y)(x',y') = (xx', xy'+yx')$$

for any $(x,y), (x',y') \in k[\epsilon]$. Then $k[\epsilon]$ is a k-algebra with a unity element $(1,0)$. We write ϵ for the element $(0,1)$. Thus $\epsilon^2 = 0$ and $k[\epsilon] = k \oplus k\epsilon$.

Now let (V,A) be an affine variety over K and assume $\alpha: A \to K[\epsilon]$ be a K-algebra homomorphism, then we have

(4.2). $\alpha = \beta + \gamma\epsilon$, i.e., $\alpha(a) = \beta(a) + \gamma(a)\epsilon$ for any $a \in A$, where

$\beta: A \to K$ is a K-algebra homomorphism

and $\gamma: A \to K$ is a K-linear map satisfying

(4.3). $\gamma(ab) = \beta(a)\gamma(b) + \gamma(a)\beta(b)$ for any a and $b \in A$.

Conversely for any couple of K-linear maps β and γ of A into K such that β is a K-algebra homomorphism and γ satisfies (4.3), the map $\alpha: A \to K[\epsilon]$ defined by (4.2) is a K-algebra homomorphism.

Let $\alpha: A \to K[\epsilon]$ be a K-algebra homomorphism and β, γ be as in (4.2), then we have $\beta = \epsilon_v$ for some $v \in V$. Hence (4.3) has the following form:

(4.3)'. $\gamma(ab) = a(v)\gamma(b) + \gamma(a)b(v)$ for any $a,b \in A$.

<u>(4.4) Definition</u>. Let (V,A) be an affine variety over K. A K-linear map $\gamma : A \to K$ satisfying (4.3)' for some $v \in V$ is called a <u>tangent vector</u> to V at v. The set of all tangent vectors at v forms a vector space over K called the <u>tangent space</u> $T(V)_v$ of V at the point v.

The operations of $T(V)_v$ are as follows:

$$(\gamma+\gamma')(a) = \gamma(a) + \gamma'(a)$$
$$(c\gamma)(a) = c\gamma(a) ,$$

where $\gamma, \gamma' \in T(V)_v$, $a \in A$ and $c \in K$.

<u>Exercise 10</u>. Let $(V,A) \in \mathscr{A}(K)$ and $v \in V$. Verify that

(1) $\gamma(1) = 0$ for any $\gamma \in T(V)_v$,

(2) each tangent vector γ to V at v is determined by the values at generators of A.

<u>(4.5) Example</u>. Let $(K^n, K[X_1,X_2,\ldots,X_n])$ be an affine n-space, and v be any fixed element of K^n. Let γ_i be a map of $K[X_1,X_2,\ldots,X_n]$ into K such that

$$\gamma_i(f) = \frac{\delta f}{\delta X_i}(v) \quad \text{for any} \quad f \in K[X_1,\ldots,X_n] .$$

Then the set $(\gamma_1,\gamma_2,\ldots,\gamma_n)$ is a K-basis of $T(K^n)_v$.

<u>Proof</u>. Let γ_i be a map of $K[X_1,X_2,\ldots,X_n]$ into K such that $\gamma_i(f) = \frac{\delta f}{\delta X_i}(v)$ for any $f \in K[X_1,X_2,\ldots,X_n]$. Then for any $f,g \in K[X_1,X_2,\ldots,X_n]$ and $c \in K$ we have

$$\gamma_i(f+g) = \frac{\delta(f+g)}{\delta X_i}(v) = \frac{\delta f}{\delta X_i}(v) + \frac{\delta g}{\delta X_i}(v)$$

and

$$\gamma_i(cf) = \frac{\delta(cf)}{\delta X_i}(v) = c\frac{\delta f}{\delta X_i}(v) .$$

Hence γ_i is K-linear. Now let $a,b \in K[X_1,X_2,\ldots,X_n]$, then

$$\frac{\delta(ab)}{\delta X_i} = \frac{\delta a}{\delta X_i} b + a\frac{\delta b}{\delta X_i} .$$

Thus we have $\gamma_i(ab) = b(v)\gamma_i(a) + a(v)\gamma_i(b)$.
Therefore, $\gamma_i \in T(K^n)_v$ for any $1 \leq i \leq n$. Next we shall prove that $\{\gamma_i \mid 1 \leq i \leq n\}$ span $T(K^n)_v$. Let γ be an arbitrary element of $T(K^n)_v$. Put $\gamma' = \sum_{i=1}^{n} \gamma(X_i)\gamma_i$, then $\gamma' \in T(K^n)_v$ and $\gamma'(X_i) = \gamma(X_i)$. Since a tangent vector is determined by the values at generators of $K[X_1,X_2,\ldots,X_n]$, we have shown that $T(K^n)_v$ is generated by $\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$ as a K-space.

Finally assume that $\sum_{i=1}^{n} c_i\gamma_i = 0$ for some $\{c_i\} \subset K$, then

$$\sum_{i=1}^{n} c_i\gamma_i(f) = \sum_{i=1}^{n} c_i \frac{\delta f}{\delta X_i}(v) = 0$$

for all $f \in K[X_1,X_2,\ldots,X_n]$. By taking $f = X_j$ we have $c_j = 0$ for all $1 \leq j \leq n$. Q.E.D.

Remark. $\gamma_i(X_j) = \frac{\delta X_j}{\delta X_i}(v) = \delta_{ij}\cdot 1$ for all $1 \leq i,j \leq n$, where $\{\gamma_i\}$ are as in Example 4.5.

Now let $\varphi : U \to V$ be a morphism of affine varieties (U,A) into (V,B) . If $\alpha : A \to K[\epsilon]$ is a K-algebra homomorphism, then

$$\alpha \circ \varphi^* : B \xrightarrow{\varphi^*} A \xrightarrow{\alpha} K[\epsilon]$$

is also a K-algebra homomorphism. If $\alpha = \beta+\gamma\epsilon$ with $\beta = \epsilon_u$ for some $u \in U$, then

$$\alpha \circ \varphi^* = \beta \circ \varphi^* + (\gamma \circ \varphi^*)\epsilon$$

and $\beta \circ \varphi^* = \epsilon_u \circ \varphi^* = \epsilon_{\varphi(u)}$. Hence we can conclude that if $\gamma \in T(U)_u$ for some $u \in U$, then $\gamma \circ \varphi^* \in T(V)_{\varphi(u)}$ and the map

$$d\varphi_u : T(U)_u \longrightarrow T(V)_{\varphi(u)}$$

$$(d\varphi_u : \gamma \longrightarrow \gamma\circ\varphi^*)$$

is K-linear.

<u>(4.6) Definition</u>. Let $\varphi : U \to V$ be a morphism of affine varieties (U,A) into (V,B) . Let u be an element of U , then we call the map

$$d\varphi_u : T(U)_u \longrightarrow T(V)_{\varphi(u)}$$

$$(d\varphi_u : \quad \gamma \longrightarrow \gamma \circ \varphi^*)$$

the <u>differential</u> of φ at u .

The following proposition is clear.

<u>(4.7) Proposition</u>. Let (U,A), (V,B) and $(W,C) \in \mathcal{A}(K)$ and $\varphi : U \to V$ and $\psi : V \to W$ be two morphisms. Then

(1) $d(\psi \circ \varphi)_u = d\psi_{\varphi(u)} \circ d\varphi_u$ for any $u \in U$.

$$\begin{array}{ccc} T(U)_u & \xrightarrow{d\varphi_u} & T(V)_{\varphi(u)} \\ & {\scriptstyle d(\psi\circ\varphi)_u} \searrow \quad \circlearrowleft \quad \swarrow {\scriptstyle d\psi_{\varphi(u)}} & \\ & T(W)_{\psi\circ\varphi(u)} & \end{array} .$$

(2) $(d1_U)_u = 1_{T(U)_u}$ for any $u \in U$.

(3) If $\varphi : (U,A) \cong (V,B)$ is an isomorphism of affine varieties, then $d\varphi_u : T(U)_u \cong T(V)_{\varphi(u)}$ for any $u \in U$.

<u>(4.8) Proposition</u>. Let (U,A) and $(V,B) \in \mathcal{A}(K)$, and $\pi_1 : U \times V \to U$ and $\pi_2 : U \times V \to V$ be the projections. Then for any pair $(u,v) \in U \times V$ the map φ of $T(U \times V)_{(u,v)}$ into $T(U)_u \dot{+} T(V)_v$ (the external direct sum) which takes $\eta \in T(U \times V)_{(u,v)}$ to $(\eta \circ \pi_1^*, \eta \circ \pi_2^*) \in T(U)_u \dot{+} T(V)_v$ is a K-linear isomorphism.

<u>Proof</u>. Since π_1 and π_2 are morphisms, we have $(\eta \circ \pi_1^*, \eta \circ \pi_2^*) \in T(U)_u \dot{+} T(V)_v$ and the map φ is well-defined and

K-linear. Let

$$\iota_1 : U \longrightarrow U \times V \quad \text{and} \quad \iota_2 : V \longrightarrow U \times V$$
$$(\iota_1 : x \longrightarrow (x,v)) \qquad (\iota_2 : y \longrightarrow (u,y))$$

be injective maps of U and V into $U \times V$ respectively, then ι_1 and ι_2 are morphisms and we have

$$\pi_1 \circ \iota_1 = 1_U \quad \text{and} \quad \pi_2 \circ \iota_2 = 1_V .$$

From Proposition 4.7 we have

$d(\pi_1 \circ \iota_1)_u = (d\pi_1)_{(u,v)} \circ (d\iota_1)_u = 1_{T(U)_u}$ and

$d(\pi_2 \circ \iota_2)_v = (d\pi_2)_{(u,v)} \circ (d\iota_2)_v = 1_{T(V)_v}$. Let

$(\eta_1,\eta_2) \in T(U)_u \dot{+} T(V)_v$, then from the above facts

$$\eta_1 = (d\pi_1)_{(u,v)}(\eta_1 \circ \iota_1^*)$$

and
$$\eta_2 = (d\pi_2)_{(u,v)}(\eta_2 \circ \iota_2^*) .$$

Since $\eta_1 \circ \iota_1^*$ and $\eta_2 \circ \iota_2^*$ are elements of $T(U \times V)_{(u,v)}$ and

$$(d\pi_1)_{(u,v)}(\eta_2 \circ \iota_2^*) = (d\pi_1)_{(u,v)}(d\iota_2)_v(\eta_2)$$
$$= d(\pi_1 \circ \iota_2)_v(\eta_2) = \eta_2 \circ (\pi_1 \circ \iota_2)^* = 0 ,$$

we have $\varphi(\eta_1 \circ \iota_1^* + \eta_2 \circ \iota_2^*) = (\eta_1,\eta_2)$.

Finally assume that $\varphi(\eta) = 0$ for some $\eta \in T(U \times V)_{(u,v)}$, then $\eta \circ \pi_1^* = 0 = \eta \circ \pi_2^*$. Since

$$\pi_1^* : A \longrightarrow A \otimes B$$
$$(\pi_1^* : f \longrightarrow \pi_1^*(f) = f \otimes 1 : (x,y) \rightarrow f(x)) ,$$

where $(x,y) \in U \times V$, $\eta \circ \pi_1^* = \eta \circ \pi_2^* = 0$ implies $\eta = 0$. Q.E.D.

<u>Exercise 11</u>. Let (U,A), (U',A'), (V,B) and $(V',B') \in \mathcal{A}(K)$ and $\mu : U \rightarrow U'$ and $\rho : V \rightarrow V'$ be morphisms of affine varieties. Let h be a map of $U \times V$ into $U' \times V'$ which takes $(x,y) \in U \times V$ to $(\mu(x), \rho(y)) \in U' \times V'$, then h is a morphism of affine varieties and

$$(dh)_a(\eta_1 \circ \iota_1^* + \eta_2 \circ \iota_2^*) = (d\mu)_u(\eta_1) \circ \iota_1'^* + (d\rho)_v(\eta_2) \circ \iota_2'^*$$

for any point $a = (u,v) \in U \times V$, where $\eta_1 \in T(U)_u$, $\eta_2 \in T(V)_v$,

$$\iota_1 : U \longrightarrow U \times V\ , \quad \iota_2 : V \longrightarrow U \times V\ , \quad \iota_1' : U' \longrightarrow U' \times V'$$
$$(\iota_1 : x \longrightarrow (x,v)) \quad (\iota_2 : y \longrightarrow (u,y)) \quad (\iota_1' : x \longrightarrow (x,\rho(v)))$$

and $\iota_2' : V' \longrightarrow U' \times V'$.
$$(\iota_2' : y \longrightarrow (\mu(u),y))$$

- Noetherian ring -

Let R be a ring and M a left R-module. We call M a <u>Noetherian module</u> if it satisfies any of the following three equivalent conditions:

(1) Every submodule of M is finitely generated.
(2) Every ascending sequence of submodules of M ,
$$M_1 \subset M_2 \subset M_3 \subset \ldots$$
terminates, i.e., there exists i such that $M_i = M_{i+1} = \ldots$
(3) Every non-empty set S of submodules of M has a maximal element, i.e., a submodule M_o such that for any element N of S which contains M_o we have $N = M_o$.

We shall say that a commutative ring R is <u>Noetherian</u> if R is a Noetherian module as a left R-module.

<u>(4.9) Examples</u>.

(1) <u>Hilbert's Basis Theorem</u> (see e.g. Lang [1, Th.2.1 on p.226]). If A is a commutative Noetherian ring, then the polynomial ring $A[X]$ in one variable with coefficient in A is also Noetherian.

(2) From (1), the polynomial ring $k[X_1,X_2,\ldots,X_n]$ in n-variables over a field k is Noetherian.

(3) From (2), any finitely generated commutative algebra over a field k is Noetherian.

Now let V be a subvariaty of K^n , i.e., V is a non-empty closed subset of K^n . Let $\iota : V \longrightarrow K^n$ be an inclusion map, then ι is a morphism. Since $\iota^*(K[X_1,X_2,\ldots,X_n]) = K[V]$, we have

$$K[X_1,X_2,\ldots,X_n]/\mathcal{J}(V) \cong K[V] \; .$$

Since $K[X_1,X_2,\ldots,X_n]$ is Noetherian, $\mathcal{J}(V)$ is finitely generated.

<u>(4.10) Proposition</u>. Let V be a subvariaty of K^n and $\Phi_v = \{\gamma \in T(K^n)_v \mid \gamma(\mathcal{J}(V)) = 0\}$, where $v \in V$. Then

(1) the map $\rho : \Phi_v \longrightarrow T(V)_v$ is a K-linear isomorphism, where

$$(\rho : \gamma \longrightarrow \bar{\gamma})$$

$\bar{\gamma}(\iota^*(f)) = \gamma(f)$ for any $f \in K[X_1,X_2,\ldots,X_n]$, and

(2) $$\dim_K \Phi_v = n - \operatorname{rank} \begin{bmatrix} \frac{\delta f_1}{\delta X_1}(v) & ,\ldots, & \frac{\delta f_1}{\delta X_n}(v) \\ \vdots & & \vdots \\ \frac{\delta f_r}{\delta X_1}(v) & ,\ldots, & \frac{\delta f_r}{\delta X_n}(v) \end{bmatrix} ,$$

where $\{f_1,f_2,\ldots,f_r\}$ is a set of generators of $\mathcal{J}(V)$.

<u>Proof</u>. (1) Straightforward.
(2) From the proof of Example 4.5, if $\gamma \in T(K^n)_v$, then

$$\gamma = \sum_{j=1}^{n} \gamma(X_j)\gamma_j \; .$$

Thus we have: γ belongs to Φ_v .

$$\Leftrightarrow \gamma\Big(\sum_{i=1}^{r} a_i f_i\Big) = 0 \qquad \text{for any } a_i \in K[X_1,X_2,\ldots,X_n]$$

$$\Leftrightarrow \sum_{i=1}^{r} a_i(v)\gamma(f_i) = 0 \quad \text{for any } a_i \in K[X_1,X_2,\ldots,X_n]$$

$$\Leftrightarrow \gamma(f_i) = 0 \qquad \text{for any } 1 \leq i \leq r$$

$$\Leftrightarrow \begin{cases} \gamma(X_1)\gamma_1(f_1) + \ldots + \gamma(X_n)\gamma_n(f_1) = 0 \\ \vdots \\ \gamma(X_1)\gamma_1(f_r) + \ldots + \gamma(X_n)\gamma_n(f_r) = 0 \; . \end{cases}$$

Hence

$$\dim_K \Phi_v = \dim_K \{ (\chi_1, \chi_2, \ldots, \chi_n) \in K^n \;\Big|\; \begin{bmatrix} \gamma_1(f_1), \ldots, \gamma_n(f_1) \\ \vdots \qquad\qquad \vdots \\ \gamma_1(f_r), \ldots, \gamma_n(f_r) \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix} = 0 \}$$

$$= n - \operatorname{rank} \begin{bmatrix} \frac{\delta f_1}{\delta X_1}(v) & , \ldots , & \frac{\delta f_1}{\delta X_n}(v) \\ \vdots & & \vdots \\ \frac{\delta f_r}{\delta X_1}(v) & , \ldots , & \frac{\delta f_r}{\delta X_n}(v) \end{bmatrix} .$$

Q.E.D.

<u>Exercise 12</u>. Prove Proposition 4.10.1.

5. Noetherian spaces

In this section we show that an affine variety is a Noetherian space, which has a finite number of irreducible components.

Let (V,A) be an affine variety over K, then A is Noetherian from Examples 4.9. Let

$$W_1 \supset W_2 \supset \ldots \supset W_r \supset W_{r+1} \supset \ldots$$

be a descending chain of closed sets in V. Then we have

$$\mathscr{I}(W_1) \subset \mathscr{I}(W_2) \subset \ldots \subset \mathscr{I}(W_r) \subset \mathscr{I}(W_{r+1}) \subset \ldots$$

Since A is Noetherian there exists an integer n such that

$$\mathscr{I}(W_n) = \mathscr{I}(W_{n+1}) = \mathscr{I}(W_{n+2}) = \ldots$$

Thus we have shown that

$$W_n = \mathscr{V}(\mathscr{I}(W_n)) = \mathscr{V}(\mathscr{I}(W_{n+1})) = W_{n+1} = W_{n+2} \ldots$$

(see Proposition 1.7). Hence we have

(5.1) Proposition. Any affine variety (V,A) has the descending chain condition (D.C.C.) on its closed sets or equivalently it has the ascending chain condition (A.C.C.) on its open sets.

(5.2) Definition. We shall say that a topological space X is Noetherian if it has A.C.C. on its open sets, i.e., any non-empty set of open sets of X has a maximal element.

Thus an affine variety is a Noetherian space.

(5.3) Definition. A topological space X is irreducible if for any closed sets X_1 and X_2 such that $X = X_1 \cup X_2$ we always have $X = X_1$ or $X = X_2$. A subset S of a topological space X is irreducible if S is irreducible with its induced topology.

From the definition X is an irreducible topological space if and only if any two non-empty open sets in X have non-empty intersection. Hence a topological space X is irreducible if and only if any non-empty open set of X is dense. Thus we have

<u>(5.4) Proposition</u>. A subset S of a topological space X is irreducible if and only if the closure $\overline{S}$ of S is irreducible.

<u>Remark</u>. Let X be a topological space and Y be a subspace of X. Then it is clear that a subset W of Y is irreducible in X if and only if W is irreducible in Y with respect to the relative topology on Y.

<u>(5.5) Proposition</u>. Let $(V,A) \in \mathcal{A}(K)$, then a non-empty closed subset W of V is irreducible if and only if $\mathcal{J}(W)$ is prime. Particularly V is irreducible if and only if A is an integral domain.

<u>Proof</u>. First we assume that W is irreducible and $ab \in \mathcal{J}(W)$ for some $a,b \in A$. Let

$$W_1 = \{w \in W \mid a(w) = 0\} \quad \text{and} \quad W_2 = \{w \in W \mid b(w) = 0\} .$$

Since $(ab)(w) = 0$ for any element $w \in W$, we have $a(w) = 0$ or $b(w) = 0$, which implies $W = W_1 \cup W_2$. Since W is irreducible, $W_1 = W$ or $W_2 = W$. Thus $a \in \mathcal{J}(W)$ or $b \in \mathcal{J}(W)$.

Conversely assume that $\mathcal{J}(W)$ is prime and $W = W_1 \cup W_2$ with some closed sets W_1 and W_2. Suppose that $W \neq W_1$, i.e., $\mathcal{J}(W) \neq \mathcal{J}(W_1)$. then $\mathcal{J}(W) \subsetneq \mathcal{J}(W_1)$ and there exists

$$a_o \in \mathcal{J}(W_1) - \mathcal{J}(W) .$$

Since $\mathcal{J}(W) = \mathcal{J}(W_1) \cap \mathcal{J}(W_2)$ from the assumption, we have $a_o b \in \mathcal{J}(W)$ for any $b \in \mathcal{J}(W_2)$. Hence $b \in \mathcal{J}(W)$, because $a_o \notin \mathcal{J}(W)$ and $\mathcal{J}(W)$ is prime. Thus $\mathcal{J}(W_2) \subset \mathcal{J}(W)$ and further $\mathcal{J}(W_2) = \mathcal{J}(W)$. Therefore, $W = \mathcal{V}(\mathcal{J}(W)) = \mathcal{V}(\mathcal{J}(W_2)) = W_2$ and W is irreducible. Q.E.D.

(5.6) Lemma. Let X and Y be topological spaces and assume that X is irreducible. Let $\varphi: X \to Y$ be a continuous map, then $\varphi(X)$ is irreducible.

Exercise 13. Prove Lemma 5.6.

(5.7) Proposition. Let (U,A) and (V,B) be irreducible affine varieties over K, then $(U \times V,\ A \otimes B)$ is also irreducible.

Proof. Assume that $U \times V = Z_1 \cup Z_2$, where Z_1 and Z_2 are closed subsets of $U \times V$. Let

$$j_{v_0}: U \longrightarrow U \times V \qquad \text{and} \qquad j_{u_0}: V \longrightarrow U \times V$$

$$(j_{v_0}: u \longrightarrow (u, v_0)) \qquad (j_{u_0}: v \longrightarrow (u_0, v))$$

be maps of U and V into $U \times V$ respectively where $v_0 \in V$ and $u_0 \in U$ are any fixed elements of V and U. Then from the proof of Proposition 3.3.2 j_{v_0} and j_{u_0} are morphisms of affine varieties. Thus $j_{v_0}(U) = U \times \{v_0\}$ is irreducible from Lemma 5.6. Hence $U \times \{v_0\} \subset Z_1$ or Z_2. Let $V_i = \{v \in V \mid (u,v) \in Z_i$ for all $u \in U\}$ where $i = 1,2$, then $V = V_1 \cup V_2$. Further since

$$V_i = \bigcap_{u \in U} \{v \in V \mid (u,v) \in Z_i\} = \bigcap_{u \in U} j_u^{-1}(Z_i) ,$$

V_i is closed for each $i = 1,2$. Thus $V = V_1$ or $V = V_2$, because V is irreducible. Hence $U \times V = Z_1$ or Z_2. Q.E.D.

Finally we shall show that a Noetherian space is the union of its finitely many maximal irreducible subsets.

Let X be a Noetherian space. Let $\mathscr{W} = \{W \mid W \subset X$ and W is closed and W cannot be expressed as a union of finitely many closed irreducible subsets of $X\}$. Assume that $\mathscr{W} \neq \emptyset$, and let W_0 be a minimal element in $\mathscr{W}$ (notice that X is Noetherian). Thus W_0 itself is not irreducible. Hence $W_0 = W_1 \cup W_2$ where W_1 and W_2 are proper closed subsets of W_0. By minimality of W_0 we have $W_i \notin \mathscr{W}$

for $i = 1,2$, which implies that $W_o = W_1 \cup W_2$ is also a finite union of closed irreducible subsets contradicting to the assumption $W_o \in \mathcal{W}$. Hence we have proved:

<u>(5.8) Lemma</u>. Assume that X is a Noetherian space, then X is the union of a finite number of closed irreducible subsets.

Now we can prove the following theorem:

<u>(5.9) Theorem</u>. Let X be a Noetherian space. Then

(1) any irreducible subset of X is contained in a maximal irreducible subset (which is closed from Proposition 5.4),

(2) X has only finitely many maximal irreducible subsets and is the union of them, and

(3) if $X = X_1 \cup X_2 \cup \ldots \cup X_t = X'_1 \cup X'_2 \cup \ldots \cup X'_s$ where $\{X_i \mid i = 1,2,\ldots,t\}$ and $\{X'_j \mid j = 1,2,\ldots,s\}$ are maximal irreducible subsets of X such that $X_i \neq X_k$ if $i \neq k$ and $X'_j \neq X'_l$ if $j \neq l$, then we have $\{X_1, X_2, \ldots, X_t\} = \{X'_1, X'_2, \ldots, X'_s\}$.

<u>Proof</u>. From Lemma 5.8 X is the union of a finite number of closed irreducible subsets $X_1, X_2, \ldots, X_n$. We can assume that $X_i \not\subseteq X_j$, when $i \neq j$. Let S be any irreducible subset of X , then $S = (S \cap X_1) \cup (S \cap X_2) \cup \ldots \cup (S \cap X_n)$. Hence $S = S \cap X_i$ for some i , which implies that $S \subset X_i$. Thus each X_j is a maximal irreducible subset of X , and $X_1, X_2, \ldots, X_n$ are precisely the maximal irreducible subsets of X .

(3) also follows from the same argument. Q.E.D.

<u>(5.10) Definition</u>. We call the maximal irreducible subsets of a Noetherian space X the <u>irreducible components</u> of X .

Exercise 14. Let X be a Noetherian space. Then

(1) any open subset of X is Noetherian with respect to the relative topology;

(2) any closed subset of X is Noetherian with respect to the relative topology;

(3) let F be a non-empty closed subset of X and $F = F_1 \cup \ldots \cup F_1$ be irreducible components of F. Let O be an open subset of X such that $F_1 \cap O \neq \emptyset$, then $F_1 \cap O$ is an irreducible component of $F \cap O$.

6. Some results from commutative algebras

In this section we shall explain two main theorems from commutative algebras which are necessary for further development, that is, Hilbert's Nullstellensatz and Noether Normalization Theorem.

- Transcendence degree -

Let L be an extension field of a field k, and S be a finite subset of L, i.e., $S = \{x_1, x_2, \ldots, x_n\}$. Let T_n be the direct product of n $\mathbb{N}$'s, i.e.,

$$T_n = \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{n},$$

where $\mathbb{N}$ is the set of natural numbers including 0. Then S is said to be <u>algebraically independent</u> over k, if whenever we have a relation

$$\sum_{\substack{\lambda_t \in k \\ t \in T_n}} \lambda_t \, x_1^{t_1} \, x_2^{t_2} \ldots x_n^{t_n} = 0$$

with alsmost all $\lambda_t = 0$, where $t = (t_1, t_2, \ldots, t_n) \in T_n$, then $\lambda_t = 0$ for all $t \in T_n$.

A subset Δ of L is algebraically independent over k if every finite subset of Δ is algebraically independent. Since the set of all algebraically independent subsets of L is inductively ordered by ascending inclusion, there exist maximal elements in that set. Thus a subset Δ of L which is algebraically independent over k and is maximal with respect to the inclusion ordering is called a <u>transcendence base</u> of L over k. It is clear that if Δ is a transcendence base of L over k, then L is algebraic over the field $k(\Delta)$ generated by Δ over k.

Similarly we have the following proposition.

<u>(6.1) Proposition</u>. If Γ is a set of generators of L over k, i.e., $L = k(\Gamma)$ and $\mathcal{A}$ is an algebraically independent subset of Γ, then there exists a transcendence base $\mathfrak{B}$ of L such that $\mathcal{A} \subset \mathfrak{B} \subset \Gamma$, because

$\{ T \mid \mathcal{A} \subset T \subset \Gamma$ and T is algebraically independent over $k \}$

is also inductively ordered.

<u>(6.2) Proposition</u>. Let k be a field and L be an extension field of k. Then if L has a transcendence base S with a finite number of elements $x_1, x_2, \ldots, x_n$, then any other transcendence base has n elements.

<u>Proof</u> (see the proof of Lang [1, Th.1.1, p.373]). Assume that L has a finite transcendence base $\{x_1, x_2, \ldots, x_n\}$. It is enough to show that if $w_1, w_2, \ldots, w_m$ are elements of L which are algebraically independent over k, then $m \leq n$. Since $\{w_1, x_1, x_2, \ldots, x_n\}$ is not algebraically independent, there exists a non-zero polynomial f_1 in $n+1$ variables with coefficients in k such that

$$f_1(w_1, x_1, x_2, \ldots, x_n) = 0 .$$

Furthermore by assumption w_1 occurs in f_1, and some x_i also occurs in f_1, say x_1, because $\{w_1, w_2, \ldots, w_m\}$ is algebraically independent over k. Then x_1 is algebraic over $k(w_1, x_2, \ldots, x_n)$. Suppose inductively that after a suitable renumbering of $x_1, x_2, \ldots, x_n$ we have found $w_1, w_2, \ldots, w_r$ $(1 \leq r < m)$ such that L is algebraic over $k(w_1, w_2, \ldots, w_r, x_{r+1}, \ldots, x_n)$. Then there exists a non-zero polynomial f in $n+1$ variables with coefficients in k such that

$$f(w_{r+1}, w_1, \ldots, w_r, x_{r+1}, \ldots, x_n) = 0$$

and such that w_{r+1} actually occurs in f. Since $\{w_1, w_2, \ldots, w_m\}$ is algebraically independent over k, some $x_j (r+1 \leq j \leq n)$ also occurs in f. After renumbering we may assume $j = r+1$. Then x_{r+1} is algebraic over

$$k(w_1, w_2, \ldots, w_{r+1}, x_{r+2}, \ldots, x_n) .$$

Since a tower of algebraic extensions is algebraic, it follows that L is algebraic over $k(w_1,w_2,\dots,w_{r+1},x_{r+2},\dots,x_n)$. We can repeat this procedure, and if $m > n$ we can replace all the $x_1,x_2,\dots,x_n$ by $w_1,w_2,\dots,w_n$, to see that L is algebraic over $k(w_1,w_2,\dots,w_n)$. This contradicts the assumption that $(w_1,\dots,w_n,w_{n+1},\dots,w_m)$ is algebraically independent. Thus we have $m \leq n$. Q.E.D.

<u>(6.3) Definition</u>. Let k be a field and L be an extension field of k . When L has a finite transcendence base with n elements over k , then we call n the <u>transcendence degree</u> of L over k and write

$$\mathrm{tr.deg}_k\, L = n \,.$$

<u>Exercise 15</u>. Let Γ be as in Proposition 6.1. If Γ is a finite set and contains no algebraically independent subset of Γ , then L is algebraic over k . When a field L is algebraic over its subfield k , then $\mathrm{tr.deg}_k\, L = 0$.

<u>Exercise 16</u>. Let E, F and k be fields such that $E \supset F \supset k$. Assume that E has a finite transcendence base over k , then we have

$$\mathrm{tr.deg}_k\, E = \mathrm{tr.deg}_F\, E + \mathrm{tr.deg}_k\, F \,.$$

More precisely if E has a finite transcendence base $(x_1,x_2,\dots,x_s)$ over F and F has a finite transcendence base $(y_1,y_2,\dots,y_t)$ over k , then $(x_1,\dots,x_s,y_1,\dots,y_t)$ is a transcendence base of E over k .

- Integral extensions - (see e.g. Lang [1, Chap.IX])

Let A be a commutative ring and B be a subring of A .

(6.4) Definition. An element $a \in A$ is said to be integral over B if one of the following four equivalent conditions holds.

(1) a satisfies a monic polynomial with coefficients in B, i.e., $a^n + b_1 a^{n-1} + \dots + b_n = 0$ for some $b_1, b_2, \dots, b_n \in B$.

(2) $B[a]$ is a finitely generated B-module.

(3) There exists a subring A' of A containing $B[a]$ which is a finitely generated B-module.

(4) There exists a faithful module over $B[a]$ which is a finitely generated B-module.

When every element of A is integral over B, we shall say that A is integral over B.

Remark. We say that an A-module M is faithful if, whenever $a \in A$ is such that $aM = 0$, then $a = 0$.

Proof of the equivalence of the four above conditions.

(1) ⇒ (2) Since the polynomial $X^n + b_1 X^{n-1} + \dots + b_n \in B[X]$ is monic, we can easily see that

$$B[a] = Ba^{n-1} + \dots + Ba + B.$$

(2) ⇒ (3) is clear.

(3) ⇒ (4) is also clear, because A' is a faithful $B[a]$-module.

(4) ⇒ (1) Let M be the faithful module over $B[a]$ which is finitely generated over B, say by elements

$$w_1, w_2, \dots, w_n.$$

Since $aM \subset M$, there exist elements $a_{ij} \in B$ such that

$$a\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

Hence

$$\begin{bmatrix} a-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & a-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \\ -a_{n1} & -a_{n2} & \cdots & a-a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = 0.$$

Now let

$$X = \begin{bmatrix} a-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & a-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \\ -a_{n1} & -a_{n2} & \cdots & a-a_{nn} \end{bmatrix},$$

then from Lang [1, Cor.4.17, p.456] we have $(\det X)M = 0$. Thus $\det X = 0$, because $\det X \in B[a]$ and M is faithful over $B[a]$. Since $\det X$ is a monic polynomial of a over B , we have proved that (4) implies (1). Q.E.D.

Exercise 17. Let B be a subring of an integral domain A , and S be a multiplicative subset of B such that $0 \notin S$. Prove: If A is integral over B , then $S^{-1}A$ is integral over $S^{-1}B$.

(6.5) Proposition. Let A be a commutative ring and B be a subring of A .

(1) Assume that $x_1, x_2, \ldots, x_n \in A$ are integral over B , then $B[x_1, x_2, \ldots, x_n]$ (the subring of A generated by B and $x_1, x_2, \ldots, x_n$) is finitely generated as a B-module.

(2) The set of all elements of A which are integral over B forms a subring of A .

Proof. (1) We prove this by induction on n . When $n = 1$, from Definition 6.4 $B[x_1]$ is a finitely generated B-module. Assume that $B[x_1, x_2, \ldots, x_{n-1}]$ is a finitely generated B-module, i.e., $B[x_1, x_2, \ldots, x_{n-1}] = Ba_1 + Ba_2 + \ldots + Ba_k$ for some $a_1, a_2, \ldots, a_k \in A$. Since x_n is integral over $B[x_1, x_2, \ldots, x_{n-1}]$,

$B[x_1, x_2, \ldots, x_n] = B[x_1, x_2, \ldots, x_{n-1}][x_n]$

$= B[x_1, x_2, \ldots, x_{n-1}]b_1 + \ldots + B[x_1, x_2, \ldots, x_{n-1}]b_l$ for some $b_1, b_2, \ldots, b_l \in B[x_1, x_2, \ldots, x_n]$. Hence $B[x_1, x_2, \ldots, x_n]$ is generated by $\{a_i b_j \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq l\}$ as a B-module.

(2) We only have to show that $x-y$ and xy are integral over B if $x, y \in A$ are integral over B . Assume that $x, y \in A$ are integral over B , then $B[x,y]$ is a finitely generated B-module from (1). Thus for any $a \in B[x,y]$ a is integral over B , because $B[a] \subset B[x,y]$ (see Definition 6.4.3). Q.E.D.

(6.6) Proposition. Assume that A, B and C are commutative rings such that $A \supset B \supset C$ and A is integral over B and B is integral over C , then A is integral over C .

Proof. Since $a \in A$ is integral over B , there exists some $b_1, b_2, \ldots, b_n \in B$ such that

$$a^n + b_1 a^{n-1} + \ldots + b_n = 0 .$$

Hence a is integral over $C[b_1, b_2, \ldots, b_n]$ and $C[b_1, b_2, \ldots, b_n, a]$ is a finitely generated $C[b_1, b_2, \ldots, b_n]$-module. Thus we have

$$C[b_1, b_2, \ldots, b_n, a] = C[b_1, b_2, \ldots, b_n]x_1 + \ldots + C[b_1, b_2, \ldots, b_n]x_s$$

for some $x_1, \ldots, x_s \in C[b_1, b_2, \ldots, b_n, a]$ and
$C[b_1, b_2, \ldots, b_n] = Cy_1 + \ldots + Cy_t$ for some
$y_1, \ldots, y_t \in C[b_1, b_2, \ldots, b_n]$ from Proposition 6.5. Hence $C[b_1, b_2, \ldots, b_n, a]$ is a finitely generated C-module and a is integral over C from Definition 6.4.3. Q.E.D.

- Extension of homomorphisms -

Nakayama's Lemma. Let A be a commutative ring and M be a finitely generated A-module. Let $\mathfrak{a}$ be an ideal of A . Then

(1) If $\mathfrak{a}M = M$, then there exists $x \in A$ such that $x-1 \in \mathfrak{a}$ and $xM = \{0\}$.
(2) If $\mathfrak{a}M = M$ and $\mathfrak{a}$ is contained in all maximal ideals of A , then $M = \{0\}$.

Proof. (1) Assume that M is generated by $w_1, w_2, \ldots, w_n$. Since $\mathfrak{a}M = \{a_1 m_1 + a_2 m_2 + \ldots + a_l m_l \mid a_i \in \mathfrak{a},\ m_i \in M \text{ and } l \in \mathbb{N}\}$ by definition, we have $w_i = \sum_{j=1}^{n} a_{ij} w_j$ for all $i = 1, 2, \ldots, n$ where $a_{ij} \in \mathfrak{a}$. Since

$$\begin{bmatrix} 1-a_{11}, & -a_{12},\ldots, & -a_{1n} \\ -a_{21}, & 1-a_{22},\ldots, & -a_{2n} \\ \vdots & & \\ -a_{n1}, & -a_{n2},\ldots, & 1-a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = 0 ,$$

we have $(\det X)M = \{0\}$ from Lang [1, Cor.4.17 on P.456], where

$$X = \begin{bmatrix} 1-a_{11}, & -a_{12},\ldots, & -a_{1n} \\ -a_{21}, & 1-a_{22},\ldots, & -a_{2n} \\ \vdots & & \\ -a_{n1}, & -a_{n2},\ldots, & 1-a_{nn} \end{bmatrix} .$$

It is clear that $(\det X)-1 \in \mathfrak{a}$.

(2) Let $a \in A$ such that $a-1 \in \mathfrak{a}$ and $aM = 0$. If a is not a unit in A , then it is contained in some maximal ideal $\mathfrak{m}$. Since $a-1 \in \mathfrak{a} \subset \mathfrak{m}$ by hypothesis, we have a contradiction $1 \in \mathfrak{m}$. Hence a is a unit and $M = \{0\}$. Q.E.D.

<u>(6.7) Proposition</u> (see Lang [1, Prop.1.10 on P.360 and Prop.3.1 on P.369]). Let A be a commutative ring and B be a subring of A . Assume that A is integral over B .

(1) Let $\mathfrak{p}$ be a prime ideal of B and $\mathfrak{p}A = \{p_1a_1 + p_2a_2 +\ldots+ p_na_n \mid p_i \in \mathfrak{p},\ a_i \in A \text{ and } n \in \mathbb{N}\}$, then $\mathfrak{p}A \neq A$ and there exists a prime ideal $\mathfrak{P}$ of A such that

$$\mathfrak{P} \cap B = \mathfrak{p} .$$

(2) Let $\varphi: B \to L$ be a homomorphism of B into an algebraically closed field L , then φ has an extension to a homomorphism of A into L .

<u>Proof</u>. (1) Let $S = B-\mathfrak{p}$, then S is a multiplicative subset of B . We shall write $B_{\mathfrak{p}}$ for the local ring of B at $\mathfrak{p}$, i.e., $S^{-1}B$. Since $B \subset A$, we have the inclusion map

$$S^{-1}B \longrightarrow S^{-1}A$$

$$(b/s \longrightarrow b/s) \quad (s \in S,\ b \in B) .$$

We write $\Lambda_{\mathfrak{p}} = S^{-1}\Lambda$. From Exercise 17 on P.37 $\Lambda_{\mathfrak{p}}$ is integral over $B_{\mathfrak{p}}$.

Let $\mathfrak{m}_{\mathfrak{p}} = S^{-1}\mathfrak{p}$ be the maximal ideal of $B_{\mathfrak{p}}$. Assume that we proved the first assertion in case B were a local ring, then since

$$\mathfrak{m}_{\mathfrak{p}}\Lambda_{\mathfrak{p}} = (\mathfrak{p}B_{\mathfrak{p}})\Lambda_{\mathfrak{p}} = \mathfrak{p}\Lambda_{\mathfrak{p}} \neq \Lambda_{\mathfrak{p}} ,$$

we have $1/1 \notin \mathfrak{p}\Lambda_{\mathfrak{p}}$. Hence $1 \notin \mathfrak{p}\Lambda$, which implies $\mathfrak{p}\Lambda \neq \Lambda$. Thus we only have to show that in case B is a local ring. In this case, if $\mathfrak{p}\Lambda = \Lambda$, then $1 \in \mathfrak{p}\Lambda$ and we have

$$1 = p_1a_1 + p_2a_2 + \dots + p_na_n$$

for some $p_i \in \mathfrak{p}$ and $a_i \in \Lambda$. Let $\Lambda_o = B[a_1,a_2,\dots,a_n]$, i.e., the subring of Λ generated by B and $(a_1,a_2,\dots,a_n)$, then from Proposition 6.5 Λ_o is a finitely generated B-module. Since $\mathfrak{p}\Lambda_o = \Lambda_o$ and $\mathfrak{p}$ is contained in the unique maximal ideal of B , we have $\Lambda_o = (0)$ by Nakayama's lemma, contradiction.

To prove our second assertion, note the following commutative diagram:

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda_{\mathfrak{p}} \quad b/s \\ \cup & \circlearrowleft & \cup \quad\ \uparrow \\ B & \longrightarrow & B_{\mathfrak{p}} \quad b/s \ . \end{array}$$

$$(b \longrightarrow b/1)$$

We have just shown $\mathfrak{m}_{\mathfrak{p}}\Lambda_{\mathfrak{p}} \neq \Lambda_{\mathfrak{p}}$. Hence $\mathfrak{m}_{\mathfrak{p}}\Lambda_{\mathfrak{p}}$ is contained in a maximal ideal $\mathfrak{M}$ of $\Lambda_{\mathfrak{p}}$. Since the inverse image of $\mathfrak{M}$ in $B_{\mathfrak{p}}$ is $\mathfrak{M} \cap B_{\mathfrak{p}}$, which contains $\mathfrak{m}_{\mathfrak{p}}$. Since $\mathfrak{m}_{\mathfrak{p}}$ is maximal, we have $\mathfrak{M} \cap B_{\mathfrak{p}} = \mathfrak{m}_{\mathfrak{p}}$. Let $\mathfrak{P}$ be the inverse image of $\mathfrak{M}$ in Λ , i.e., $\mathfrak{P} = \mathfrak{M} \cap \Lambda$. Then $\mathfrak{P}$ is a prime ideal of Λ . The inverse image of $\mathfrak{m}_{\mathfrak{p}}$ in B is simply $\mathfrak{p}$. Taking the inverse image of $\mathfrak{M}$ going around both ways in the diagram, we have

$$\mathfrak{P} \cap B = \mathfrak{p}$$

as was to be shown.

(2) Let $\mathfrak{p}$ be the kernel of φ and let S be the complement of $\mathfrak{p}$ in B. Then we have a commutative diagram:

$$\begin{array}{ccccc} \Lambda & \longrightarrow & S^{-1}\Lambda & b/s & \\ \cup & \circlearrowright & \cup & \uparrow & \\ B & \longrightarrow & S^{-1}B & b/s\,, & \text{where } b \in B \text{ and } s \in S\,. \\ (b & \longrightarrow & b/1) & & \end{array}$$

Let $\hat{\varphi}: S^{-1}B \to L$ be a map which takes b/s to $\varphi(b)/\varphi(s)$, then $\hat{\varphi}$ is a well-defined homomorphism and we have

$$\varphi : B \longrightarrow S^{-1}B \xrightarrow{\ \hat{\varphi}\ } L$$
$$(\varphi : b \longrightarrow b/1 \longrightarrow \varphi(b))\,.$$

Notice that $\operatorname{Ker}\hat{\varphi} = S^{-1}\mathfrak{p}$. Thus we only have to prove the assertion in case B is a local ring, because $S^{-1}\Lambda$ is integral over $S^{-1}B$. Therefore, we now assume that B is a local ring. Let $\mathfrak{m}$ be the kernel of φ. We assume that $\mathfrak{m}$ is the maximal ideal of B. Since $\mathfrak{m}\Lambda \neq \Lambda$, there exists a maximal ideal $\mathfrak{M}$ of Λ such that $\mathfrak{M} \supset \mathfrak{m}\Lambda$. It is clear that $\mathfrak{M} \cap B = \mathfrak{m}$. Thus $\Lambda/\mathfrak{M}$ is a field which is an algebraic extension of $B/\mathfrak{m}$. Let $\tilde{\varphi}: B/\mathfrak{m} \longrightarrow \varphi(B)$ be the isomorphism
$$(\tilde{\varphi}: b+\mathfrak{m} \longrightarrow \varphi(b))$$
of $B/\mathfrak{m}$ onto $\varphi(B)$ induced from φ, then there exists an embedding Φ of $\Lambda/\mathfrak{M}$ into L which makes the following diagram commutative:

$$\begin{array}{ccccccc} \Lambda & \longrightarrow & \Lambda/\mathfrak{M} & b+\mathfrak{M} & & \searrow^{\Phi} & \\ \cup & \circlearrowright & \cup & \uparrow & \circlearrowright & & L\,. \\ B & \longrightarrow & B/\mathfrak{m} & b+\mathfrak{m} & & \nearrow_{\tilde{\varphi}} & \\ (b & \longrightarrow & b+\mathfrak{m}) & & & & \end{array}$$

Thus we get an extension of φ to homomorphism of Λ into L.

Q.E.D.

<u>(6.8) Theorem</u> (see Lang [1, Th.2.1 on P.374]). Let $\Lambda = k[x_1, x_2, \ldots, x_m]$ be a finitely generated commutative algebra over a field k.

(1) Then any ring homomorphism $\varphi: k \to K$ where K is an algebraically closed field extends to a ring homomorphsim $\tilde{\varphi}: \Lambda \to K$.
(2) If Λ is a field then Λ is algebraic over k.
(3) If Λ is an integral domain and $y_1, y_2, \ldots, y_l$ are non-zero elements of Λ then there is a k-algebra map $\psi: \Lambda \to \bar{k}$ such that

$\psi(y_i) \neq 0$ for all $i = 1,2,\ldots,l$, where $\bar{k}$ is an algebraic closure of k .

<u>Proof</u>. (1) Let $\mathcal{M}$ be a maximal ideal of A . Let σ be the canonical homomorphism $\sigma : A \to A/\mathcal{M}$. Then $\sigma(A) = \sigma(k)[\sigma(x_1),\ldots,\sigma(x_m)]$ is a field, and is in fact an extension field of $\sigma(k)$.

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & A/\mathcal{M} = \sigma(k)[\sigma(x_1),\ldots,\sigma(x_m)] \\ & & | \\ & & \sigma(k) \end{array}$$

If we can prove our theorem when the finitely generated ring is in fact a field, then we apply $\varphi \circ \sigma^{-1}$ on $\sigma(k)$ and extend this to a homomorphism of $\sigma(k)[\sigma(x_1),\ldots,\sigma(x_m)]$ into K to get what we want.

$$\begin{array}{ccccccc} \varphi\circ\tilde{\sigma}^{-1}\circ\sigma : A = k[x_1,\ldots,x_m] & \xrightarrow{\sigma} & A/\mathcal{M} = \sigma(k)[\sigma(x_1),\ldots,\sigma(x_m)] & \xrightarrow{\varphi\circ\tilde{\sigma}^{-1}} & K \\ \cup & & \cup & \circlearrowright & \\ k & \cong & \sigma(k) & \xrightarrow{\varphi\circ\sigma^{-1}} & K \end{array}$$

Without loss of generality, we therefore assume that A is a field. If A is algebraic over k , we are done. Otherwise let $t_1, t_2, \ldots, t_r$ be a transcendence basis of A over k , $r \geq 1$. Without loss of generality, we may assume that φ is the identity on k , that is, we assume that K is an algebraic closure of k .

$$\begin{array}{ccccc} \bar{\varphi} \circ \tilde{i} : A = k[x_1,\ldots,x_m] & \xrightarrow{\tilde{i}} & \bar{k} & \xrightarrow{\bar{\varphi}} & K \\ \cup & \circlearrowright & \cup & \circlearrowright & \cup \\ k & \xrightarrow{i} & k & \xrightarrow{\varphi} & \varphi(k) \end{array}$$

where $\bar{k}$ is an algebraic closure of k and i is an identity map on k . Each element $x_1, x_2, \ldots, x_m$ is algebraic over $k(t_1,\ldots,t_r)$. If we multiply the irreducible polynomial $\mathrm{Irr}(x_i, k(t), X)$ by a suitable non-zero element of $k[t] = k[t_1,\ldots,t_r]$, where $k(t) = k(t_1,\ldots,t_r)$, then we get a polynomial all of whose coefficients lie in $k[t]$. Let $a_1(t),\ldots,a_m(t)$ be the set of the leading coefficients of these polynomials, and let $a(t)$ be their product

$$a(t) = a_1(t) \cdot \ldots \cdot a_m(t) \ .$$

Since $a(t) \neq 0$, there exist elements $t_1', \ldots, t_r' \in \bar{k}$ such that $a(t_1',\ldots,t_r') \neq 0$ (see Lang [1, Cor.4.6 on P.192]), and hence

$a_i(t_1',\ldots,t_r') \neq 0$ for any i . Each x_i is integral over the ring

$$(k(t_1,\ldots,t_r) \supset)k[t_1,\ldots,t_r, \frac{1}{a_1(t)}, \ldots, \frac{1}{a_m(t)}] .$$

Consider the homomorphism $\psi : k[t_1,\ldots,t_r] \to \bar{k}$ such that ψ is the identity on k , and $\psi(t_j) = t_j'$ $(j = 1,2,\ldots,r)$. Let $\mathfrak{p}$ be its kernel. Then $a(t_1,\ldots,t_r) \notin \mathfrak{p}$.

$$\begin{array}{l} k[x_1,\ldots,x_m] \\ \quad\cup \\ k(t_1,\ldots,t_r) \\ \quad\cup \\ k[t_1,\ldots,t_r]_{\mathfrak{p}} = \{a/b \mid a,b \in k[t], \quad b \notin \mathfrak{p}\} \\ \quad\cup \\ k[t_1,\ldots,t_r] \xrightarrow{\psi} \bar{k} . \end{array}$$

The homomorphism ψ extends uniquely to the local ring $k[t_1,\ldots,t_r]_{\mathfrak{p}}$. Since

$$k[x_1,\ldots,x_m] = k[t_1,\ldots,t_r]_{\mathfrak{p}}[x_1,\ldots,x_m]$$

is integral over $k[t_1,\ldots,t_r]_{\mathfrak{p}}$, ψ extends to a homomorphism of $k[x_1,\ldots,x_m]$ into $\bar{k}$ from Proposition 6.7.2. Thus we have proved (1).

(2) Let $\bar{k}$ be an algebraic closure of k , then from (1) there exists a ring homomorphism of A into $\bar{k}$ which is identity on k . Since the homomorphism is injective, A is algebraic over k .

(3) Let $k[x_1,\ldots,x_m, y_1^{-1},\ldots,y_1^{-1}]$ be the subring of the quotient field of A generated by $x_1,\ldots,x_m, y_1^{-1},\ldots,y_1^{-1}$, then from (1) there exists a ring homomorphism of $k[x_1,\ldots,x_m, y_1^{-1},\ldots,y_1^{-1}]$ into $\bar{k}$ which is identity of k . The restriction of this homomorphism to the ring $k[x_1,\ldots,x_m]$ is a desired map.

Q.E.D.

Now let A be a commutative ring and I an ideal of A . We define the <u>radical $\sqrt{I}$ of I</u> to be the set

$$\sqrt{I} = \{f \in A \mid f^m \in I \text{ for some } m \in \mathbb{N} \},$$

where $\mathbb{N}$ is the set of natural numbers including 0 . If $f,g \in \sqrt{I}$

with $f^m, g^n \in I$, then $(f+g)^{m+n} \in I$. Thus we can easily see that $\sqrt{I}$ is also an ideal. Further we have $\sqrt{\sqrt{I}} = \sqrt{I}$. We call the radical of (0) , i.e.,

$$\{f \in A \mid f^m = 0 \text{ for some } m \in \mathbb{N}\}$$

the <u>nilradical of A</u> . It is clear the $\sqrt{I} \supset I$.

<u>Hilbert's Nüllstellensatz</u>. Let (V,A) be an affine variety over K and I an ideal of A , then we have

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I} .$$

<u>Proof</u>. Let $g \in \sqrt{I}$, that is $g^m \in I$ for some $m \in \mathbb{N}$. Hence for any $v \in \mathcal{V}(I)$ we have $g^m(v) = g(v)^m = 0$. If $m = 0$ and $g \neq 0$, then $I = A$ and $\mathcal{I}(\mathcal{V}(I)) = A = \sqrt{A}$. Thus we can assume that $m \neq 0$. Hence $g(v) = 0$ and therefore $g \in \mathcal{I}(\mathcal{V}(I))$, i.e., $\sqrt{I} \subset \mathcal{I}(\mathcal{V}(I))$.

Let $f \in \mathcal{I}(\mathcal{V}(I))$ and assume that $f \notin \sqrt{I}$. (Hence $I \subsetneq A$.) Let S be the set of ideals of A which contain I but do not contain any power of f . For example $I \in S$ but $A \notin S$. Since A is Noetherian (see Examples 4.9), S has a maximal element $\mathfrak{p}$. Suppose there exists $x, y \in A$ such that $x \notin \mathfrak{p}$, $y \notin \mathfrak{p}$ and $xy \in \mathfrak{p}$. Since $\mathfrak{p} \subsetneq \mathfrak{p}+Ax$ and $\mathfrak{p}$ is maximal in S , $f^m \in \mathfrak{p}+Ax$ for some $m \in \mathbb{N}$. Similarly we have $f^n \in \mathfrak{p}+Ay$ for some $n \in \mathbb{N}$. Hence

$$f^{m+n} = f^m f^n \in (\mathfrak{p}+Ax)(\mathfrak{p}+Ay) \subset \mathfrak{p} ,$$

which contradicts to the fact that $\mathfrak{p} \in S$. Hence $\mathfrak{p}$ is a prime ideal .

Since $A/\mathfrak{p}$ is an integral domain, there exists a K-algebra map $\theta_o : A/\mathfrak{p} \to K$ such that $\theta_o(f+\mathfrak{p}) \neq 0$ from Theorem 6.8.3. Let

$$\theta : A \longrightarrow A/\mathfrak{p} \xrightarrow{\theta_0} K$$
$$(a \longrightarrow a+\mathfrak{p})$$

be a map which takes $a \in A$ to $\theta_o(a+\mathfrak{p})$. Since θ is a K-algebra map and $(V,A) \in \mathcal{A}(K)$, $\theta = \epsilon_v$ for some $v \in V$. Since $I \subset \mathfrak{p}$, we have $c(v) = \epsilon_v(c) = \theta(c) = \theta_o(c+\mathfrak{p}) = 0$ for any $c \in I$. Thus

$v \in \mathscr{V}(I)$. Hence $f(v) = 0$, because $f \in \mathscr{I}(\mathscr{V}(I))$. However, we also have

$$f(v) = \epsilon_v(f) = \theta(f) = \theta_o(f+\mathfrak{p}) \neq 0 ,$$

a contradiction. Hence $\mathscr{I}(\mathscr{V}(I)) \subset \sqrt{I}$, and we have shown that $\mathscr{I}(\mathscr{V}(I)) = \sqrt{I}$. Q.E.D.

Exercise 18. (1) Let $(V,A) \in \mathscr{A}(K)$. Show that $\mathscr{V}(I) = \mathscr{V}(\sqrt{I})$ for any ideal I in A .

(2) Let $(V,A) \in \mathscr{A}(K)$ and I and J be ideals of A . Show that

$$\mathscr{V}(I) = \mathscr{V}(J) \Leftrightarrow \sqrt{I} = \sqrt{J} ,$$

and the operator $\mathscr{V}$ is an inclusion reversing bijection with the inverse $\mathscr{I}$ between the set of ideals I such that $\sqrt{I} = I$ and the set of closed sets of V .

Exercise 19. Let R be a commutative ring with unity element 1. Let $\mathfrak{a}$ be an ideal of R such that $R \supsetneq \mathfrak{a}$. Let $\mathrm{Rad}\, \mathfrak{a}$ be the intersection of all prime ideals containing $\mathfrak{a}$. (Since $R \supsetneq \mathfrak{a}$, $\mathfrak{a}$ is contained in some maximal ideal of R .) Prove:

$$\sqrt{\mathfrak{a}} = \mathrm{Rad}\, \mathfrak{a} .$$

Hints. Assume that $x \notin \sqrt{\mathfrak{a}}$, i.e., $x^m \notin \mathfrak{a}$ for any $m \in \mathbb{N}$. Then the set $\mathscr{F} = \{I \mid I$ is an ideal of R such that $I \supset \mathfrak{a}$ and $I \cap \{x^m \mid m \in \mathbb{N}\} = \emptyset\}$ is non-empty and inductively ordered by the inclusion relation. Let $\mathfrak{p}$ be a maximal element in $\mathscr{F}$. By the same argument as in the proof of Nullstellensatz $\mathfrak{p}$ is prime. Hence there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \supset \mathfrak{a}$ but $x \notin \mathfrak{p}$, which implies $x \notin \mathrm{Rad}\, \mathfrak{a}$.

(6.9) Proposition. Let A be a finitely generated commutative K-algebra with trivial nilradical. Let $V = \mathrm{Hom}_{K\text{-alg}}(A,K)$ and define

$$\iota : A \longrightarrow M(V,K)$$
$$(\iota : a \longrightarrow \iota(a) : v \to v(a))$$

where $v \in V$. Then ι is an injective K-algebra map and hence the pair is an affine variety.

<u>Proof</u> From Theorem 6.8.1 $V \neq \emptyset$. It is easy to verify that ι is a well-defined K-algebra map. Assume that $\iota(a)(v) = 0$ for some $a \in A$ and for all $v \in V$, then $v(a) = 0$ for any $v \in V$. Suppose that $\mathfrak{p}$ is a prime ideal of A such that $a \notin \mathfrak{p}$. Then there exists a K-algebra map

$$\theta_o : A/\mathfrak{p} \longrightarrow K$$

such that $\theta_o(a+\mathfrak{p}) \neq 0$ from Theorem 6.8.3. Let

$$\theta : A \longrightarrow A/\mathfrak{p} \xrightarrow{\theta_o} K$$
$$(a \longrightarrow A/\mathfrak{p})$$

be a map which takes $a \in A$ to $\theta_o(a+\mathfrak{p})$. Since θ is a K-algebra map such that $\theta(a) \neq 0$, we have got a contradiction. Thus a is contained in any prime ideal of A . Hence we have

$$a \in \mathrm{Rad}(0) = \sqrt{(0)}$$

from Exercise 19. Since A has a trivial nilradical, $a = 0$. Therefore, ι is injective. It is clear that the evaluation map:

$$V \longrightarrow V = \mathrm{Hom}_{K\text{-alg}}(A,K) \quad \text{where} \quad \epsilon_v : A \longrightarrow K \quad \text{is bijective.}$$
$$(v \longrightarrow \epsilon_v) \qquad\qquad (\epsilon_v : a \longrightarrow v(a))$$

Q.E.D.

<u>(6.10) Theorem</u>. Let $\mathscr{C}(K)$ be the category of finitely generated K-algebras with trivial nilradicals. Let Ψ be the contravariant representation functor from $\mathscr{C}(K)$ into the category of sets which takes each algebra A in $\mathscr{C}(K)$ to $\Psi(A) = \mathrm{Hom}_{K\text{-alg}}(A,K)$ and a K-algebra map $f : A \to B$, where A and B are objects of $\mathscr{C}(K)$, to

$$\Psi(f) : \mathrm{Hom}_{K\text{-alg}}(B,K) \longrightarrow \mathrm{Hom}_{K\text{-alg}}(A,K) \ .$$
$$(\Psi(f) : \alpha \longrightarrow \alpha \circ f)$$

Then from Proposition 6.9 we have

(1) Ψ is a contravariant functor $\mathscr{C}(K)$ into $\mathscr{A}(K)$,
(2) let Φ be a rule which associates to each object (V,A) in $\mathscr{A}(K)$ an object

$$\Phi(V) = A \quad \text{in} \quad \mathscr{C}(K) \ ,$$

and to each morphism $\varphi : V \to U$ associates a K-algebra map

$$\Phi(\varphi) = \varphi^* \ ,$$

where $(U,B) \in \mathscr{A}(K)$, then Φ is a contravariant functor from $\mathscr{A}(K)$ into $\mathscr{C}(K)$,

(3) $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identity functors of $\mathscr{C}(K)$ into $\mathscr{C}(K)$ and $\mathscr{A}(K)$ into $\mathscr{A}(K)$ respectively.

<u>Exercise 20</u>. Prove Theorem 6.10.

- Perfect field -

<u>(6.11) Definition</u>. A field k is called <u>perfect</u> if $k^p = k$, where $p > 0$ is the characteristic of k and

$$k^p = (x^p \mid x \in k) .$$

We also call a field of characteristic 0 perfect.

<u>Example</u>. An algebraically closed field K is perfect.

<u>(6.12) Definition</u>. An element α (of an extension field of a field k) which is algebraic over k is said to be <u>separably algebraic</u> over k if the monic minimal polynomial of α has no multiple roots.

<u>Example</u>. If the characteristic of k is zero, then $\alpha \in \bar{k}$ is always separably algebraic over k where $\bar{k}$ is an algebraic closure of k .

<u>(6.13) Definition</u>. Let E be an algebraic extension of a field k . We define E to be separably algebraic over k if each element of E is separably algebraic over k .

<u>(6.14) Definition</u>. Let F be a finitely generated extension field of a field k , i.e., $F = k(x_1,\ldots,x_q)$. Then we shall say that F is <u>separably generated</u> if we can find a transcendence base $(t_1,t_2,\ldots,t_r)$ of F/k such that F is separably algebraic over $k(t_1,t_2,\ldots,t_r)$.

<u>(6.15) Definition</u>. Let L and F be extension fields of a field k both contained in one field Ω , which is algebraically closed, then L is said to be <u>linearly disjoint</u> from F over k if every finite set of elements of L that is linearly independent over k is still such over F .

<u>Exercise 21</u>. Let Ω, L, F and k be as in Definition 6.15. Prove that F is linearly disjoint from L over k if L is linearly disjoint from F over k .

```
    Ω
   / \
  L   F
   \ /
    k
```

<u>Criterion for linear disjointness</u> (see Lang [1, Chap.X, §5]). Suppose that L is the quotient field of an integral domain R and R is a vector space over a field k , $L \supset R \supset k$. Let $\{u_\alpha\}$ be a basis of R considered as a vector space over k . Assume that F is an extension field of k and L and F are contained in an algebraically closed field Ω .

If $\{u_\alpha\}$ remain linearly independent over F , then L and F are linearly disjoint over k .

<u>Proof</u>. Let $x_1, x_2, \ldots, x_m \in R$ be linearly independent over k . Since each x_i is a finite linear combination of $\{u_\alpha\}$, $\{x_1, x_2, \ldots, x_m\}$ is contained in a finite dimensional vector space generated by some of the $\{u_\alpha\}$, say $u_1, u_2, \ldots, u_n$.

$$\{x_1, x_2, \ldots, x_m\} \subset \sum_{i=1}^{n} ku_i \ , \quad \dim_k \sum_{i=1}^{n} ku_i = n \ .$$

Since $\{x_1, x_2, \ldots, x_m\}$ is linearly independent over k , $\sum_{i=1}^{n} ku_i$ has also a k-basis $\{x_1, \ldots, x_m, z_1, \ldots, z_t\}$. Since $\dim_F \sum_{i=1}^{n} Fu_i = n$ from the assumption and

$$\sum_{i=1}^{n} Fu_i = \sum_{i=1}^{m} Fx_i + \sum_{i=1}^{t} Fz_i \ ,$$

$\{x_1,x_2,\ldots,x_m\}$ is also linearly independent over F .

Now assume that $x_1',x_2',\ldots,x_m' \in L$ are linearly independent over k . Then there exists $y \in R$ such that $y \neq 0$ and $yx_1',\ldots,yx_m' \in R$. Since $\{yx_1',\ldots,yx_m'\}$ is also linearly independent over k , $\{yx_1',\ldots,yx_m'\}$ is linearly independent over F from the above argument. Hence also so is $\{x_1',x_2',\ldots,x_m'\}$ over F . Q.E.D.

<u>(6.16) Definition</u>. Let k be a field of characteristic $p > 0$. The field obtained from k by adjoining all p^m-th roots of all elements of k will be denoted by $k^{\frac{1}{p^m}}$. The compositum of all such fields for $m = 1,2,\ldots$ is denoted by $k^{\frac{1}{p^\infty}}$, i.e., $k^{\frac{1}{p^\infty}}$ is the smallest subfield of $\bar{k}$ containing all $k^{\frac{1}{p^m}}$, $m = 1,2,\ldots$ where $\bar{k}$ is an algebraic closure of k .

<u>(6.17) Definition</u>. An <u>extension</u> field L of k is called <u>separable</u> if every subfield of L containing k and finitely generated over k is separably generated.

<u>Exercise 22</u>. Show that an extension field L of k is separably algebraic over k if and only if L is separable over k in the sense of Definition 6.17 and L is algebraic over k .

In the proof of the Theorem 6.18 we refer to the following propositions from Lang [1] and Jacobson [1]. For the convenience of the reader we refer to them with their proofs.

<u>Lang</u> [1, Prop.5.3 on P.382]: Let K be an extension field of a field k both contained in an algebraically closed field Ω . Let $\{u_1,u_2,\ldots,u_r\}$ be a subset of Ω which is algebraically independent over K . Then the field $k(u_1,u_2,\ldots,u_r)$ is linearly disjoint from K over k .

Proof. Since the set of monomials $\{M(u_1,u_2,\ldots,u_r)\}$ forms a k-basis for $k[u_1,u_2,\ldots,u_r]$, it is enough to prove that $\{M(u_1,u_2,\ldots,u_r)\}$ remains linearly independent over K from the Criterion for linear disjointness. Since linear dependence of $\{M(u_1,u_2,\ldots,u_r)\}$ over K implies algebraic dependence of $\{u_1,u_2,\ldots,u_r\}$ over K , $\{M(u_1,u_2,\ldots,u_r)\}$ remains linearly independent over K . Q.E.D.

Jacobson [1, Lemma 2 on P.48]: Let K be an extension field of a field k of characteristic $p > 0$. Let ρ be an element of K which is algebraic over k . Then ρ is separably algebraic over k if and only if we have

$$k(\rho) = k(\rho^p) = K(\rho^{p^2}) = k(\rho^{p^3}) = \ldots$$

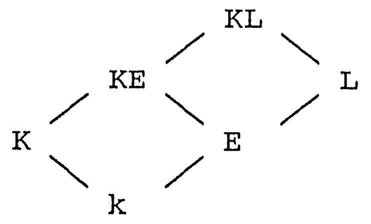
Proof. Let $g(X)$ be the monic minimal polynomial of ρ over k . We first assume that ρ is not separable over k . Since $g'(\rho) = 0$ and $\deg g'(X) < \deg(X)$, $g'(X) = 0$. Hence we have $g(X) = h(X^p)$ for some $h(X) \in k[X]$. Thus we have $[k(\rho^p):k] \leqq \deg h(X) < \deg g(X) = [h(\rho):k]$, which implies

$$k(\rho^p) \subsetneqq k(\rho) .$$

Now assume that ρ is separably algebraic over k . Let $h(x)$ be the monic minimal polynomial of ρ over $k(\rho^p)$. Since $h(X) \mid g(X)$ and $g(X) = 0$ has no multiple roots, $h(X)$ has distinct roots. Since ρ satisfies the equation $X^p - \rho^p = (X-\rho)^p = 0$ over $k(\rho^p)$, we have $h(X) = X-\rho$. Hence $\rho \in k(\rho^p) = k[p^\rho]$, which implies $k[\rho^p] = k[\rho]$. Since $\rho^p \in (k[\rho^p])^p \subset k[\rho^{p^2}]$, we have $k[\rho^{p^2}] = k[\rho^p]$. Similarly we have

$k[\rho] = k[\rho^p] = k[\rho^{p^2}] = k[\rho^{p^3}] = \ldots$ Q.E.D.

Lang [1, Prop.5.1 on P.380]: Let K be a field containing another field k , and let $L \supset E$ be two other extensions of k . Assume that K and L are contained in an algebraically closed field Ω . If K and E are linearly disjoint over k and KE , L are linearly disjoint over E , then K and L are linearly disjoint over k .



Proof: Let $\{\kappa\}$ be a basis of K as vector space over k and let $\{\alpha\}$ be a basis of E over k. Let $\{\lambda\}$ be a basis of L over E. It is clear that $\{\alpha\lambda\}$ is a basis of L over k. Assume that $\{\alpha\lambda\}$ is not linearly independent over K, then there exists a relation

$$\sum_{\lambda,\alpha} \left(\sum_{\kappa} c_{\kappa\lambda\alpha}\kappa\right)\lambda\alpha = 0$$

with some $c_{\kappa\lambda\alpha} \neq 0$ where $c_{\kappa\lambda\alpha} \in k$. Thus we have

$$\sum_{\lambda} \left(\sum_{\kappa,\alpha} c_{\kappa\lambda\alpha}\kappa\alpha\right)\lambda = 0$$

contradicting to the linear disjointness of L and KE over E, because $\sum_{\kappa,\alpha} c_{\kappa\lambda\alpha}\kappa\alpha = \sum_{\alpha} \left(\sum_{\kappa} c_{\kappa\lambda\alpha}\kappa\right)\alpha$ with some $c_{\kappa\lambda\alpha} \neq 0$ and K and E are linearly disjoint over k.

Now let $\{v_1,\dots,v_t\}$ be a finite subset of L which is linearly independent over k. Then $\{v_1,\dots,v_t\}$ is contained in a finite dimensional k-subspace of L generated by finite elements $\{\alpha_i\lambda_i\}$ from $\{\alpha\lambda\}$. Since $\{\alpha_i\lambda_i\}$ is linearly independent over K, $\{v_1,\dots,v_t\}$ is also linearly independent over K. Thus K and L are linearly disjoint over k. Q.E.D.

Now we state Theorem 6.18.

<u>(6.18) Theorem</u> (see Lang [1, Prop.6.1 on P.382]). Let L be an extension field of a field k both contained in an algebraically closed field Ω of characteristic $p > 0$. Then the following conditions are equivalent:

(1) L is linearly disjoint from $k^{\frac{1}{p^\infty}}$ over k.

(2) L is linearly disjoint from $k^{\frac{1}{p^m}}$ for some $m \geqq 1$ over k.

(3) Every subfield of L containing k and finitely generated over k is separably generated, i.e., L is separable over k.

<u>Proof</u>. (1) ⇒ (2) Since $\Omega \supset k^{\frac{1}{p^\infty}} \supset k^{\frac{1}{p^m}} \supset k$, it is clear that L is linearly disjoint from $k^{\frac{1}{p^m}}$ for some $m \geqq 1$.

(2) ⇒ (3) Let $k(x_1,\ldots,x_n)$ be a finitely generated subfield over k contained in L. It is clear that we can assume $L = k(x_1,\ldots,x_n)$. Let $r = \mathrm{tr.deg}_k L$. If $r = n$, then L is clearly separably algebraic over itself. Thus we assume that $r < n$ and $(x_1,x_2,\ldots,x_r)$ is a transcendence base of L over k. Then x_{r+1} is algebraic over $k(x_1,\ldots,x_r)$. Let $f(X_1,\ldots,X_{r+1}) \in k[X_1,\ldots,X_{r+1}]$ be a polynomial of lowest degree such that

$$f(x_1,\ldots,x_{r+1}) = 0 .$$

Since $(x_1,\ldots,x_r)$ is algebraically independent over k, f is irreducible. We show that not all x_i $(i = 1,2,\ldots,r+1)$ appear to the p-th power throughout. If they did, we could write

$$f(X_1,\ldots,X_{r+1}) = \Sigma\, c_\alpha M_\alpha(X_1,\ldots,X_{r+1})^p ,$$

where $M_\alpha(X_1,\ldots,X_{r+1})$ are monomials in $X_1,\ldots,X_{r+1}$ and $c_\alpha \in k$. Since $f(x_1,\ldots,x_{r+1}) = \Sigma\, c_\alpha M_\alpha(x_1,\ldots,x_{r+1})^p = 0$, we have $\Sigma c_\alpha^{\frac{1}{p}} M_\alpha(x_1,\ldots,x_{r+1}) = 0$ for some p-th roots $c_\alpha^{\frac{1}{p}} \in k^{\frac{1}{p}}$, which implies that the $M_\alpha(x_1,\ldots,x_{r+1})$ are linearly dependent over $k^{\frac{1}{p}}$. However, the $M_\alpha(x_1,\ldots,x_{r+1})$ are linearly independent over k, because otherwise we would get an equation for $x_1,\ldots,x_{r+1}$ of lower degree. Thus we get a contradiction to the linear disjointness of L and $k^{\frac{1}{p}}$ $(\subset k^{\frac{1}{p^m}}$, $m = 1,2,\ldots)$. Say X_{i_0} does not appear to the p-th power throughout but actually appear in $f(X_1,\dots,X_{r+1})$ $(1 \leqq i_o \leqq r+1)$. Since $f(X_1,\dots,X_{r+1})$ is a polynomial of lowest degree such that $f(x_1,\dots,x_{r+1}) = 0$, we have

$$\frac{df}{dX_{i_o}}(x_1,\dots,x_{r+1}) \neq 0 .$$

Hence x_{i_o} is not a multiple root of an equation $f(x_1,\dots,X_{i_o},\dots,x_{r+1}) = 0$ over $k(x_1,\dots,x_{i_o-1},x_{i_o+1},\dots,x_{r+1})$. Thus x_{i_o} is separably algebraic over $k(x_1,\dots,x_{i_o-1},x_{i_o+1},\dots,x_{r+1})$. Changing the suffices of the x , we may assume x_1 is separably algebraic over $k(x_2,\dots,x_{r+1})$. Since $k(x_2,\dots,x_n) \supset k(x_2,\dots,x_{r+1})$, x_1 is separably algebraic over $k(x_2,\dots,x_n)$. If $\{x_2,\dots,x_n\}$ is a transcendence base, the proof is complete. If not, say x_2 is separably algebraic over $k(x_3,\dots,x_n)$, then $k(x_1,\dots,x_n)$ is separably algebraic over $k(x_3,\dots,x_n)$. Proceeding inductively, we can get down to a transcendence base. This proves that (2) implies (3).

(3) $\Rightarrow$ (1) We would like to show that every finite set of elements $\{x_1,x_2,\dots,x_n\}$ of L which is linearly independent over k is still such over $k^{\frac{1}{p^\infty}}$. Thus it is enough to prove the assertion in case L is finitely generated over k , i.e., $L = k(x_1,x_2,\dots,x_n)$. Let $\{u_1,\dots,u_r\}$ be a transcendence base for L over k such that L is separably algebraic over $k(u_1,\dots,u_r)$. By Lang [1, Prop.5.3 on P.382] $k(u_1,\dots,u_r)$ and $k^{\frac{1}{p^\infty}}$ are linearly disjoint over k , because $\{u_1,u_2,\dots,u_r\}$ is algebraically independent over $k^{\frac{1}{p^\infty}}$. Let $K = k^{\frac{1}{p^\infty}}$. Then $k(u_1,\dots,u_r)K$ is purely inseparable over $k(u_1,\dots,u_r)$, i.e.,

$$\forall\ x \in k(u_1,\dots,u_r)K\ \exists n \in \mathbb{N}$$

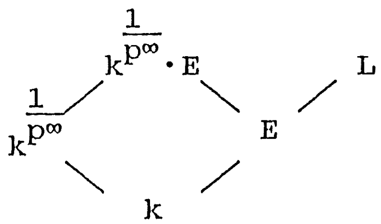
such that $x^{p^n} \in k(u_1,\ldots,u_r)$. Assume that $(\rho_1,\ldots,\rho_s) \subset L$ is linearly independent over $k(u_1,\ldots,u_r)$. Then there exists a finite extension field E of $k(u_1,\ldots,u_r)$ which contains $(\rho_1,\ldots,\rho_s)$ and has a $k(u_1,\ldots,u_r)$-basis $(\rho_1,\ldots,\rho_s, \rho_{s+1},\ldots,\rho_t)$. Since $(\epsilon^p,\epsilon^{2p},\epsilon^{3p},\ldots) \subset \sum_{i=1}^{t} k(u_1,\ldots,u_r)\rho_i^p$ for any element $\epsilon \in E$ and $\epsilon \in k(u_1,\ldots,u_r)[\epsilon^p]$ from Jacobson [1, Lemma 2 on P.48], $(\rho_1^p,\ldots,\rho_t^p)$ is also a $k(u_1,\ldots,u_r)$-basis of E . Thus $(\rho_1,\ldots,\rho_t)$ will never be linearly dependent over $k(u_1,\ldots,u_r)K$. From Lang [1, Prop.5.1 on P.380] L is linearly disjoint from K $(= k^{\frac{1}{p^\infty}})$ over k . Q.E.D.

(6.19) Corollary (see Lang [1, Cor.6.3 on P.383]).

(1) Let E be a separable extension of a field k and L be a separable extension of E . Then L is separable over k .

(2) Let L be an extension field of k . Assume that L is finitely generated over k . Then L is separable over k if and only if L is separably generated over k .

Proof. (1) We only have to show this in case $\mathrm{ch}\, k = p > 0$.



Since E is separable over k , E is linearly disjoint from $k^{\frac{1}{p^\infty}}$ over k . Since L is separable over E , L is linearly disjoint from $E^{\frac{1}{p^\infty}}$ over E . Hence L is linearly disjoint from $k^{\frac{1}{p^\infty}}\cdot E$ over E , because $E^{\frac{1}{p^\infty}} \supset k^{\frac{1}{p^\infty}}\cdot E$. From Lang [1, Prop.5.1 on P.380] (see

P.50) L is linearly disjoint from $k^{\frac{1}{p^\infty}}$ over k . thus L is separable over k .

(2) From the definition of separable extension it is clear that L is separably generated over k if L is separable over k . Assume that ch k = p > 0 and L is separably generated over k , i.e., there exists a transcendence base $\{t_1,\ldots,t_n\}$ of L/k such that L is separably algebraic over $k(t_1,\ldots,t_n)$. Since $\{t_1,\ldots,t_n\}$ is also algebraically independent over $k^{\frac{1}{p^m}}$ for any $m \geqq 1$, from Lang [1, Prop.5.3 on P.382] (see P.49) $k^{\frac{1}{p^m}}$ and $k(t_1,\ldots,t_n)$ are linearly disjoint over k . Hence $k(t_1,\ldots,t_n)$ is separable over k from Theorem 6.18. Thus L is separable over k from (1). Q.E.D.

<u>(6.20) Corollary</u> (see Lang [1, Cor.6.4 on P.383]). If k is a perfect field (of characteristic p > 0 or 0), then every extension field of k is separable.

<u>Proof</u>. It is clear in case ch k = 0 . Now we assume that ch k = p > 0 and L is an extension field of k . Since k is perfect, i.e., $k = k^p = k^{p^2} = k^{p^3} = \ldots$, we have $k^{\frac{1}{p^m}} = k$ for any integer m > 0 . Hence L is separable over k from Theorem 6.18.

Q.E.D.

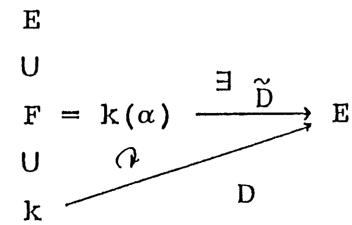
<u>(6.21) Proposition</u> (see Zariski & Samuel [1, Cor.2 on P.124]). Let k be a field and $F = k(\alpha)$ be a separably algebraic extension of k with primitive element α . Let E be an extension field of F , then any derivation $D: k \to E$, i.e., D is a map of k into E such that

$$D(x+y) = D(x) + D(y) \quad \text{and} \quad D(xy) = xD(y) + yD(x)$$

for any $x,y \in k$, can be extended to a derivation

$$\tilde{D}: F \to E .$$

Proof.



Let $f(X)$ be a monic minimal polynomial of α over k. Let $g^D(X)$ be the polynomial obtained by applying D to all coefficients of $g(X) \in k[X]$. Since α is separable over k, we have $f'(\alpha) \neq 0$ and the equation $f^D(\alpha) + Xf'(\alpha) = 0$ has a unique solution $X = -f^D(\alpha)/f'(\alpha)$. Now let $g(\alpha) \in F$ where $g(X) \in k[X]$ and define

$$\tilde{D}(g(\alpha)) = g^D(\alpha) - \left(\frac{f^D(\alpha)}{f'(\alpha)}\right) g'(\alpha) ,$$

then $\tilde{D}$ is a well-defined derivation of F into E such that $\tilde{D}|k = D$. Q.E.D.

(6.22) Proposition (see Zariski & Samuel [1, Cor.4 and 4' on P.125]). Let k be a field of characteristic $p > 0$ and L be an extension field of k. Then

$$k^p = \{x \in k \mid D(x) = 0 \text{ for any derivation } D:k \to L\} .$$

(For the definition of derivation see Proposition 6.21).

Proof. Let $z \in k - k^p$ and u be an arbitrary element of L, then $D(g(z)) = ug'(z)$, where $g(X) \in k^p[X]$, defines a derivation

$$D:k^p[z] \to L$$

such that $D(z) = u$ and $D(k^p) = \{0\}$. Let I be the set of all pairs (F',D') composed of a field F' such that $k^p[z] \subset F' \subset k$ and a derivation D' of F' extending D. We define

$$(F',D') < (F'',D'')$$

if $F' \subset F''$ and D'' extends D'. This relation $<$ defines an inductive order in I. Thus by Zorn's lemma there exists a maximal element (F_o,D_o) of I. Assume that $F_o \subsetneqq k$ and let $y \in k-F_o$ and $\tilde{u}$ be an arbitrary element of L. Then

$$\tilde{D}(g(y)) = g^{D_o}(y) + \tilde{u}g'(y) , \text{ where } g(X) \in F_o[X] ,$$

defines a derivation $\tilde{D}:F_o[y] \to L$ such that $\tilde{D}(y) = \tilde{u}$, because X^p-y^p is the monic minimal polynomial of y over F_o (see Lang [1,

§7 of Ch.VII]). Since $(F_o, D_o) \lneq (F_o[y], \tilde{D})$, we have got a contradiction. Hence $F_o = k$ and we have shown that for any elements $z \in k-k^p$ and $u \in L$ there exists a derivation

$$D_o : k \to L \quad \text{such that} \quad D_o(z) = u \ .$$

Therefore we have

$$k^p = \{x \in k \mid D(x) = 0 \quad \text{for any derivation} \quad D: k \to L\} \ .$$

Q.E.D.

<u>(6.23) Proposition</u> (see Zariski & Samuel [1, Th.42 on P.128]). Let k be a field of characteristic $p > 0$ and L be an extension field of k . If every derivation D of k into L , i.e., D is a map of k into L such that

$$D(x+y) = D(x) + D(y) \quad \text{and} \quad D(xy) = xD(y) + yD(x)$$

for any $x, y \in k$, can be extended to a derivation $\tilde{D}$ of L into itself, then L is separable over k .

$$\begin{array}{ccc} L & \xrightarrow{\exists \tilde{D}} & L \\ \cup & \circlearrowleft \nearrow_{D} & \\ k & & \end{array}$$

<u>Proof</u>. We shall show that if $\{x_1, \dots, x_t\}$ is a subset of L which is linearly independent over k , then the p-th powers of $x_1, \dots, x_t$ are also linearly independent over k (see Theorem 6.18). Conversely, assume that $\{x_1^p, \dots, x_t^p\}$ are linearly dependent over k .

Let

(*) $$a_1 x_{j_1}^p + \dots + a_n x_{j_n}^p \quad (a_i \in k-\{0\})$$

be one of the shortest non-trivial relations satisfied by $\{x_i^p\}$. We shall write $x_1 = x_{j_1}, \dots, x_n = x_{j_n}$ and assume that $a_1 = 1$.

Now let D be a derivation of k into k and $\tilde{D}$ be its extension, then we have

$$\tilde{D}(a_1 x_1^p + \dots + a_n x_n^p) = D(a_2) x_2^p + \dots + D(a_n) x_n^p = 0 \ ,$$

because $\tilde{D}(x_i^p) = 0 \quad (1 \leqq i \leqq n)$ and $\tilde{D}(a_1) = \tilde{D}(1) = 0$. Hence

$D(a_1) = D(a_2) = \ldots = D(a_n) = 0$ for any derivation $D: k \to k$, because (*) is one of the shortest non-trivial relations satisfied by $\{x_i^p\}$. From Proposition 6.22 we have

$$\{a_1, \ldots, a_n\} \subset k^p .$$

Thus $\{x_1, \ldots, x_t\}$ are not linearly independent over k, which contradicts to the assumption. Q.E.D.

Noether Normalization Theorem:

1. (see e.g. Lang [1, Theorem 4.1 on P.378]). Let $A = k[x_1, x_2, \ldots, x_m]$ be a finitely generated commutative algebra over a field k. Assume that A is an integral domain. Then there exist algebraically independent elements $z_1, z_2, \ldots, z_n$ over k in A such that A is integral over $k[z_1, z_2, \ldots, z_n]$.

2. Let A be a finitely generated commutative algebra over a perfect field K, and assume that A is an integral domain. Then $A = K[x_1, x_2, \ldots, x_n]$, with $\{x_1, x_2, \ldots, x_d\}$ algebraically independent over K, and for each $i > d$, x_i is separably algebraic over $K(x_1, x_2, \ldots, x_d)$, with monic minimal polynomial $F_i(X_i)$ which has coefficients in $K[x_1, x_2, \ldots, x_d]$.

Proof of 1. (see Step 1 of Proof of 2).

Proof of 2.

Step 1: (We follow the proof of Lang [1, Theorem 4.1 on P.378].) We shall prove the following assertion: "Let A be a finitely generated commutative algebra over a field K (Notice that K is not necessarily perfect), and assume that A is an integral domain. Then $A = K[x_1, x_2, \ldots, x_n]$, with $\{x_1, x_2, \ldots, x_d\}$ algebraically independent over K, and for each $i > d$, x_i is integral over $K[x_1, x_2, \ldots, x_{i-1}]$."

Proof of Step 1: Assume that A is generated by $(x_1, x_2, \dots, x_n)$. If $x_1, x_2, \dots, x_n$ is algebraically independent over K, then we are done.

Assume that $d < n$, where $d = \text{tr.deg}_K K(x_1, x_2, \dots, x_n)$, then there exists a non-trivial relation

$$(*) \qquad \sum_{\substack{\lambda_t \in K \\ t \in T_n}} \lambda_t x_1^{t_1} x_2^{t_2} \dots x_n^{t_n} = 0$$

with almost all $\lambda_t = 0$. Let $m_1, m_2, \dots, m_{n-1}$ be positive integers, and put

$$y_1 = x_1 - x_n^{m_1}, \dots, y_{n-1} = x_{n-1} - x_n^{m_{n-1}} .$$

Substitute

$$x_i = y_i + x_n^{m_i} \quad (i = 1,2,\dots,n-1) \quad \text{in } (*) .$$

Using vector notation, we put

$$(m) = (m_1, m_2, \dots, m_{n-1}, 1)$$

and use the dot product

$$(t)\cdot(m) = t_1 m_1 + t_2 m_2 + \dots + t_{n-1} m_{n-1} + t_n .$$

If we expand (*) after the substitution, we get

$$f(y_1, y_2, \dots, y_{n-1}, x_n) + \sum_t c_t x_n^{(t)\cdot(m)} = 0$$

where f is a polynomial in which no pure power of x_n appears. We now choose q to be a large integer which is greater than any component of a vector (t) such that $c_t \neq 0$, and take

$$(m) = (q, q^2, \dots, q^{n-1}, 1) .$$

Then all $(t)\cdot(m)$ are distinct for those (t) such that $c_t \neq 0$. Thus we obtain an integral equation for x_n over $K[y_1, y_2, \dots, y_{n-1}]$. Hence we have found a set of new generators $(y_1, y_2, \dots, y_{n-1}, x_n)$ of A such that x_n is integral over $K[y_1, y_2, \dots, y_{n-1}]$.

Now we can just proceed inductively, using the transitivity of integral extensions, to shrink the number of y's until we reach an algebraically independent set of y's.

Remark to Step 1. Let $\{x_1, x_2, \ldots, x_n\}$ be a set of generators of A obtained in Step 1. Then $K[x_1, x_2, \ldots, x_i]$ is integral over $K[x_1, x_2, \ldots, x_{i-1}]$ for any $i > d$, and thus x_i is integral over $K[x_1, x_2, \ldots, x_d]$ for any $i > d$. Hence each x_i $(i > d)$ has the monic minimal polynomial over the quotient field of $K[x_1, x_2, \ldots, x_d]$ with coefficients in $K[x_1, x_2, \ldots, x_d]$. $\{x_1, x_2, \ldots, x_d\}$ is a transcendence base of the quotient field of A .

From this remark we have done the proof of 2 in case of K is a field of characteristic 0.

Step 2 (We follow the proof of Lang [1, Prop.6.1 on P.382]). We shall prove the following: "Let K be a perfect field of characteristic $p > 0$ and A be a finitely generated commutative algebra over K , say $A = K[x_1, x_2, \ldots, x_n]$. Assume that A is an integral domain and $\{x_1, x_2, \ldots, x_n\}$ is not algebraically independent over K , then we can rearrange $\{x_1, x_2, \ldots, x_n\}$ such that x_n is separably algebraic over $K(x_1, x_2, \ldots, x_{n-1})$, the quotient field of $K[x_1, x_2, \ldots, x_{n-1}]$."

Proof of Step 2. Since $\{x_1, x_2, \ldots, x_n\}$ is not algebraically independent, we can take a transcendence base $\{x_t, x_{t+1}, \ldots, x_n\}$ with $t > 1$ from $\{x_1, x_2, \ldots, x_n\}$ after rearranging $\{x_1, x_2, \ldots, x_n\}$. Then x_{t-1} is algebraic over $K(x_t, x_{t+1}, \ldots, x_n)$, the quotient field of $K[x_t, x_{t+1}, \ldots, x_n]$. Let $f(X_{t-1}, X_t, \ldots, X_n) \in K[X_{t-1}, \ldots, X_n]$ be a non-trivial polynomial of lowest degree such that

$$f(x_{t-1}, x_t, \ldots, x_n) = 0 .$$

Then f is irreducible in $K[X_{t-1}, X_t, \ldots, X_n]$. We show that not all X_i $(i = t-1, \ldots, n)$ in f appear to the p-th power throughout. If they did we would write

$$f(X_{t-1}, X_t, \ldots, X_n) = \Sigma\, c_\alpha M_\alpha (X_{t-1}, \ldots, X_n)^p ,$$

where $M_\alpha(X)$ are monomials in $X_{t-1}, X_t, \ldots, X_n$ and $c_\alpha \in K$. This would imply that $M_\alpha(x_{t-1},\ldots,x_n)$ are linearly dependent over K, because

$$f(x_{t-1},x_t,\ldots,x_d) = \Sigma\, c_\alpha M_\alpha(x_{t-1},\ldots,x_n)^p$$
$$= (\Sigma\, c'_\alpha M_\alpha(x_{t-1},\ldots,x_n))^p = 0 ,$$

where c'_α is an element of K such that $c'^p_\alpha = c_\alpha$.

However, the $M_\alpha(x_{t-1},x_t,\ldots,x_n)$'s are linearly independent over K, otherwise we would get an equation for $x_{t-1},x_t,\ldots,x_n$ of lower degree.

Say X_n does not appear to the p-th power throughout, but actually appears in $f(x)$. Since $f(X_{t-1},X_t,\ldots,X_n)$ is a polynomial of lowest degree such that $f(x_{t-1},x_t,\ldots,x_n) = 0$, we have

$$\frac{df}{dX_n}(x_{t-1},x_t,\ldots,x_n) \neq 0 .$$

Hence x_n is not a multiple root of an equation $f(x_{t-1},x_t,\ldots,x_n) = 0$ over $K(x_{t-1},x_t,\ldots,x_{n-1})$. Thus x_n is separably algebraic over $K(x_1,x_2,\ldots,x_{n-1})$.

<u>Step 3</u>. "Assume that K is a perfect field of characteristic $p > 0$. Let $A = K[x_1,x_2,\ldots,x_n]$ be a finitely generated commutative algebra over K. Suppose that A is an integral domain and $\{x_1,x_2,\ldots,x_n\}$ is not algebraically independent over K, then we can replace $\{x_1,x_2,\ldots,x_n\}$ by a set of new generators $\{y_1,y_2,\ldots,y_{n-1},x_n\}$ including x_n such that x_n is integral over $K[y_1,y_2,\ldots,y_{n-1}]$ and separably algebraic over $K(y_1,y_2,\ldots,y_{n-1})$."

<u>Proof of Step 3</u>. From Step 2 we can assume that x_n is separably algebraic over $K(x_1,x_2,\ldots,x_{n-1})$. Hence there exists a minimal separable polynomial $g(X_n)$ of x_n over $K(x_1,x_2,\ldots,x_{n-1})$. Thus we get a non-trivial relation

(*) $$\sum_{\substack{\lambda_t \in K \\ t \in T_n}} \lambda_t x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n} = 0$$

with almost all $\lambda_t = 0$ such that

$$\sum_{\substack{\lambda_t \in K \\ t \in T_n}} \lambda_t X_1^{t_1} X_2^{t_2} \cdots X_n^{t_n} = h(X_1, X_2, \ldots, X_{n-1}) g(X_n)$$

for some $h(X_1, \ldots, X_{n-1}) \in K[X_1, X_2, \ldots, X_{n-1}]$ such that $h(x_1, \ldots, x_{n-1}) \neq 0$. Following the same argument and the notation of the proof of Step 1, we let $q = p^e$ and

$$m = (q, q^2, \ldots, q^{n-1}, 1) \, ,$$

where e is an enough large positive integer, and put

$$y_1 = x_1 - x_n^{m_1}, \ldots, y_{n-1} = x_{n-1} - x_n^{m_{n-1}} \, ,$$

and subsitute $x_i = y_i + x_n^{m_i}$ $(i = 1, 2, \ldots, n-1)$ in (*). Expanding (*) after the substitution, we get

(**) $$\Sigma c_t x_n^{(t)\cdot(m)} + f(y_1, y_2, \ldots, y_{n-1}, x_n) = 0 \, ,$$

where f is a polynomial in which no pure power of x_n appears. Thus we obtain an integral equation for x_n over $K[y_1, \ldots, y_{n-1}]$. Now since

$$\frac{\delta}{\delta X_n} \Big(\sum_{\substack{\lambda_t \in K \\ t \in T_n}} \lambda_t X_1^{t_1} X_2^{t_2} \cdots X_n^{t_n} \Big)$$

$$= \frac{\delta}{\delta X_n} \sum_{\substack{\lambda_t \in K \\ t \in T_n}} \lambda_t (Y_1 + X_n^{m_1})^{t_1} (Y_2 + X_n^{m_2})^{t_2} \cdots (Y_{n-1} + X_n^{m_{n-1}})^{t_{n-1}} X_n^{t_n} \, ,$$

where $X_i = Y_i + X_n^{m_i}$ for $i = 1, 2, \ldots, n-1$, we see that x_n is separably algebraic over $K(y_1, \ldots, y_{n-1})$.

Thus repeating the same procedure of Step 3 we finally reach the set of generators $\{x_1, x_2, \ldots, x_n\}$ of A with algebraically independent subset $\{x_1, x_2, \ldots, x_d\}$ such that for $i > d$, x_i is separably algebraic over $K(x_1, x_2, \ldots, x_d)$ with monic minimal polynomial $F_i(X_i)$

which has coefficients in $K[x_1,x_2,\ldots,x_d]$, because $K[x_1,x_2,\ldots,x_d]$ is a unique factorization domain. Q.E.D.

Finally we add several remarks on ideals which will be used for proving Lemma 7.14 by Krull.

(6.24) Definition. Let R be a commutative ring with unity element 1 and $\mathfrak{a}$ be an ideal in R . Then $\mathfrak{a}$ is said to be primary if the conditions $a,b \in R$, $ab \in \mathfrak{a}$ and $a \notin \mathfrak{a}$ imply the existence of an integer $m > 0$ such that $b^m \in \mathfrak{a}$.

(6.25) Proposition. Let R be a commutative ring with unity element 1 and $\mathfrak{a}$ be a primary ideal of R . Then

(1) the radical of $\mathfrak{a}$, $\sqrt{\mathfrak{a}} = \{a \in R \mid a^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$, is a prime ideal;

(2) for any $x,y \in R$ if $xy \in \mathfrak{a}$ and $x \notin \mathfrak{a}$, then $y \in \sqrt{\mathfrak{a}}$;

(3) if I and J are ideals in R such that $IJ \subset \mathfrak{a}$ and $I \not\subset \mathfrak{a}$, then $J \subset \sqrt{\mathfrak{a}}$.

Proof. (1) Assume that $xy \in \sqrt{\mathfrak{a}}$ and $x \notin \sqrt{\mathfrak{a}}$, where $x,y \in R$. Since $xy \in \sqrt{\mathfrak{a}}$, we have $(xy)^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. Since $x^n \notin \mathfrak{a}$ and $\mathfrak{a}$ is primary, $(y^n)^m \in \mathfrak{a}$ for some $m > 0$. Hence $y \in \sqrt{\mathfrak{a}}$ and $\sqrt{\mathfrak{a}}$ is prime.

(2) is clear from the definitions of primary ideal and radical.

(3) Let a be an element of I such that $a \notin \mathfrak{a}$. Since $aJ \subset \mathfrak{a}$ and $a \notin \mathfrak{a}$, we have $J \subset \sqrt{\mathfrak{a}}$ from (2). Q.E.D.

(6.26) Definition. Let R be a commutative ring with unity element 1 and $\mathfrak{a}$ be an ideal of R . We call $\mathfrak{a}$ irreducible if $\mathfrak{a}$ is not a finite intersection of ideals of R strictly containing $\mathfrak{a}$.

<u>(6.27) Lemma</u>. Let R be a commutative Noetherian ring with unity element 1, then every ideal is a finite intersection of irreducible ideals.

<u>Proof</u>. Suppose that the family S of all ideals of R which are not finite intersections of irreducible ideals is non-empty. Since R is Noetherian, S has a maximal element M . M is an intersection of a finite set of ideals $I_1, I_2, \ldots, I_t$ strictly containing M , because M is not irreducible. Since M is maximal in S , $I_1, I_2, \ldots, I_t$ are not in S . Hence $I_1, I_2, \ldots, I_t$ are finite intersections of irreducible ideals. Therefore, M is also a finite intersection of irreducible ideals, a contradiction. Q.E.D.

<u>(6.28) Lemma</u>. Let R be a commutative Noetherian ring with unity element 1, then every irreducible ideal of R is primary.

<u>Proof</u>. Let I be an ideal of R and assume that I is not primary. We shall show that I is not irreducible. Since I is not primary, there exist elements b and c in $R-I$ such that $bc \in I$ and no power of b lies in I . Let

$$Q_t = \{x \in R \mid xb^t \in I\} ,$$

where t is a positive integer, then Q_t's are ideals in R and we have the following increasing sequence

$$Q_1 \subset Q_2 \subset Q_3 \subset \ldots \subset Q_t \subset Q_{t+1} \subset \ldots$$

Since R is Noetherian, there exists $n > 0$ such that $Q_n = Q_{n+1}$. We claim that

$$I = (I + Rb^n) \cap (I + Rc) .$$

Clearly, $I \subset (I + Rb^n) \cap (I + Rc)$. Conversely, let $x \in (I + Rb^n) \cap (I + Rc)$, then we have

$$x = u+yb^n = v+zc , \text{ where } u,v \in I \text{ and } y,z \in R .$$

Since $bc \in I$, we have $bx \in I$. Hence $yb^{n+1} = bx-bu \in I$. Since $Q_n = Q_{n+1}$, $yb^n \in I$. Therefore, $x \in I$.

Now since $c \notin I$ and $b^n \notin I$, we have

$$I+Rc \supsetneqq I \quad \text{and} \quad I + Rb^n \supsetneqq I \ .$$

Hence I is an intersection of two ideals $I + Rb^n$ and $I + Rc$ strictly containing I. Thus I is not irreducible.

<u>(6.29) Proposition</u> (see Zariski & Samuel [1, P.216]). Let R be a commutative Noetherian ring with unity element 1 and $\mathcal{M}$ be an ideal of R, then

$$\mathcal{M} \cdot \bigcap_{n=1}^{\infty} \mathcal{M}^n = \bigcap_{n=1}^{\infty} \mathcal{M}^n \ .$$

<u>Proof</u>. Let $\mathfrak{a} = \bigcap_{n=1}^{\infty} \mathcal{M}^n$ and $\mathcal{M} \cdot \mathfrak{a} = \bigcap_{i=1}^{t} Q_i$ where Q_i's are primary ideals in R and t is a positive integer (see Lemma 6.27 and Lemma 6.28). Since $\mathcal{M} \cdot \mathfrak{a} \subset \mathfrak{a}$, it is enough to show that $\mathfrak{a} \subset Q_i$ for all i in order to establish the equality. Since $\mathcal{M} \cdot \mathfrak{a} \subset Q_i$, $\mathfrak{a} \subset Q_i$ if $\mathcal{M} \not\subset \sqrt{Q_i}$ (see Proposition 6.25). If $\mathcal{M} \subset \sqrt{Q_i}$ then $\mathcal{M}^l \subset Q_i$ for some integer $l > 0$, because $\mathcal{M}$ is a finitely generated R-module. Thus in either case we have $\mathfrak{a} \subset Q_i$. Hence $\mathcal{M} \cdot \mathfrak{a} = \mathfrak{a}$. Q.E.D.

<u>(6.30) Corollary</u>. Let R be a Noetherian local ring with maximal ideal M, then

$$\bigcap_{n=1}^{\infty} M^n = (0) \ .$$

<u>Proof</u>. It is clear from Nakayama's Lemma. Q.E.D.

<u>(6.31) Definition</u>. Let R be a Noetherian local ring with unique maximal ideal M. A subset U of R is said to be open if and only if for any $x \in U$ there exists a positive integer $n > 0$ such that $U \supset M^n+x$. These open sets in R define a topology, which is Hausdorff and is called the <u>M-adic topology</u> on R.

<u>Justification of the definition</u>. Clearly the M-adic topology is well-defined. Let $x,y \in R$ and assume $x \neq y$, then there exists an integer $n_o > 0$ such that $x-y \notin M^{n_o}$ from Corollary 6.30. Since $(M^{n_o} + x) \cap (M^{n_o} + y) = \emptyset$, R is Hausdorff.

7. Dimension of affine algebraic varieties

(7.1) Definition. Let (V,A) be an irreducible affine variety over K, then the dimension of V is the transcendence degree of the quotient field of A over K. We write $\dim V$ for the dimension of V. In general case we define

$$\dim V = \max \{\dim V_i \mid i = 1,2,\ldots,l\} ,$$

where $V_1, V_2, \ldots, V_l$ are the irreducible components of a given affine variety (V,A).

Exercise 23. Let $(V,A) \in \mathcal{A}(K)$. Then $\dim V = 0$ if and only if V is a non-empty finite set.

One can see at once that the dimension of affine n-space is n. In this section we shall explain the deeper geometrical meaning of dimension.

(7.2) Proposition. Let (V,A) be an irreducible affine variety over K and W be a proper non-empty closed irreducible subset of V. Then we have

$$\dim W < \dim V .$$

Proof. From Noether Normalization Theorem we can take algebraically independent elements $\{x_1,\ldots,x_m\}$ over K in A such that A is integral over $K[x_1,x_2,\ldots,x_m]$. Then $\{x_1,x_2,\ldots,x_m\}$ is a transcendence base of the quotient field of A and $\dim V = m$.

Now let $\mathfrak{p} = \mathcal{I}(W)$, which is a prime ideal from Proposition 5.5, and let $\upsilon : A \to A/\mathfrak{p}$ be the natural map. Then $A/\mathfrak{p}$ is integral over $K[\upsilon(x_1),\ldots,\upsilon(x_m)]$ and we can choose a maximal algebraic independent subset of $\{\upsilon(x_1),\upsilon(x_2),\ldots,\upsilon(x_m)\}$ as a transcendence base of the quotient field of $A/\mathfrak{p}$. Since

$$\dim W = \text{tr.deg}_K \text{ (the quotient field of } A/\mathfrak{p}) \leqq m ,$$

it is enough to show that $(\upsilon(x_1),\ldots,\upsilon(x_m))$ is not algebraically independent.

Let b be a non-zero element of $\mathfrak{p}$. Since A is integral over $K[x_1,x_2,\ldots,x_m]$, we have

$$b^n+f_1b^{n-1}+\ldots+f_n = 0$$

for some $f_1,\ldots,f_n \in K[x_1,x_2,\ldots,x_m]$. We can assume that n is minimal among those equations. Thus $f_n \neq 0$. Let

$$f_n = \sum_{\substack{t=(t_1,\ldots,t_m)\in T_m \\ \lambda_t\in K}} \lambda_t\, x_1^{t_1}\ldots x_m^{t_m} ,$$

where $T_m = \underbrace{\mathbb{N}\times\mathbb{N}\times\ldots\times\mathbb{N}}_{m}$ and almost all λ_t's are zero. Since $\upsilon(b) = 0$ and $\upsilon(b^n+f_1b^{n-1}+\ldots+f_{n-1}b) + \upsilon(f_n) = 0$, we have

$$\upsilon(f_n) = \sum_{\substack{t\in T_m \\ \lambda_t\in K}} \lambda_t\, \upsilon(x_1)^{t_1}\ldots\upsilon(x_m)^{t_m} = 0 ,$$

which implies $(\upsilon(x_1),\ldots,\upsilon(x_m))$ is not algebraically independent. Hence $\dim W < \dim V$. Q.E.D.

<u>(7.3) Corollary</u>. Let (V,A) be an irreducible affine variety over K . Then

(1) any set S of closed irreducible subsets of V has a maximal element, and

(2) if W is a maximal set among closed non-empty proper irreducible subsets of V , then W is a component of $\mathcal{V}(Af)$ for some non-zero non-unit $f \in A$. For the definition of units of A see Lang [1, P.61].

<u>Proof</u>. (1) We assume that S contains a non-empty closed irreducible subset of V . Then we can choose $Y_o \in S$ such that

$$\dim Y_o = \max\,(\dim Y \mid Y \in S \text{ and } Y \neq \emptyset) .$$

Y_o is a maximal element in S .

(2) Assume that $W = \mathscr{V}(I)$ for some ideal I . Since W is a proper subset of V , $I \supsetneqq (0)$, i.e., there exists $f \in I-(0)$. Then $W = \mathscr{V}(I) \subset \mathscr{V}(Af)$, because $Af \subset I$. Thus W is contained in some irreducible component Y of $\mathscr{V}(Af)$. Since W is maximal among closed non-empty irreducible proper subsets of V , we have $W = Y$.

Q.E.D.

(7.4) Definition. Let f be a non-constant polynomial, i.e., non-zero non-unit element in $K[X_1,X_2,\ldots,X_n]$, then the subset $\mathscr{V}(Af)$ in affine n-space K^n is called a hypersurface.

(7.5) Lemma. Any irreducible component of a hypersurface in affine n-space has dimension $n-1$.

Proof. Let f be a non-constant polynomial in $A = K[X_1,X_2,\ldots,X_n]$ and $\mathscr{V}(Af) \subset K^n$ be a hypersurface. Since A is a unique factorization domain (see Lang [1, Corollary 6.3 on P.199]), we have

$$f = f_1 f_2 \ldots f_m ,$$

where $f_1,f_2,\ldots,f_m$ are irreducible elements in A . Now let $f_1,f_2,\ldots,f_{m_o}$ be a full set of different elements from $\{f_1,f_2,\ldots,f_m\}$, then we have

$$\mathscr{V}(Af) = \mathscr{V}(Af_1) \cup \mathscr{V}(Af_2) \cup \ldots \cup \mathscr{V}(Af_{m_o}) .$$

Assume that $g \in \sqrt{Af_i}$, then $g^l \in Af_i$ for some $l \in \mathbb{N}$ and $f_i \mid g$, because $l \geqq 1$. Hence we have

$$Af_i = \sqrt{Af_i} = \mathscr{I}(\mathscr{V}(Af_i))$$

from the Nullstellensatz. If $gh \in Af_i$, i.e., $f_i \mid gh$ for some $g,h \in A$, then we have $f_i \mid g$ or $f_i \mid h$, which implies that Af_i is a prime ideal. Hence each $\mathscr{V}(Af_i)$ is an irreducible closed subset. It is easy to see that $\mathscr{V}(Af_i)$ $(i = 1,2,\ldots,m_o)$ are irreducible components of $\mathscr{V}(Af)$, because $\mathscr{V}(Af_i) \not\subset \mathscr{V}(Af_j)$ when $i \neq j$ (see the proof of Theorem 5.9).

Now we shall show that

$$\dim \mathscr{V}(Af_i) = \text{tr.deg}_K \text{ (the quotient field of } A/Af_i)$$
$$= n-1 \text{ , where } i = 1,2,\dots,m_o \text{ .}$$

We can assume that f_i involves X_n . Suppose that

$$\{X_1 + Af_i,\dots,X_{n-1} + Af_i\}$$

is not algebraically independent over K , then there exists a non-trivial polynomial in $n-1$ variables over K such that

$$g(X_1 + Af_i,\dots,X_{n-1} + Af_i) = 0 \text{ .}$$

Thus $g(X_1,X_2,\dots,X_{n-1}) \in Af_i$, i.e., $f_i \mid g$, which implies g involves X_n , a contradiction. Hence

$$n-1 \leq \dim \mathscr{V}(Af_i) < \dim K^n = n$$

(see Proposition 7.2). Q.E.D.

- Norm -

<u>(7.6) Definition</u>. Let E be a finite extension field of a field F . Let $\alpha \in E$ be an arbitrary element of E , then we define its <u>norm</u> $N_{E/F}(\alpha)$ to be the determinant of the F-linear transformation given by the multiplication by α . Thus if $\{e_1,e_2,\dots,e_n\}$ is an F-basis of E and

$$\alpha e_i = \sum_{j=1}^{n} f_{ji}e_j \quad (1 \leq i \leq n) \text{ ,}$$

then $N_{E/F}(\alpha) = \det(f_{ij})$. It is clear that $N_{E/F}(\alpha)$ is independent of choice of basis and the map

$$N_{E/F}:E^* \longrightarrow F^*$$
$$(N_{E/F}:\alpha \longrightarrow N_{E/F}(\alpha))$$

is a multiplicative homomorphism where $E^* = E-\{0\}$ and $F^* = F-\{0\}$.

Now let $F_1 = F[\alpha]$ where $\alpha \in E$ and $\{x_1,x_2,\dots,x_m\}$ be an F_1-basis of E , i.e.,

$$E = F_1x_1 \oplus F_1x_2 \oplus\dots\oplus F_1x_m \text{ .}$$

Since we can choose an F-basis of E of which each basis element lies in some F_1x_i , we have

$$N_{E/F}(\alpha') = \{N_{F_1/F}(\alpha')\}^{[E:F_1]} \quad \text{for any} \quad \alpha' \in F[\alpha] .$$

Next assume that $X^l+\lambda_1X^{l-1}+\ldots+\lambda_l$ $(\lambda_i \in F\ ,\ \ i = 1,2,\ldots,l)$ is the minimal polynomial of α over F , then $\{1,\alpha,\ldots,\alpha^{l-1}\}$ is an F-basis of F_1 . Hence we have

$$N_{F_1/F}(\alpha) = (-1)^l\lambda_l .$$

Thus we have got the following proposition.

<u>(7.7) Proposition</u>. Let $\alpha \in E$ and $F_1 = F[\alpha]$. If $X^l+\lambda_1X^{l-1}+\ldots+\lambda_l$ $(\lambda_i \in F\ ,\ \ i = 1,2,\ldots,l)$ is the minimal polynomial of α over F , then we have

$$N_{E/F}(\alpha) = \{(-1)^l\lambda_l\}^{[E:F_1]} .$$

Now we shall show the first main theorem of this section.

<u>(7.8) Theorem</u>. Let (V,A) be an irreducible affine variety over K , and f be a non-zero non-unit element of A . Let Y be an irreducible component of $\mathcal{V}(Af)$, then

$$\dim Y = (\dim V)-1 .$$

<u>Proof</u>. Let $\mathfrak{p} = \mathcal{I}(Y)$ and $Y_1,Y_2,\ldots,Y_t$ be the irreducible components of $\mathcal{V}(Af)$ other than Y .

We assume that $t \geq 1$. Since $Y \not\subseteq Y_1 \cup Y_2 \cup\ldots\cup Y_t$, i.e. $\mathcal{I}(Y_1 \cup Y_2 \cup\ldots\cup Y_t) \not\subseteq \mathcal{I}(Y)$, we can choose $g \in \mathcal{I}(Y_1 \cup Y_2 \cup\ldots\cup Y_t)-\mathcal{I}(Y)$. Let $A_g = A[\frac{1}{g}]$ $(= \{\frac{a}{g^r} \mid a \in A\ ,$ $r \in \mathbb{N}\})$ in the quotient field of A and

$$V_g = \{v \in V \mid g(v) \neq 0\} \quad \text{and}$$

$$\mathfrak{p}_g = \{\frac{a}{g^r} \mid a \in \mathfrak{p}\ ,\ \ r \in \mathbb{N}\} .$$

From Proposition 2.8 (V_g,A_g) is an affine variety over K .

(*) We shall show that

$$" Y \cap V_g = \{v \in V_g \mid f(v) = 0\} \text{ and}$$

$$\mathfrak{p}_g = \{h \in A_g \mid h(y) = 0 \text{ for all } y \in Y \cap V_g\}$$

$$= \mathcal{I}_{A_g}(Y \cap V_g) \ . "$$

<u>Proof of (*)</u>. It is clear that $Y \cap V_g \subset \{v \in V_g \mid f(v) = 0\}$. Assume that $v \in V_g$ and $f(v) = 0$, then we have $v \in \mathcal{V}_V(Af)$. Hence $v \in Y$ or $v \in Y_i$ for some $1 \leq i \leq t$. In case $v \in Y_i$ for some $1 \leq i \leq t$, then $g(v) = 0$, because $g \in \mathcal{I}(Y_1 \cup Y_2 \cup \ldots \cup Y_t)$, a contradiction. Hence $v \in Y$ and $Y \cap V_g = \{v \in V_g \mid f(v) = 0\}$.

It is also clear that $\mathfrak{p}_g \subset \mathcal{I}_{A_g}(Y \cap V_g)$. Assume that

$h = \frac{a}{g^r} \in \mathcal{I}_{A_g}(Y \cap V_g)$ then $hg^{r+1} = ag$ is zero on Y . Hence $ag \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime and $g \notin \mathfrak{p}$, we have $a \in \mathfrak{p}$. Thus we have $h \in \mathfrak{p}_g$.

Since $\mathfrak{p}_g$ is prime in A_g , $Y \cap V_g$ is an irreducible closed set in V_g . Furthermore,

$$\begin{aligned} \dim V_g &= \text{tr.deg}_K \text{ (the quotient field of } A_g \text{)} \\ &= \text{tr.deg}_K \text{ (the quotient field of } A \text{)} \\ &= \dim V \end{aligned}$$

and

$$\begin{aligned} \dim Y &= \text{tr.deg}_K \text{ (the quotient field of } A/\mathcal{I}(Y) \text{)} \\ &= \text{tr.deg}_K \text{ (the quotient field of } A_g/\mathfrak{p}_g \text{)} \\ &= \dim Y \cap V_g \ , \end{aligned}$$

where $Y \cap V_g$ is considered as a closed subset of V_g . Thus we can now assume that $\mathcal{V}(Af)$ is irreducible and $\mathcal{V}(Af) = Y$.

Now from the Noether Normalization Theorem 1 we can take algebraically independent elements $x_1, x_2, \ldots, x_d$ from A such that A is integral over $B = K[x_1, x_2, \ldots, x_d]$.

Let E be the quotient field of A and F be the quotient field of B. Let $a \in A$ be an arbitrary element of A, then the minimal polynomial of a over F lies in $B[X]$, because B is a unique factorization domain. Hence

$$N_{E/F}(a) \in B$$

from Proposition 7.7 and the map $N_{E/F}$ takes A into B.

Let $f_o = N_{E/F}(f)$ and assume that $X^l+b_1X^{l-1}+\ldots+b_l$ is the minimal polynomial of f over F. Then we have

$$\begin{aligned} N_{E/F}(f) &= \{(-1)^l b_l\}^{[E:F[f]]} \\ &= (-1)^{[E:F]}\, b_l^{[E:F[f]]} . \end{aligned}$$

Since $f^l+b_1f^{l-1}+\ldots+b_l = 0$, we have

$$(-1)^{[E:F]} b_l^{s-1}(f^l+b_1f^{l-1}+\ldots+b_l) = 0 ,$$

where $s = [E:F[f]]$. Thus

$$\begin{aligned} f_o = N_{E/F}(f) &= (-1)^{[E:F]} b_l^s \\ &= (-f)(-1)^{[E:F]}(b_l^{s-1}f^{l-1}+\ldots+b_l^{s-1}b_{l-1}) \in Af . \end{aligned}$$

From the Nullstellensatz we have

$$\mathfrak{p} = \mathcal{I}(Y) = \mathcal{I}(\mathcal{V}(Af)) = \sqrt{Af} .$$

Hence $f_o \in Af \subset \mathfrak{p}$.

(**) Now we show that $I = B \cap \mathfrak{p}$, where I is the radical of Bf_o in B.

<u>Proof of (**)</u>. Since $Bf_o \subset \mathfrak{p}$, $I \subset B \cap \mathfrak{p}$. Assume that $h \in B \cap \mathfrak{p}$, then $h^r = gf$ for some $g \in A$ and $r \in \mathbb{N}$, because $\mathfrak{p} = \sqrt{Af}$. Thus we have

$$N_{E/F}(h)^r = N_{E/F}(g)N_{E/F}(f) = f_oN_{E/F}(g) \in Bf_o .$$

Since $h \in B$, the minimal polynomial of h over F is $X-h$. Thus $N_{E/F}(h) = h^{[E:F]}$. Hence $h \in I$ and we have $I = B \cap \mathfrak{p}$.

Hence I is prime in B. The only possibility that I can be prime is if $f_o = p^r$ for some irreducible element p in

$B = K[x_1, x_2, \ldots, x_d]$ and some $r \geq 1$. Thus we have $I = Bp$. Since A is integral over B , $A/\mathfrak{p}$ is integral over $B+\mathfrak{p}/\mathfrak{p}$ and

$$\begin{aligned} & \text{tr.deg}_K \text{ (the quotient field of } A/\mathfrak{p}\text{)} \\ = & \text{tr.deg}_K \text{ (the quotient field of } B+\mathfrak{p}/\mathfrak{p}\text{)} \\ = & \text{tr.deg}_K \text{ (the quotient field of } B/B\cap\mathfrak{p}\text{)} \\ = & \text{tr.deg}_K \text{ (the quotient field of } B/Bp\text{)} . \end{aligned}$$

From Lemma 7.5 we have $\text{tr.deg}_K(B/Bp) = d-1$. Therefore,

$$\begin{aligned} \dim Y &= \text{tr.deg}_K \text{ (the quotient field of } A/\mathfrak{p}\text{)} \\ &= d-1 \\ &= (\dim V)-1 . \end{aligned}$$

Q.E.D.

<u>(7.9) Corollary</u>.

(1) Let (V,A) be an irreducible affine variety over K and

$$V = V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_o \supsetneq \emptyset$$

be any maximal sequence of closed irreducible subsets, i.e., each V_i is closed irreducible and there is no proper closed irreducible subset between V_{i+1} and V_i , for $i = 0,1,2,\ldots,d-1$, and V_o has no proper closed irreducible subset. (Because of Proposition 7.2 such a sequence exists with finite terms.)

Then we have $\dim V = d$.

(2) Let (U,A) and (V,B) be irreducible affine varieties over K . Then we have

$$\dim(U \times V) = \dim U + \dim V .$$

(3) Let (V,A) be an irreducible affine variety over K and Y be a closed irreducible subset of <u>codimension</u> $r \geq 1$, i.e., $\dim V - \dim Y = r \geq 1$. Then there exist closed irreducible subsets Y_i of codimension $1 \leq i \leq r$ such that

$$V \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_r = Y .$$

(4) Let (V,A) be an irreducible affine variety over K and $f_1, f_2, \ldots, f_t$ be elements in A. Assume that

$$\mathcal{V}(Af_1+\ldots+Af_t) \neq \emptyset ,$$

then each irreducible component of $\mathcal{V}(Af_1+\ldots+Af_t)$ has codimension at most t.

(5) Let (V,A) be an irreducible affine variety over K and Y be a closed irreducible subset of codimension $r \geq 1$. Then Y is an irreducible component of $\mathcal{V}(Af_1+\ldots+Af_r)$ for some $f_1, f_2, \ldots, f_r \in A$.

Proof. (1) We follow the induction on $\dim V$. Assume that $\dim V = 0$, then $|V| = 1$ where $|V|$ is the cardinality of V, and the assertion is correct. Hence it is enough to show that

$$\dim V_{d-1} = (\dim V)-1 .$$

Since $V \supsetneq V_{d-1}$, $\mathcal{I}(V_{d-1}) \supsetneq \mathcal{I}(V) = 0$. Hence we can take a non-zero non-unit element f in $\mathcal{I}(V_{d-1})$. Since $\mathcal{V}(Af) \supset \mathcal{V}(\mathcal{I}(V_{d-1})) = V_{d-1}$ and V_{d-1} is an irreducible component of $\mathcal{V}(Af)$, we have $\dim V_{d-1} = (\dim V)-1$ from the Theorem.

(2) First we assume that $\dim V = 0$, then $|V| = 1$ and $B = K$. Since $A \otimes_K K = A$, we have $\dim(U\times V) = \dim U + \dim V$. Thus we can follow the induction on $\dim U + \dim V$. Assume that $\dim V > 0$, then we can take a non-zero non-unit element b in B. Let $\mathcal{V}_V(Bb) = V_1 \cup \ldots \cup V_t$ be a decomposition of $\mathcal{V}_V(Bb)$ as a union of irreducible components, then

$$\mathcal{V}_{U\times V}((A \otimes_K B)(1 \otimes b)) = (U \times V_1) \cup \ldots \cup (U \times V_t)$$

is a decomposition of $\mathcal{V}_{U\times V}((A \otimes_K B)(1 \otimes b))$ into irreducible components from Proposition 5.7. Thus from the Theorem we have

$$\dim(U\times V_1) = \dim(U\times V)-1 .$$

By induction we have $\dim(U\times V_1) = \dim U + \dim V_1$. Again from the Theorem we have $\dim V_1 = (\dim V)-1$. Therefore, we have proved

$$\dim(U\times V) = \dim U + \dim V .$$

(3) is clear from Corollary 7.3 and (1).

(4) Let $W_t = \mathscr{V}(Af_1+\dots+Af_t)$, then W_t is a closed subvariety of $W_{t-1} = \mathscr{V}(Af_1+\dots+Af_{t-1})$. Let Y be an irreducible component of $\mathscr{V}_{W_{t-1}}(Af_t)$ $(= W_t)$, then Y is contained in some irreducible component Y_o of W_{t-1} and $W_t \supset \mathscr{V}_{Y_o}(Af_t) \supset Y$. Hence $Y_o = Y$ or $\dim Y_o = \dim Y + 1$ from the Theorem. Since $\operatorname{codim} Y_o \leq t-1$ by induction, we have $\operatorname{codim} Y \leq t$.

(5) Let Y_i be closed irreducible subsets of codimension $1 \leq i \leq r$ such that $V \supsetneq Y_1 \supsetneq Y_2 \supsetneq \dots \supsetneq Y_r = Y$. We shall prove that for any q $(1 \leq q \leq r)$ there exists $f_i \in A$ $(1 \leq i \leq q)$ such that all irreducible components of $\mathscr{V}(Af_1+\dots+Af_q)$ have codimension q in V and Y_q is one of these components.

Assume that $q = 1$. Let $f_1 \in \mathscr{I}(Y_1)-\{0\}$. Since f_1 is non-zero and non-unit and $V \supset \mathscr{V}(Af_1) \supset Y_1$, it is clear that f_1 satisfies the condition from the Theorem.

Now suppose that there exist $f_1,\dots,f_{q-1} \in A$ which satisfy the condition. Let $Z_1 = Y_{q-1}, Z_2, \dots, Z_m$ be irreducible components of $\mathscr{V}(Af_1+\dots+Af_{q-1})$. Since any Z_j is of codimension $q-1$, none of Z_j's is contained in Y_q . Hence $\mathscr{I}(Y_q) \not\subset \mathscr{I}(Z_j)$ for any $1 \leq j \leq m$. Therefore we have

$$\mathscr{I}(Y_q) \not\subset \bigcup_{j=1}^{m} \mathscr{I}(Z_j)$$

from "Zariski & Samuel [1, P.215]: Let A be a commutative ring and $\mathfrak{a}$ an ideal of A . If $\mathfrak{a}$ is contained in a finite union $\bigcup_{i=1}^{m} \mathfrak{p}_i$ of prime ideals $\mathfrak{p}_i$ of A , then $\mathfrak{a}$ is contained in one of $\mathfrak{p}_i$'s ."

Let f_q be an element of $\mathscr{I}(Y_q)$ such that $f_q \notin \bigcup_{j=1}^{m} \mathscr{I}(Z_j)$. Let Z be an irreducible component of $\mathscr{V}(Af_1+\dots+Af_q)$, then there exists Z_j $(1 \leq j \leq m)$ such that $Z_j \supset Z$. Since

$$Z \subset \mathscr{V}(Af_q) \cap Z_j = \mathscr{V}_{Z_j}(Af_q) ,$$

$\dim Z_j - \dim Z = 1$ from the Theorem. Hence $\operatorname{codim} Z = q$. Since $Y_q \subset \mathscr{V}(Af_q)$ and $Y_q \subset Y_{q-1} \subset \mathscr{V}(Af_1+\ldots+Af_{q-1})$, there exists an irreducible component Z of $\mathscr{V}(Af_1+\ldots+Af_q)$ which contains Y_q . Since $\operatorname{codim} Y_q = \operatorname{codim} Z$, we have $Y_q = Z$. Q.E.D.

Exercise 24. Prove Corollary 7.9.3.

Exercise 25. Prove the assertion "Zariski & Samuel [1, P.215]" in the proof of Corollary 7.9.5.

(7.10) Proposition. Let A be a finitely generated commutative algebra over an algebraically closed field K . Assume that A is an integral domain and let

$$V = \mathrm{Hom}_{K\text{-alg}}(A,K)$$

and define

$$\begin{array}{c} \iota : A \longrightarrow M(V,K) \\ (\iota : a \longrightarrow [\iota(a) : v \to v(a)]) \end{array} ,$$

where $v \in V$. Then

(1) ι is an injective K-algebra map and $(V,A) \in \mathscr{A}(K)$; and

(2) the Krull dimension of A , i.e., the maximal length of d of a chain prime ideals

$$0 \subsetneqq \mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \cdots \subsetneqq \mathfrak{p}_d \subsetneqq A$$

is equal to $\dim V$ (= $\mathrm{tr.deg}_K$ {the quotient field of A}).

Proof. (1) is clear from Proposition 6.9.

(2) Let $V = \mathscr{V}(\{0\}) \supset \mathscr{V}(\mathfrak{p}_1) \supset \mathscr{V}(\mathfrak{p}_2) \supset \cdots \supset \mathscr{V}(\mathfrak{p}_d) \supset \emptyset$ be the sequence of closed subsets of V corresponding to

$$0 \subsetneqq \mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \cdots \subsetneqq \mathfrak{p}_d \subsetneqq A .$$

Since $\mathscr{I}(\mathscr{V}(\mathfrak{p}_i)) = \sqrt{\mathfrak{p}_i} = \mathfrak{p}_i$ for each $1 \leq i \leq d$ from the Hilbert's Nullstellensatz, each $\mathscr{V}(\mathfrak{p}_i)$ is irreducible and $\mathscr{V}(\mathfrak{p}_i) \supsetneqq \mathscr{V}(\mathfrak{p}_{i+1})$. Since $\mathscr{I}(\mathscr{V}(\mathfrak{p}_i)) \subset \mathscr{I}(F) \subset \mathscr{I}(\mathscr{V}(\mathfrak{p}_{i+1}))$ for any closed irreducible subset F of V such that $\mathscr{V}(\mathfrak{p}_i) \supset F \supset \mathscr{V}(\mathfrak{p}_{i+1})$, we have $\dim V = d$ from Corollary 7.9.1 as desired. Q.E.D.

<u>(7.11) Definition</u>. Let V be an irreducible affine variety over K. Let p be a point of V. We define the <u>local ring</u> $\mathcal{O}_p$ of p on V to be

$$\mathcal{O}_p = K[V]_{\mathcal{I}(\{p\})} ,$$

where $K[V]$ is the coordinate ring of V and

$$K[V]_{\mathcal{I}(\{p\})} = (K[V] - \mathcal{I}(\{p\}))^{-1}K[V] .$$

It can be easily checked that $\mathcal{O}_p$ is a Noetherian K-algebra and integral domain. Since $\mathcal{I}(\{p\})$ is a prime ideal in $K[V]$, $\mathcal{O}_p$ is really a local ring with the maximal ideal $(K[V] - \mathcal{I}(\{p\}))^{-1}\mathcal{I}(\{p\})$.

<u>Exercise 26</u>. Let A be a commutative ring with unity element 1 and S be a multiplicative subset of A.

(1) For an ideal $\mathcal{A}$ of A we define

$$S^{-1}\mathcal{A} = \{a/s \mid a \in \mathcal{A} \text{ and } s \in S\}$$

to be the subset of $S^{-1}A$ consisting of all fractions a/s with $a \in \mathcal{A}$ and $s \in S$. Then $S^{-1}\mathcal{A}$ is an $S^{-1}A$-ideal and for any ideals $\mathcal{A}$ and $\mathcal{B}$ of A we have

$$S^{-1}(\mathcal{A}+\mathcal{B}) = S^{-1}\mathcal{A}+S^{-1}\mathcal{B} ,$$

$$S^{-1}(\mathcal{A}\mathcal{B}) = (S^{-1}\mathcal{A})(S^{-1}\mathcal{B}) \quad \text{and}$$

$$S^{-1}(\mathcal{A} \cap \mathcal{B}) = S^{-1}\mathcal{A} \cap S^{-1}\mathcal{B} .$$

(2) If A is Noetherian, then $S^{-1}A$ is also a Noetherian ring.

<u>(7.12) Lemma</u>. Let R be a commutative ring with unity element 1. Let M be a maximal ideal of R. We write R_M for $(R-M)^{-1}R$ and MR_M for $(R-M)^{-1}M$. Then

(1) the operation: $\begin{array}{ccc} R/M \times M/M^2 & \longrightarrow & M/M^2 \\ ((r+M, m+M^2) & \longrightarrow & rm+M^2) \end{array}$ defines a R/M-vector space structure on M/M^2,

(2) the map: $\begin{array}{ccc} R/M & \longrightarrow & R_M/MR_M \\ (r+M & \longrightarrow & r/1+MR_M) \end{array}$ is a field-isomorphism,

(3) the map: $\begin{matrix} M/M^2 \longrightarrow MR_M/(MR_M)^2 \\ (m+M^2 \longrightarrow m/1+(MR_M)^2) \end{matrix}$ defines a vector space-isomorphism between the two R/M-vector spaces.

Since the proof of Lemma 7.12 is straightforward, we leave it as an exercise.

Exercise 27. Prove Lemma 7.12.

(7.13) Lemma. Let R be a commutative ring with unity element 1. Let $\mathfrak{a}$ be an ideal of R such that $R \supsetneq \mathfrak{a}$. Assume that $\mathfrak{a}$ is contained in some prime ideal $\mathfrak{p}$ of R . Then $\mathfrak{p}$ contains a prime ideal $\mathfrak{p}_o$ which is minimal among prime ideals containing $\mathfrak{a}$.

Proof. Let $\mathcal{P} = \{\mathcal{N} \mid \mathcal{N}$ is a prime ideal of R such that $\mathfrak{p} \supset \mathcal{N} \supset \mathfrak{a}\}$. We define an order $\geqq$ on $\mathcal{P}$ as follows: $\mathcal{M} \geqq \mathcal{N}$ $(\mathcal{M}, \mathcal{N} \in \mathcal{P})$ if and only if $\mathcal{N} \supset \mathcal{M}$. Let $\mathcal{P}_o$ be a totally ordered subset of $\mathcal{P}$. Let

$$\mathcal{N}_o = \bigcap_{\mathcal{N} \in \mathcal{P}_o} \mathcal{N} ,$$

then $\mathcal{N}_o$ is an ideal of R such that $\mathfrak{p} \supset \mathcal{N}_o \supset \mathfrak{a}$.

Now we show that $\mathcal{N}_o$ is a prime ideal. Assume that $ab \in \mathcal{N}_o$ for some $a,b \in R$. If there exists $\mathcal{N} \in \mathcal{P}_o$ such that $a \notin \mathcal{N}$, then we have $b \in \mathcal{N}$. Suppose that b is not contained in some $\mathcal{M} \in \mathcal{P}_o$, then $\mathcal{N} \supsetneq \mathcal{M}$. Thus $\mathcal{M}$ does not contain both a and b , which contradicts the fact that $\mathcal{M}$ is prime and contains the product ab . Thus b is contained in all $\mathcal{M} \in \mathcal{P}_o$, i.e., $b \in \mathcal{N}_o$. Hence $\mathcal{N}_o$ is prime and $\mathcal{P}$ is inductively ordered.

From the Zorn's lemma $\mathcal{P}$ has a maximal element $\mathfrak{p}_o$ and it is the desired ideal. Q.E.D.

<u>(7.14) Lemma</u> (Krull) (see Nagata [1, (9.2)]). Let R be a commutative Noetherian ring with unity element 1. Assume that R is an integral domain. Let a be a non-zero non-unit element of R. Let $\mathfrak{p}$ be a prime ideal of R which is minimal among prime ideals of R containing Ra, then $\mathfrak{p}$ contains no prime ideals except $\{0\}$ and $\mathfrak{p}$ itself.

<u>Proof</u>. Since $R \supsetneqq Ra$, certainly such a $\mathfrak{p}$ exists from Lemma 7.13. Let $S = R-\mathfrak{p}$, then $S^{-1}\mathfrak{p}$ is a unique maximal ideal of the local ring $S^{-1}R$. We denote $S^{-1}R = R_{\mathfrak{p}}$ and $S^{-1}\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}}$. It is clear that $\mathfrak{p}R_{\mathfrak{p}}$ contains $S^{-1}(Ra)$ and $a/1$ is a non-zero non-unit element in $R_{\mathfrak{p}}$, which also generated $S^{-1}(Ra) = (S^{-1}R)a/1$.

Assume that $\mathfrak{p}_o$ is a non-zero prime ideal of R properly contained in $\mathfrak{p}$, then $S^{-1}\mathfrak{p}_o$ is also a non-zero prime ideal of $S^{-1}R$ properly contained in $\mathfrak{p}R_{\mathfrak{p}}$.

Now let $\mathfrak{a}$ be a prime ideal of $R_{\mathfrak{p}}$ such that

$$\mathfrak{p}R_{\mathfrak{p}} \supset \mathfrak{a} \supset R_{\mathfrak{p}}a/1 .$$

Since $\mathfrak{p}R_{\mathfrak{p}} \cap R = \mathfrak{p} \supset \mathfrak{a} \cap R \supset R_{\mathfrak{p}}a/1 \cap R \supset Ra$ and $\mathfrak{a} \cap R$ is a prime ideal in R, we have $\mathfrak{p} = \mathfrak{a} \cap R$. Hence

$$\mathfrak{p}R_{\mathfrak{p}} = S^{-1}(\mathfrak{a} \cap R) = \mathfrak{a} .$$

Therefore, $\mathfrak{p}R_{\mathfrak{p}}$ is also minimal among prime ideals of $R_{\mathfrak{p}}$ containing $R_{\mathfrak{p}}a/1$. Thus we may assume that $\mathfrak{p}$ is the unique maximal ideal in R.

Let $R' = R/Ra$ and $\upsilon: R \longrightarrow R/Ra$ $(\upsilon: r \longrightarrow r+Ra)$ be the natural map, then every prime ideal of R' is equal to $\upsilon(\mathfrak{p})$.

Now we show that R' satisfies the minimal condition for ideals. Since every prime ideal of R' is equal to $\upsilon(\mathfrak{p})$, we have $\sqrt{\{0\}} = \upsilon(\mathfrak{p})$ in R'. Since R' is Noetherian, $\upsilon(\mathfrak{p})$ is generated by finite number of nilpotent elements $p_1, p_2, \ldots, p_m$ as R'-module. Then there exists a natural number $N \neq 0$ such that $p_i^N = 0$ for every i . Since $\upsilon(\mathfrak{p})^{Nm} = \{0\}$, $\upsilon(\mathfrak{p})$ is a nilpotent ideal. Let $r \neq 0$ be the smallest natural number such that $\upsilon(\mathfrak{p})^r = \{0\}$. Then we have the following sequence of ideals:

$$R' \supset \upsilon(\mathfrak{p}) \supset \upsilon(\mathfrak{p})^2 \supset \ldots \supset \upsilon(\mathfrak{p})^{r-1} \supset \upsilon(\mathfrak{p})^r = \{0\} .$$

We can consider all the factor modules

$$R'/\upsilon(\mathfrak{p}),\ \upsilon(\mathfrak{p})/\upsilon(\mathfrak{p})^2, \ldots,\ \upsilon(\mathfrak{p})^{r-1}/\upsilon(\mathfrak{p})^r$$

to be $R'/\upsilon(\mathfrak{p})$-modules.

$$R'/\upsilon(\mathfrak{p}) \times \upsilon(\mathfrak{p})^i/\upsilon(\mathfrak{p})^{i+1} \longrightarrow \upsilon(\mathfrak{p})^i/\upsilon(\mathfrak{p})^{i+1}$$

$$(r+\upsilon(\mathfrak{p}),\ x+\upsilon(\mathfrak{p})^{i+1}) \longrightarrow rx+\upsilon(\mathfrak{p})^{i+1}$$

Since R' is Noetherian, $R'/\upsilon(\mathfrak{p}),\ \upsilon(\mathfrak{p})/\upsilon(\mathfrak{p})^2, \ldots, \upsilon(\mathfrak{p})^{r-1}/\upsilon(\mathfrak{p})^r$ are finitely generated R'-modules. Hence they are also finitely generated $R'/\upsilon(\mathfrak{p})$-modules. Since $R'/\upsilon(\mathfrak{p})$ is a field, the factor modules

$$R'/\upsilon(\mathfrak{p}),\ \upsilon(\mathfrak{p})/\upsilon(\mathfrak{p})^2, \ldots,\ \upsilon(\mathfrak{p})^{r-1}/\upsilon(\mathfrak{p})^r$$

have finite $R'/\upsilon(\mathfrak{p})$-basis as follows

$$R'/\upsilon(\mathfrak{p}) = R'/\upsilon(\mathfrak{p})(1+\upsilon(\mathfrak{p}))$$

$$\upsilon(\mathfrak{p})^i/\upsilon(\mathfrak{p})^{i+1} = R'/\upsilon(\mathfrak{p})(x_1^{(i)}+\upsilon(\mathfrak{p})^{i+1}) \oplus \ldots \oplus R'/\upsilon(\mathfrak{p})(x_{l_i}^{(i)}+\upsilon(\mathfrak{p})^{i+1})$$

$(i = 1,2,\ldots,r-1)$.

Since we can also consider $R'/\upsilon(\mathfrak{p})(x_j^{(i)}+\upsilon(\mathfrak{p})^{i+1})$ $(j = 1,\ldots,l_i)$ to be an R'-module and they are all irreducible R'-modules, we see that R' has a composition series. Thus $R' = R/Ra$ satisfies the minimal condition for ideals.

Let $\mathfrak{q}$ be a prime ideal of R such that $\mathfrak{q} \subsetneq \mathfrak{p}$. Let $\mathfrak{q}^{(i)} = \mathfrak{q}^i R_{\mathfrak{q}} \cap R$, i.e., $\mathfrak{q}^{(i)} = \{(R-\mathfrak{q})^{-1}\mathfrak{q}^i\} \cap R$ $(i = 1,2,\ldots)$, then it is clear that $\mathfrak{q}^{(i)}$ are ideals of R and $\mathfrak{q}^{(1)} = \mathfrak{q} \supset \mathfrak{q}^{(2)} \supset \mathfrak{q}^{(3)} \supset \ldots$

Put $\mathfrak{a}_i = \mathfrak{q}^{(i)}+Ra$. Since R/Ra satisfies the minimal condition, there exists $n \in \mathbb{N}$ such that $\mathfrak{a}_i = \mathfrak{a}_n$ for any $i \geq n$. Let $\mathfrak{q}^{(n)}:Ra = \{x \mid x \in R \text{ and } xRa \subset \mathfrak{q}^{(n)}\}$. Then we have $\mathfrak{q}^{(n)}:Ra = \mathfrak{q}^{(n)}$, because $a \notin \mathfrak{q}$ by the minimality of $\mathfrak{p}$.

Now we shall show that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(i)}$ for all $i \geq n$. Notice that $R/\mathfrak{q}^{(i)}$ is a Noetherian local ring with maximal ideal $\mathfrak{p}/\mathfrak{q}^{(i)}$. Since $\mathfrak{q}^{(n)}/\mathfrak{q}^{(i)} \subset Ra+\mathfrak{q}^{(i)}/\mathfrak{q}^{(i)}$ and

$$\mathfrak{q}^{(n)}/\mathfrak{q}^{(i)} = \mathfrak{q}^{(n)}/\mathfrak{q}^{(i)} : Ra+\mathfrak{q}^{(i)}/\mathfrak{q}^{(i)}$$
$$= \{x+\mathfrak{q}^{(i)} \in R/\mathfrak{q}^{(i)} \mid x(Ra+\mathfrak{q}^{(i)}/\mathfrak{q}^{(i)}) \subset \mathfrak{q}^{(n)}/\mathfrak{q}^{(i)}\},$$

for any $x+\mathfrak{q}^{(i)} \in \mathfrak{q}^{(n)}/\mathfrak{q}^{(i)}$ we have

$$x+\mathfrak{q}^{(i)} = ra+\mathfrak{q}^{(i)} \quad (r \in R)$$

and
$$r+\mathfrak{q}^{(i)} \in \mathfrak{q}^{(n)}/\mathfrak{q}^{(i)} .$$

Hence
$$(a+\mathfrak{q}^{(i)})(\mathfrak{q}^{(n)}/\mathfrak{q}^{(i)}) = (\mathfrak{q}^{(n)}/\mathfrak{q}^{(i)}) .$$

Thus $\mathfrak{q}^{(n)}/\mathfrak{q}^{(i)} = \{0\}$, i.e., $\mathfrak{q}^{(n)} = \mathfrak{q}^{(i)}$ from the Nakayama's Lemma. Therefore, we have $\bigcap_{i=1}^{\infty} \mathfrak{q}^{(i)} = \mathfrak{q}^{(n)}$.

From the definition we have

$$\mathfrak{q}^{(i)} = \mathfrak{q}^{i}R_{\mathfrak{q}} \cap R = \{(R-\mathfrak{q})^{-1}\mathfrak{q}^{i}\} \cap R .$$

Since $R_{\mathfrak{q}}$ is a Noetherian local ring with maximal ideal $\mathfrak{q}R_{\mathfrak{q}}$,

$$\bigcap_{i=1}^{\infty} (\mathfrak{q}R_{\mathfrak{q}})^{i} = \{0\}$$

from Corollary 6.30. Since

$$(R-\mathfrak{q})^{-1}\mathfrak{q}^{i} = \{(R-\mathfrak{q})^{-1}\mathfrak{q}\}^{i} = (\mathfrak{q}R_{\mathfrak{q}})^{i} ,$$

we have $\bigcap_{i=1}^{\infty} \mathfrak{q}^{(i)} = \{0\}$, which implies $\mathfrak{q}^{(n)} = \{0\}$. Hence $\mathfrak{q} = \{0\}$. Thus we have shown that $\mathfrak{p}$ contains no prime ideal except $\{0\}$ and $\mathfrak{p}$. Q.E.D.

We supplement the following lemma.

<u>(7.15) Lemma</u>. Let R be a Noetherian local ring with unique maximal ideal M. Then M is generated by its elements $\{f_1, f_2, \ldots, f_n\}$ as R-module if and only if M/M^2 is generated by $\{f_1+M^2, \ldots, f_n+M^2\}$ as R/M-module.

Especially $(f_1, f_2, \ldots, f_n)$ is a minimal set of generators of R-modules M, i.e., any proper subset of $(f_1, f_2, \ldots, f_n)$ does not generate M if and only if $(f_1+M^2, \ldots, f_n+M^2)$ forms a R/M-basis for M/M^2.

Proof. Assume that M/M^2 is generated by $(f_1+M^2, \ldots, f_n+M^2)$ as R/M-module. Let $N = Rf_1+\ldots+Rf_n$, then we have

$$M(M/N) = M/N .$$

From Nakayama's Lemma, $M/N = (0)$, i.e., $M = N$. Q.E.D.

(7.16) Definition. Let R be a commutative ring with unity element 1. For a prime ideal $\mathfrak{p}$ of R we define the height of $\mathfrak{p}$ to be the maximal of length of descending chains of prime ideals which begin with $\mathfrak{p}$. The length of a chain is defined to be one less than the number of terms of the chain.

(7.17) Theorem (Krull). Let R be a commutative Noetherian ring with unity element 1. We assume that R is an integral domain. Let $\mathfrak{a}$ be a non-zero proper ideal of R generated by r elements, i.e., $\exists\ a_1, a_2, \ldots, a_r \in \mathfrak{a}$ such that

$$\mathfrak{a} = Ra_1+Ra_2+\ldots+Ra_r .$$

Let $\mathfrak{p}$ be a prime ideal of R which is minimal among prime ideals of R containing $\mathfrak{a}$, then the height $\mathfrak{p}$ is not greater than r, i.e.,

$$\text{height } \mathfrak{p} \leq r .$$

Proof. We follow the proof of Nagata [1, (9.3)]. Assume that $r = 1$, i.e., $\mathfrak{a} = Ra_1$, then from Lemma 7.14 we have height $\mathfrak{p} = 1 \leq r$.

Next assume that $r > 1$. Let $\mathfrak{p} = \mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \ldots \supsetneqq \mathfrak{p}_s$ be a chain of prime ideals $\mathfrak{p}_i$ of R. It is enough to show that

$$s \leq r .$$

Since R is Noetherian, we can assume that there is no prime ideal between $\mathfrak{p}_0$ and $\mathfrak{p}_1$. Considering $R_{\mathfrak{p}} = (R-\mathfrak{p})^{-1}R$ instead of R, we may assume that $\mathfrak{p}$ is the unique maximal ideal of R (see the proof of Lemma 7.14).

We can assume that $a_1 \notin \mathfrak{p}_1$, because $\mathfrak{a} \not\subset \mathfrak{p}_1$. Then there is no prime ideal except $\mathfrak{p}$ which contains $\mathfrak{p}_1+Ra_1$. Hence $\sqrt{\mathfrak{p}_1+Ra_1} = \mathfrak{p}$ from Exercise 19 on p.45. Since $a_i \in \mathfrak{p}$ $(i = 1,2,\dots,r)$, we have $a_i^{t_i} \in \mathfrak{p}_1+Ra_1$ for some $t_i \in \mathbb{N}$ $(i = 1,2,\dots,r)$. Thus there exists a natural number $t \in \mathbb{N}$ such that

$$a_i^t \in \mathfrak{p}_1+Ra_1$$

for all $1 \leq i \leq r$. We write $a_i^t = a_1b_i+c_i$ with $b_i \in R$ and $c_i \in \mathfrak{p}_1$. Let

$$\mathfrak{a}' = Rc_2+Rc_3+\dots+Rc_r .$$

Let $\mathfrak{p}'$ be a minimal prime ideal between $\mathfrak{p}_1$ and $\mathfrak{a}'$. Since

$$a_i \in \sqrt{\mathfrak{a}'+Ra_1}$$

for all $1 \leq i \leq r$, $\sqrt{\mathfrak{a}'+Ra_1} \supset \mathfrak{a}$. Since $\mathfrak{p}_1+Ra_1 \supset \mathfrak{a}'+Ra_1$, we have

$$\mathfrak{p} = \sqrt{\mathfrak{p}_1+Ra_1} \supset \sqrt{\mathfrak{a}'+Ra_1} \supset \mathfrak{a} .$$

Thus $\sqrt{\mathfrak{a}'+Ra_1} = \sqrt{\sqrt{\mathfrak{a}'+Ra_1}} = \mathfrak{p}$, because any prime ideal of R is contained in the unique maximal ideal $\mathfrak{p}$, which is minimal among prime ideals of R containing $\mathfrak{a}$. Hence we have

$$\mathfrak{p} = \sqrt{\mathfrak{p}_1+Ra_1} \supset \sqrt{\mathfrak{p}'+Ra_1} \supset \sqrt{\mathfrak{a}'+Ra_1} = \mathfrak{p} ,$$

because $\mathfrak{p}_1+Ra_1 \supset \mathfrak{p}'+Ra_1 \supset \mathfrak{a}'+Ra_1$. Therefore in the ring $R/\mathfrak{p}'$, which is Noetherian local ring with maximal ideal $\mathfrak{p}/\mathfrak{p}'$, $\mathfrak{p}/\mathfrak{p}'$ is a prime ideal which is minimal among prime ideals of $R/\mathfrak{p}'$ containing $Ra_1+\mathfrak{p}'/\mathfrak{p}'$. Hence from Lemma 7.14 we have

$$\mathfrak{p}_1/\mathfrak{p}' = (0) , \text{ i.e., } \mathfrak{p}_1 = \mathfrak{p}' ,$$

because $\mathfrak{p}/\mathfrak{p}' \supsetneqq \mathfrak{p}_1/\mathfrak{p}'$. Thus $\mathfrak{p}_1$ is a prime ideal which is minimal among prime ideals of R containing $\mathfrak{a}' = Rc_2+Rc_3+\dots+Rc_r$. Therefore from the induction we have

$$\text{height } \mathfrak{p}_1 \leq r-1 .$$

Hence $s \leq r$. Q.E.D.

Finally we shall prove the main theorem of this section.

<u>(7.18) Theorem</u>. Let (V,A) be an irreducible affine variety over K . Then

(1) $\dim_K T(V)_v \geq \dim V$ for all $v \in V$.

(2) Let $S = \{v \in V \mid \dim_K T(V)_v = \dim V\}$, then S is a non-empty open subset of V (hence S is dense).

<u>Proof</u>. We write $A = K[x_1,x_2,\ldots,x_n]$ according to the Noether Normalization Theorem 2. Let

$$\{0\} \to \operatorname{Ker}\theta \to K[X_1,X_2,\ldots,X_n] \xrightarrow{\theta} A \to \{0\}$$
$$X_i \longrightarrow x_i$$

be the same exact sequence defined in the proof of Theorem 2.3, then there exists a morphism

$$\varphi : V \to K^n$$

such that $\theta = \varphi^*$ and V is isomorphic to the subvariety $\varphi(V)$ of K^n . Thus we have

$$\mathcal{I}(\varphi(V)) = \operatorname{Ker}\theta$$

and $$K[\varphi(V)] = K[X_1,X_2,\ldots,X_n]/\mathcal{I}(\varphi(V)) \cong A .$$

Let $F_i(X_1,\ldots,X_d,X_i)$ be a monic minimal polynomial of x_i over $K(x_1,x_2,\ldots,x_n)$ obtained in the Noether Normalization Theorem 2, where $i > d$, then each $F_i(X_1,\ldots,X_d,X_i) \in \mathcal{I}(\varphi(V))$. Hence we can choose a set of generators of $\mathcal{I}(\varphi(V))$ containing $\{F_i \mid i > d\}$, i.e., $\{F_{d+1},\ldots,F_n\} \cup \{P_1,P_2,\ldots,P_l\}$.

(1) We follow the proof of Hartshorne [1, Theorem 5.3 on P.33]. Let $P = \varphi(v)$, where $v \in V$ and assume that

$$P = (a_1,a_2,\ldots,a_n) \in K^n .$$

Let
$$\upsilon : K[X_1, X_2, \ldots, X_n] \longrightarrow K$$
$$(\upsilon : X_i \longrightarrow a_i)$$
be a K-algebra map which takes X_i to a_i for each $1 \leq i \leq n$, then we have
$$\operatorname{Ker} \upsilon = \langle X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n \rangle .$$
Put $\mathfrak{a}_P = \operatorname{Ker} \upsilon$. Let $\tilde{\theta}$ be a linear map from $K[X_1, X_2, \ldots, X_n]$ into K^n such that
$$\tilde{\theta} : K[X_1, \ldots, X_n] \longrightarrow K^n$$
$$(\tilde{\theta} : f \longrightarrow (\frac{\delta f}{\delta X_1}(P), \ldots, \frac{\delta f}{\delta X_n}(P))) .$$

It is easy to check that
$$\tilde{\theta}(X_i - a_i) = (0, \ldots, 0, \overset{\overset{i}{\downarrow}}{1}, 0, \ldots, 0)$$
for any $1 \leq i \leq n$ and
$$\tilde{\theta}(\mathfrak{a}_P^2) = 0 .$$
Thus we can define a K-isomorphism $\tilde{\theta}'$ from $\mathfrak{a}_P / \mathfrak{a}_P^2$ onto K^n such that
$$\tilde{\theta}' : \mathfrak{a}_P / \mathfrak{a}_P^2 \longrightarrow K^n$$
$$(\tilde{\theta}' : f + \mathfrak{a}_P^2 \longrightarrow \tilde{\theta}(f)) .$$

Now let
$$J = \begin{bmatrix} \frac{\delta F_{d+1}}{\delta X_1} & , \ldots\ldots\ldots\ldots , & \frac{\delta F_{d+1}}{\delta X_n} \\ \vdots & & \\ \frac{\delta F_n}{\delta X_1} & , \ldots\ldots\ldots\ldots , & \frac{\delta F_n}{\delta X_n} \\ \frac{\delta P_1}{\delta X_1} & , \ldots\ldots\ldots\ldots , & \frac{\delta P_1}{\delta X_n} \\ \vdots & & \\ \frac{\delta P_1}{\delta X_1} & , \ldots\ldots\ldots\ldots , & \frac{\delta P_1}{\delta X_n} \end{bmatrix} .$$

Then $\dim_K T(\varphi(V))_P = n - \operatorname{rank} J(P)$ (see Proposition 4.10).

It is clear that $\operatorname{rank} J(P) = \dim_K \tilde{\theta} \mathcal{I}(\varphi(V))$. Since $\mathcal{I}(\varphi(V)) \subset \mathfrak{a}_P$, we have

$$\tilde{\theta}'(\mathcal{I}(\varphi(V))+a_P^2/a_P^2) = \tilde{\theta}(\mathcal{I}(\varphi(V)) .$$

Thus $$\text{rank } J(P) = \dim_K(\mathcal{I}(\varphi(V))+a_P^2/a_P^2) .$$

Since $K[\varphi(V)] = K[K^n]/\mathcal{I}(\varphi(V)) \supset a_P/\mathcal{I}(\varphi(V))$, the local ring $\mathcal{O}_P$ of P on $\varphi(V)$ is from the definition

$$K[\varphi(V)]_{(a_P/\mathcal{I}(\varphi(V)))} .$$

Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}_P$, i.e.,

$$\mathcal{M} = (a_P/\mathcal{I}(\varphi(V)))K[\varphi(V)]_{(a_P/\mathcal{I}(\varphi(V)))} ,$$

then from Lemma 7.12 we have

$$\mathcal{M}/\mathcal{M}^2 \cong \{a_P/\mathcal{I}(\varphi(V))\}/\{a_P^2+\mathcal{I}(\varphi(V))/\mathcal{I}(\varphi(V))\} \cong a_P/a_P^2+\mathcal{I}(\varphi(V))$$

as K-spaces. Hence $\dim_K\mathcal{M}/\mathcal{M}^2+\text{rank } J(P) = n$.

Now let $t_1,t_2,\ldots,t_l$ be elements of $\mathcal{M}$ whose images in $\mathcal{M}/\mathcal{M}^2$ form a K-basis of this vector space, then from Lemma 7.15 $\mathcal{M}$ is generated by $t_1,t_2,\ldots,t_l$ as $\mathcal{O}_P$-ideal. From Theorem 7.17 we have

$$\text{height } \mathcal{M} \leq l = \dim_K\mathcal{M}/\mathcal{M}^2 .$$

Since height $\mathcal{I}(\{P\}) \leq$ height $\mathcal{M}$ and
tr.deg_K {the quotient field of $K[\varphi(V)]$} = dim $\varphi(V)$ = height $\mathcal{I}(\{P\})$
from Proposition 7.10, we have shown that
$n\text{-rank } J(P) = \dim_K T(\varphi(V))_P \geq \dim \varphi(V)$ where $P = \varphi(v)$.

(2) We follow the proof of Steinberg [2, Proposition 3 on P.60]. Since $\dim_K T(\varphi(V))_{\varphi(v)} = d$ $(= \dim\varphi(V))$ if and only if rank $J(P) = n-d$, we have $\dim_K T(\varphi(V))_{\varphi(v)} = d$ if and only if some (n-d)th order minor of $J(P)$ is non-zero. Hence

$$S' = \{P \in \varphi(V) \mid \dim_K T(\varphi(V))_P = \dim \varphi(V)\}$$

forms an open set of $\varphi(V)$. Since

$$J(P) = \begin{bmatrix} \frac{\delta F_{d+1}}{\delta X_1}, \ldots, & \frac{\delta F_{d+1}}{\delta X_d}, & \frac{\delta F_{d+1}}{\delta X_{d+1}}, & 0, \ldots\ldots, & 0 \\ \frac{\delta F_{d+2}}{\delta X_1}, \ldots, & \frac{\delta F_{d+2}}{\delta X_d}, & 0, & \frac{\delta F_{d+2}}{\delta X_{d+2}}, & 0, \ldots, 0 \\ \vdots & \vdots & 0, & \ddots & \vdots \\ \frac{\delta F_n}{\delta X_1}, \ldots, & \frac{\delta F_n}{\delta X_d}, & 0, \ldots & \ldots 0, & \frac{\delta F_n}{\delta X_n} \\ \frac{\delta P_1}{\delta X_1}, \ldots & & \ldots & \ldots, & \frac{\delta P_1}{\delta X_n} \\ \vdots & & & & \vdots \\ \frac{\delta P_1}{\delta X_1}, \ldots & & \ldots & \ldots, & \frac{\delta P_1}{\delta X_n} \end{bmatrix} (P)$$

and x_i is separably algebraic over $K(x_1,\ldots,x_d)$ with monic minimal polynomial F_i , for each $i > d$, certainly the minor

$$\det \begin{bmatrix} \frac{\delta F_{d+1}}{\delta X_{d+1}}, & 0, \ldots\ldots, & 0 \\ 0, & \frac{\delta F_{d+2}}{\delta X_{d+2}}, \; 0, \ldots, & 0 \\ \vdots & \ddots & \vdots \\ 0, \ldots\ldots, & 0, & \frac{\delta F_n}{\delta X_n} \end{bmatrix} = \prod_{i>d} \left[\frac{\delta F_i}{\delta X_i}\right] \neq 0 .$$

Thus $S' \neq \emptyset$. Q.E.D.

<u>(7.19) Definition</u>. Let (V,A) be an irreducible affine variety over K . A point $v \in V$ is said to be <u>simple</u> or <u>non-singular</u> if $\dim_K T(V)_v = \dim V$. If all points of V are simple, we call V <u>smooth</u> or <u>non-singular</u>.

8. Constructible sets

In this section we shall define a constructible set of a topological space, the image of which under an affine variety morphism is also constructible. From this fact we shall prove that the image of an algebraic group is also an algebraic group (see §15).

(8.1) Definition. A subset of a topological space is said to be locally closed if it is open in its closure. A constructible set is a finite union of locally closed sets.

Exercise 28. (1) Let L be a subset of a topological space X. Then L is locally closed if and only if L is the intersection of an open set with a closed set.

(2) Let C and C' are constructible sets of a topological space X, then $C \cup C'$ and $C \cap C'$ are also constructible sets.

Exercise 29. Let C be a constructible set of a topological space X. If C is a union of finite irreducible sets, then C contains an open dense subset U of $\overline{C}$.

(8.2) Definition. Let (U,A) and (V,B) be irreducible affine varieties over K. A morphism

$$\varphi : U \to V$$

is called

(1) dominant if $\varphi(U)$ is dense in V; and

(2) finite if A is integral over $\varphi^*(B)$.

(8.3) Lemma. Let (U,A) and (V,B) be irreducible affine varieties over K. A morphism $\varphi : U \to V$ is dominant if and only if $\varphi^* : B \to A$ is injective.

Proof. Since $\operatorname{Ker} \varphi^* = \{f \in B \mid f \circ \varphi(U) = 0\} = \mathcal{I}(\varphi(U)) = \mathcal{I}(\overline{\varphi(U)})$

and $\mathscr{V}(\operatorname{Ker} \varphi^*) = \overline{\varphi(U)}$, φ is dominant if and only if $\operatorname{Ker} \varphi^* = (0)$. Q.E.D.

<u>(8.4) Proposition</u>. Let (U,A) and (V,B) be irreducible affine varieties over K . Assume that $\varphi:U \to V$ is a finite morphism. Then

(1) If Z is a closed irreducible subset of U , then $\varphi_{|Z}:Z \to V$ is also a finite morphism.

(2) For any closed subset Z of U , $\varphi(Z)$ is closed in V , in particular if φ is dominant, then

$$\varphi(U) = V .$$

<u>Proof</u>. (1) Let $\iota:Z \to U$ be the inlcusion map and $\pi = \iota^*$. Let $\mathfrak{p} = \mathscr{I}_A(Z)$, then $\pi(A) \cong A/\mathfrak{p}$. Since A is integral over $\varphi^*(B)$, $A/\mathfrak{p}$ is integral over $\varphi^*(B)+\mathfrak{p}/\mathfrak{p}$. Thus $\varphi_{|Z}:Z \to V$ is finite, because

$$\begin{aligned}(\varphi_{|Z})^*(B) &= \{g_{|Z} \mid g \in \varphi^*(B)\} \\ &\cong \varphi^*(B)/\varphi^*(B) \cap \mathfrak{p} \\ &\cong \varphi^*(B)+\mathfrak{p}/\mathfrak{p} .\end{aligned}$$

(2) If Z is a closed subset of U with irreducible components $Z_1, Z_2, \ldots, Z_n$, then

$$\varphi(Z) = \varphi(Z_1) \cup \ldots \cup \varphi(Z_n) .$$

Thus we may assume that Z is irreducible. Notice

$$\varphi_{|Z_i}:Z_i \to V$$

is a finite morphism for any $1 \leq i \leq n$ from (1). Hence we may also assume that $Z = U$. We also can assume $\overline{\varphi(U)} = V$, i.e., φ is dominant.

Let $v \in V$ be any element of V and $\epsilon_v:B \to K$ be the evaluation at v . Since φ^* is injective from Lemma 8.3, we can define

$$\overline{\epsilon_v}: \varphi^*(B) \to K$$

by $\overline{\epsilon_v}(\varphi^*(b)) = b(v)$. Since A is integral over $\varphi^*(B)$, $\overline{\epsilon_v}$ extends to a K-algebra homomorphism $\theta:A \to K$ from Proposition 6.7.

Since $(U,A) \in \mathscr{A}(K)$, we have $\theta = \epsilon_u$ for some $u \in U$, which gives

$$\epsilon_u(\varphi^*(b)) = \overline{\epsilon_v}(\varphi^*(b)) \text{ , i.e., } b(\varphi(u)) = b(v)$$

for any $b \in B$. Thus we have $\epsilon_{\varphi(u)} = \epsilon_v$. Hence $\varphi(u) = v$ and φ is surjective and $\varphi(U)$ is closed. Q.E.D.

<u>Corollary to Proposition 8.4</u>. Let (U,A) and (V,B) be irreducible affine varieties over K . Assume that

$$\varphi : U \to V$$

is a surjective finite morphism and O is an open subset of U such that $O = \varphi^{-1}(W)$ for some subset W of V , then $\varphi(O)$ is open in V .

<u>Proof</u>. Since $O = \{u \in U \mid \varphi(u) \in W\}$ and $U-O = \{u \in U \mid \varphi(u) \notin W\} = \varphi^{-1}(V-W)$ and $\varphi(U-O) = V-W$ is closed in V , $\varphi(O) = W$ is open in V . Q.E.D.

<u>(8.5) Proposition</u>. Let (U,A) and (V,B) be irreducible affine varieties over K . Assume that $\varphi : U \to V$ is a dominant morphism, then $\varphi(U)$ contains a non-empty open subset of V .

<u>Proof</u>. Let $\tilde{B} = \varphi^*(B)$ and E be the quotient field of A and F be the quotient field of $\tilde{B}$. Notice $F \subset E$. Let

$$A' = \{a/b \mid a \in A , \ b \in \tilde{B}-\{0\}\} \quad (\subset E) ,$$

then A' is a finitely generated F-algebra. From the Noether Normalization Theorem 1 there exist elements $x_1, x_2, \ldots, x_r \in A'$ such that A' is integral over $F[x_1, x_2, \ldots, x_r]$ and $\{x_1, x_2, \ldots, x_r\}$ is algebraically independent over F . Then we have

$$\begin{aligned} r &= \text{tr.deg}_F \text{ (the quotient field of } A' \text{)} \\ &= \text{tr.deg}_F E \\ &= \text{tr.deg}_K E - \text{tr.deg}_K F \\ &= \dim U - \dim V \text{ , because } B \cong \tilde{B} \text{ .} \end{aligned}$$

Since $x_i = x_i'/b_i$ for some $x_i' \in A$ and $b_i \in \tilde{B}-\{0\}$, where $1 \leq i \leq r$, and

$$F[x_1, x_2, \ldots, x_r] = F[x_1', x_2', \ldots, x_r'] ,$$

we can assume that each x_i $(1 \leq i \leq r)$ is an element of A .

Let $\{a_1, a_2, \ldots, a_m\}$ be a set of generators of A as K-algebra. Since $a_j \in A'$ $(1 \leq j \leq m)$ is integral over $F[x_1, x_2, \ldots, x_r]$, a_j satisfies

$$a_j^{l(j)} + \frac{c_{j_1}}{d_{j_1}} a_j^{l(j)-1} + \ldots + \frac{c_{j_{l(j)}}}{d_{j_{l(j)}}} = 0 ,$$

where $c_{ji} \in \tilde{B}[x_1, x_2, \ldots, x_r]$ and $d_{ji} \in \tilde{B}-\{0\}$ for each $1 \leq i \leq l(j)$. Let $f_j = d_{j_1} \cdot d_{j_2} \cdots d_{j_{l(j)}}$, then the equation

$$f_j^{l(j)}(a_j^{l(j)} + \frac{c_{j_1}}{d_{j_1}} a_j^{l(j)-1} + \ldots + \frac{c_{j_{l(j)}}}{d_{j_{l(j)}}}) = 0$$

shows that $f_j a_j$ is integral over $\tilde{B}[x_1, x_2, \ldots, x_r]$ for each $1 \leq j \leq m$. Let $f = f_1 f_2 \ldots f_m$ $(\in \tilde{B}-\{0\})$, then $fa_1, fa_2, \ldots, fa_m$ are also integral over $\tilde{B}[x_1, x_2, \ldots, x_r]$. Thus $A_f = \{a/f^s \mid a \in A$ and $s \in \mathbb{N}\}$ $(\subset E)$ is integral over $\tilde{B}[x_1, x_2, \ldots, x_r]_f = \{c/f^s \mid c \in \tilde{B}[x_1, x_2, \ldots, x_r]$ and $s \in \mathbb{N}\}$ $(\subset E)$.

Now let f_o be an element of $B-\{0\}$ such that $\varphi^*(f_o) = f$. We shall show that

$$\varphi(U_f) = V_{f_o} .$$

Let ρ be a map of $\tilde{B}[x_1, x_2, \ldots, x_r]_f$ into $K[V_{f_o} \times K^r]$ $(= B_{f_o} \otimes K[X_1, X_2, \ldots, X_r])$ such that $\rho(\varphi^*(b)/\varphi^*(f_o)^l)$ takes $(y, (\lambda_1, \ldots, \lambda_r))$ to $b(y)/f_o(y)^l$ for $b \in B$ and $l \in \mathbb{N}$ and $\rho(x_i)$ takes $(y, (\lambda_1, \ldots, \lambda_r))$ to λ_i for $1 \leq i \leq r$, where $y \in V_{f_o}$ and $(\lambda_1, \ldots, \lambda_r) \in K^r$, then ρ is a well-defined bijective K-algebra homomorphism.

$$\rho : \tilde{B}[x_1, \ldots, x_n]_f \longrightarrow K[V_{f_o} \times K^r]$$
$$\frac{\varphi^*(b)}{\varphi^*(f_o)^l} \longrightarrow [\rho(\frac{\varphi^*(b)}{\varphi^*(f_o)^l}) : (y, (\lambda_1, \ldots, \lambda_r)) \longrightarrow \frac{b(y)}{f_o(y)^l}]$$
$$x_i \longrightarrow [\rho(x_i) : (y, (\lambda_1, \ldots, \lambda_r)) \longrightarrow \lambda_i] .$$

Since $\tilde{B}[x_1, \ldots, x_r]_f \subset A_f$, from Lemma 2.4 there exists a unique morphism

$$\psi : U_f \longrightarrow V_{f_0} \times K^r$$

such that $\qquad \psi^* : \tilde{B}[x_1,\ldots,x_r]_f \hookrightarrow A_f$.

Since A_f is integral over $\tilde{B}[x_1,\ldots,x_r]_f$, ψ is finite. From Proposition 8.4 $\psi(U_f)$ is closed in $V_{f_0} \times K^r$. Since $K[\psi(U_f)] \cong \tilde{B}[x_1,\ldots,x_r]_f$, we have $\dim \psi(U_f) = \dim(V_{f_0} \times K^r)$. Hence from Proposition 7.2 ψ is surjective.

Now let $\varphi|U_f : U_f \to V_{f_0}$ be the restriction of φ to U_f , then $\varphi|U_f$ is a morphism and satisfies $\varphi|U_f = \mathrm{Pr} \circ \psi$, i.e.,

$$\begin{array}{ccc} U_f & \xrightarrow{\psi} & V_{f_0} \times K^r , \\ & {\scriptstyle \varphi|U_f} \searrow \quad \circlearrowleft \quad \swarrow {\scriptstyle P_r} & \\ & V_{f_0} & \end{array}$$

where $P_r : V_{f_0} \times K^r \to V_{f_0}$ is the projection, because

$$(\varphi|U_f)^* = (P_r \circ \psi)^* .$$

Since ψ and P_r are surjective, $\varphi|U_f$ is also surjective. Therefore, we have $V_{f_0} = \varphi(U_f)$ as required. Hence $\varphi(U)$ contains a non-empty open subset V_{f_0} of V . Q.E.D.

<u>(8.6) Theorem</u>. Let (U,A) and (V,B) be affine varieties and $\varphi : U \to V$ be a morphism, then φ maps constructible sets to constructible sets.

<u>Proof</u>. We follow the induction on $\dim U$. Assume that $\dim U = 0$, then U is a finite set (see Exercise 23 on P.66). Since a set of finite points is closed in an affine variety, a finite set in V is always constructible.

Now assume that $\dim U > 0$ and S is a constructible set in U . Since S is a union of locally closed sets $S_1, S_2, \ldots, S_l$ $(\subset U)$, we have

$$S = S_1 \cup S_2 \cup \ldots \cup S_l$$

and $\qquad \varphi(S) = \varphi(S_1) \cup \varphi(S_2) \cup \ldots \cup \varphi(S_l)$.

Thus we can assume that S is locally closed (see Exercise 28 on P.88). Hence S is open in its closure. We can also assume that $\overline{S} = U$.

Let $U_1, U_2, \ldots, U_n$ be irreducible components of U , then we have

$$\varphi(S) = \varphi(S \cap U_1) \cup \ldots \cup \varphi(S \cap U_n) ,$$

where each $S \cap U_i$ is open in U_i for every $1 \leq i \leq n$. Hence we can further assume U is irreducible. Since any non-empty open set in an affine variety is a finite union of principal open sets (see Remark 2.9), S is a union of finite principal open sets $U_{f_1}, U_{f_2}, \ldots, U_{f_m}$ for some $f_1, f_2, \ldots, f_m \in A-(0)$.

$$S = U_{f_1} \cup U_{f_2} \cup \ldots \cup U_{f_m} .$$

Therefore, we can assume now that $S = U_f$ for some $f \in A-(0)$ and show $\varphi(U_f)$ is constructible. Since (U_f, A_f) is an affine variety from Proposition 2.8, restricting φ to U_f we may assume $U_f = U$. Thus it is enough to show that $\varphi(U)$ is a constructible set.

Finally we can also assume that φ is dominant, because $\varphi(U)$ is a constructible subset of $\overline{\varphi(U)}$ if and only if $\varphi(U)$ is a constructible subset of V . Let O be a non-empty open set of V such that $O \subset \varphi(U)$ (see Proposition 8.5). Let $Z = V-O$ and $W = \varphi^{-1}(Z)$. Assume that W' is an irreducible component of W , then $W' \subsetneq U$, because $\varphi(W') \neq \varphi(U)$. Thus we have $\dim W' < \dim U$ and $\dim W < \dim U$. By induction $\varphi(W)$ is constructible. Since

$$U = \varphi^{-1}(O) \cup \varphi^{-1}(Z) = \varphi^{-1}(O) \cup W ,$$

we have $\varphi(U) = O \cup \varphi(W)$. Hence $\varphi(U)$ is contructible, because O is open and $\varphi(W)$ is constructible. Q.E.D.

II. Varieties

In this chapter we introduce the notion of variety, which is a generalization of the notion of affine and projective varieties. The idea of variety is necessary for defining the homogeneous space G/H of an affine algebraic group G by its closed subgroup H .

9. Sheaves of functions

(9.1) Definition. A sheaf of functions over K on a non-empty topological space X is a function $\mathcal{S}$ which assigns to each open set $U \subset X$ a K-algebra $\mathcal{S}(U)$ consisting of K-valued functions on U such that

(0) $\mathcal{S}(\emptyset) = (0)$;

(1) if $U \subset V$ are two open sets, then $f|_U \in \mathcal{S}(U)$ for any $f \in \mathcal{S}(V)$;

(2) let U be an open set covered by open subsets U_i (i running over some index set I), i.e.,
$$U = \bigcup_{i \in I} U_i ,$$
then a function f of U into K belongs to $\mathcal{S}(U)$ if and only if $f|_{U_i} \in \mathcal{S}(U_i)$ for any $i \in I$.

We call a pair $(X,\mathcal{S})$ of a topological space and a sheaf of functions a ringed space over K .

Remark to Definition 9.1. $\mathcal{S}(U)$ is a K-subalgebra of $M(U,K)$ (see §1).

(9.2) Definition. Let $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ be ringed spaces over K . We define a morphism φ of $(X,\mathcal{S}_X)$ into $(Y,\mathcal{S}_Y)$ to be a continuous map of X into Y such that for any open set O in Y and function $f \in \mathcal{S}_Y(O)$ $f\circ(\varphi|_{\varphi^{-1}(O)})$ belongs to $\mathcal{S}_X(\varphi^{-1}(O))$.

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & & \\ \cup & & \cup & & \\ \varphi^{-1}(O) & \xrightarrow{\varphi} & O & \xrightarrow{f} & K , \quad f \in \mathcal{S}_Y(O) . \end{array}$$

When φ is homeomorphic and φ^{-1} is also a morphism of $(Y,\mathcal{S}_Y)$ into $(X,\mathcal{S}_X)$, we call φ an isomorphism of ringed spaces.

Exercise 30. Let $(X,\mathcal{S}_X)$, $(Y,\mathcal{S}_Y)$ and $(Z,\mathcal{S}_Z)$ be ringed spaces over K . Show that

(1) if $\varphi:(X,\mathcal{S}_X) \to (Y,\mathcal{S}_Y)$ and

$\psi:(Y,\mathcal{S}_Y) \to (Z,\mathcal{S}_Z)$ are morphisms of ringed spaces, then

$\psi\circ\varphi:(X,\mathcal{S}_X) \to (Z,\mathcal{S}_Z)$ is also a morphism of ringed spaces;

(2) the maps $1_X:X \to X$ $(1_X:x \to x)$ and $\theta:X \to Y$ $(\theta:x \to y_0)$ where y_o is any fixed element of Y , then 1_X and θ are morphisms of ringed spaces;

(3) if $\varphi:(X,\mathcal{S}_X) \to (Y,\mathcal{S}_Y)$ is an isomorphism of ringed spaces, then $\mathcal{S}_Y(\varphi(O)) \cong \mathcal{S}_X(O)$ $(f \longrightarrow f\circ(\varphi|O))$ as K-algebras for any open set O in X .

Now let V be an affine variety over K with coordinate ring K[V] . We define a sheaf of functions $\mathcal{O}_V$ on V .

(9.3) Definition. Let $(V,K[V]) \in \mathcal{A}(K)$. Let $v \in V$ and N_v be a neighbourhood of v , i.e., N_v is a subset of V which contains an open set O such that $v \in O \subset N_v$. A K-valued function f defined on N_v is said to be regular at v if there exists an open neighbourhood U_v $(\subset N_v)$ of v and $g,h \in K[V]$ such that $g(y) \neq 0$ and

$$f(y) = \frac{h(y)}{g(y)}$$

for all $y \in U_v$. A K-valued function f defined on a non-empty open set U of V is said to be regular if it is regular at all points of U .

(9.4) Proposition. Let $(V,K[V]) \in \mathcal{A}(K)$. Let $\mathcal{O}_V(U)$ be the set of all regular functions defined on U , where U is a non-empty open set of V , then $\mathcal{O}_V(U)$ is a K-subalgebra of M(U,K) and $\mathcal{O}_V$ defines a sheaf of functions on V .

Proof. Let $f,g \in \mathcal{O}_V(U)$ and $v \in U$, then there exist open neighbourhoods U_v and U'_v in U such that

$$f = \frac{a}{s} \quad \text{on } U_v \quad \text{for some } a,s \in K[V]$$

and

$$g = \frac{a'}{s'} \quad \text{on } U'_v \quad \text{for some } a',s' \in K[V] .$$

On $U_v \cap U'_v$ we have

$$(f-g)(y) = \left(\frac{s'a - a's}{ss'}\right)(y)$$

and

$$(fg)(y) = \left(\frac{aa'}{ss'}\right)(y) \qquad (y \in U_v \cap U'_v) .$$

Hence $f-g$, $fg \in \mathcal{O}_V(U)$. It is clear that $cf \in \mathcal{O}_V(U)$ for any $c \in K$. Thus $\mathcal{O}_V(U)$ is a K-subalgebra of $M(U,K)$. Let $U' \subset U$ be two open sets in V. Let $f \in \mathcal{O}_V(U)$, then it is clear that $f|_{U'}$ is regular at each point v in U'. Hence $f|_{U'} \in \mathcal{O}_V(U')$. Now let U be an open set of V covered by open subsets U_i $(i \in I)$, i.e., $U = \bigcup_{i \in I} U_i$. Let f be a function of U into K such that $f|_{U_i} \in \mathcal{O}_V(U_i)$ for $\forall i \in I$. Assume that $v \in U$, then there exists $i \in I$ such that $v \in U_i$. Since $f|_{U_i} \in \mathcal{O}_V(U_i)$, there exists an open subset O_v of U_i such that $v \in O_v$ and f is regular on O_v. Thus $f \in \mathcal{O}_V(U)$. Q.E.D.

Next we shall investigate how regular functions relate to the coordinate ring. Let v be an element of an affine variety V over K. Let (U_v,f) be a pair of open neighbourhood U_v of v and a regular function f on U_v, i.e., $f \in \mathcal{O}_V(U_v)$. We denote the set of all pairs such as (U_v,f) at v by $\hat{\mathcal{O}}_v$. We define an equivalence relation $\sim$ on $\hat{\mathcal{O}}_v$ as follows.

$$(U_v,f) \sim (W_v,g) \quad \text{if and only if}$$

$$f|_{T_v} = g|_{T_v}$$

for some open neighbourhood T_v of v in $U_v \cap W_v$, where $(U_v,f),\ (W_v,g) \in \hat{\mathcal{O}}_v$.

<u>(9.5) Proposition</u>. Let $(V, K[V]) \in \mathcal{A}(K)$ and v be an arbitrary point of V .

(1) Let $\hat{\mathcal{O}}_v/\sim$ be the set of all equivalence classes with respect to $\sim$, then $\hat{\mathcal{O}}_v/\sim$ has the K-algebra structure as follows. Let $(\overline{U_v, f})$, $(\overline{W_v, g}) \in \hat{\mathcal{O}}_v/\sim$ be equivalence classes represented by (U_v, f) and (W_v, g) respectively, then $\hat{\mathcal{O}}_v/\sim$ has the following algebra operation.

$$(\overline{U_v, f}) + (\overline{W_v, g}) = (\overline{U_v \cap W_v \, , \, (f+g)|_{U_v \cap W_v}})$$

$$(\overline{U_v, f}) \cdot (\overline{W_v, g}) = (\overline{U_v \cap W_v \, , \, (fg)|_{U_v \cap W_v}})$$

$$c(\overline{U_v, f}) = (\overline{U_v, cf}) \qquad (c \in K) .$$

(2) Let $K[V]_{\mathcal{I}(v)} = (K[V] - \mathcal{I}(v))^{-1}K[V]$, where $\mathcal{I}(v) = \mathcal{I}(\{v\})$. Then there exists a K-algebra isomorphism φ of $\hat{\mathcal{O}}_v/\sim$ onto $K[V]_{\mathcal{I}(v)}$ which takes

$$(\overline{U_v, f}) \in \hat{\mathcal{O}}_v/\sim \quad \text{to} \quad a/s \in K[V]_{\mathcal{I}(v)} ,$$

where $a, s \in K[V]$ and $f(y) = \frac{a(y)}{s(y)}$ $(y \in O_v)$ for some open neighbourhood O_v of v in U_v .

<u>Proof</u>. (1) is clear.

(2) Since $\mathcal{I}(v)$ is a prime ideal of $K[V]$, $K[V]_{\mathcal{I}(v)}$ is well-defined and is a local ring. Assume that $(U_v, f) \sim (U_v', f')$, then there exists an open neighbourhood W_v of v in $U_v \cap U_v'$ such that

$$f|_{W_v} = f'|_{W_v} .$$

Since $f|_{W_v}, f'|_{W_v} \in \mathcal{O}_V(W_v)$, there exists an open neighbourhood O_v of v in W_v such that for some $a, s \in K[V]$

$$f(y) = f'(y) = \frac{a(y)}{s(y)} \quad (y \in O_v) .$$

Now let T_v, T_v' be open neighbourhoods of v in U_v such that for some $a_1, s_1, a_2, s_2 \in K[V]$

$$f(y) = \frac{a_1(y)}{s_1(y)} \quad (y \in T_v)$$

and
$$f(z) = \frac{a_2(z)}{s_2(z)} \quad (z \in T'_v) .$$

Hence $a_1s_2 - a_2s_1 = 0$ on $T_v \cap T'_v$. Next let

$$V = V_1 \cup \ldots \cup V_t$$

be the decomposition of V into its irreducible components $V_1,\ldots,V_t$. We assume that $V_1,\ldots,V_i$ do not contain v but $V_{i+1},\ldots,V_t$ contain v . Since $(T_v \cap T'_v) \cap V_j \neq \emptyset$ $(i+1 \leq j \leq t)$ and

$$V_j \supset \mathcal{V}_{V_j}(K[V](a_1s_2 - a_2s_1)) \supset T_v \cap T'_v \cap V_j ,$$

we have $V_j = \mathcal{V}_{V_j}(K[V](a_1s_2 - a_2s_1))$ for any $i+1 \leq j \leq t$.

Hence $(a_1s_2 - a_2s_1) \mid (V_{i+1} \cup \ldots \cup V_t) = 0$. On the other hand since $v \notin V_1 \cup \ldots \cup V_i$, there exists $q \in \mathcal{I}(V_1 \cup \ldots \cup V_i)$ such that $q(v) \neq 0$, i.e., $q \notin \mathcal{I}(v)$ (see Proposition 1.7). Thus we have $q(a_1s_2 - a_2s_1) = 0$ in $K[V]$, which implies that $a_1/s_1 = a_2/s_2$ in $K[V]_{\mathcal{I}(v)}$. Hence φ is a well-defined map.

It is clear that φ is a K-algebra homomorphism. Assume that $\varphi(\overline{(U_v,f)}) = \frac{a}{s} = 0$, where $(U_v,f) \in \hat{\mathcal{O}}_v$, we have $ga = 0$ in $K[V]$ for some $g \in K[V] - \mathcal{I}(v)$. Let V_g be the principal open set in V defined by g . It is clear that $v \in V_g$. Since

$$f(y) = \frac{a(y)}{s(y)} = \frac{g(y)a(y)}{g(y)s(y)} = 0$$

for any $y \in O_v \cap V_g$, $(U_v,f) \sim (V_g,0)$, i.e., φ is injective. Finally we show that φ is surjective. Let $\frac{h}{g} \in K[V]_{\mathcal{I}(v)}$. Then we have

$$\varphi(\overline{(V_g,\frac{h}{g})}) = \frac{h}{g} ,$$

because $g(v) \neq 0$. Q.E.D.

We shall denote this local ring $\hat{\mathcal{O}}_v/\sim$ by $\mathcal{O}_v$.

Exercise 31. Prove Proposition 9.5.1.

(9.6) Proposition. Let $(V,K[V])$ be an irreducible affine variety over K. Let U be a non-empty open set of V. Then there exists a K-algebra isomorphism ψ of $\mathcal{O}_V(U)$ onto $\bigcap_{x\in U} K[V]_{\mathcal{J}(x)}$ $(\subset K(V)$, the quotient field of $K[V])$ which takes $f \in \mathcal{O}_V(U)$ to $\frac{h}{g} \in K(V)$ where $h,g \in K[V]$ and $f = \frac{h}{g}$ on some open neighbourhood U_x $(\subset U)$ of some point x in U.

Proof. First we show that $\psi(f)$ does not depend on x or U_x. Let x' be an arbitrary point of U and $O_{x'}$ be an open neighbourhood of x' in U such that

$$f(y) = \frac{h'(y)}{g'(y)} \quad (y \in O_{x'})$$

for some $g',h' \in K[V]$. Of course $g'(y) \neq 0$ for any $y \in O_{x'}$. Since V is irreducible, we have $U_x \cap O_{x'} \neq \emptyset$. Since $\frac{h}{g} = \frac{h'}{g'}$ on $U_x \cap O_{x'}$, we have

$$V \supset \mathcal{V}(K[V](hg' - gh')) \supset U_x \cap O_{x'},$$

which implies $hg' - gh' = 0$ on V. Hence $\frac{h}{g} = \frac{h'}{g'}$ in $K(V)$ and we have shown that $\psi(f)$ does not depend on x or U_x. It is clear from the above argument that $\psi(f)$ belongs to $\bigcap_{x\in U} K[V]_{\mathcal{J}(x)}$. One can easily check that ψ is a K-algebra homomorphism.

Now we assume that $\psi(f) = 0$ for some $f \in \mathcal{O}_V(U)$. Let $x \in U$, then there exists an open neighbourhood U_x of x in U such that

$$f(y) = \frac{h(y)}{g(y)}$$

for all $y \in U_x$. Since $\frac{h}{g} = 0$ in $K(V)$, $h = 0$ in $K(V)$. Thus $f = 0$ when $\psi(f) = 0$. Hence ψ is injective.

Next let $\frac{h}{g} \in \bigcap_{x\in U} K[V]_{\mathcal{J}(x)}$, then for each $x \in U$ there exist $g_x, h_x \in K[V]$ such that $g_x \notin \mathcal{J}(x)$ and

$$\frac{h}{g} = \frac{h_x}{g_x}.$$

Since $\frac{h_x}{g_x} = \frac{h_{x'}}{g_{x'}}$ for another point $x' \in U$, we have

$$h_x g_{x'} = g_x h_{x'}$$

on V . Hence $\frac{h_x(y)}{g_x(y)} = \frac{h_{x'}(y)}{g_{x'}(y)}$ for any $y \in V_{g_x} \cap V_{g_{x'}}$. Since

$$\frac{h_x}{g_x} \in \mathcal{O}_V(V_{g_x} \cap U) \quad \text{and} \quad U = \bigcup_{x \in U} (V_{g_x} \cap U) ,$$

there exists $f \in \mathcal{O}_V(U)$ such that

$$f|_{V_{g_x} \cap U} = \frac{h_x}{g_x}$$

(see Proposition 9.4). Hence $\psi(f) = \frac{h_x}{g_x} = \frac{h}{g}$ and ψ is surjective.

Q.E.D.

(9.7) Theorem. Let $(V,K[V]) \in \mathcal{A}(K)$ and $\mathcal{O}_V$ be as in Proposition 9.4. Let φ be a map of $K[V]$ into $\mathcal{O}_V(V)$ which takes $f \in K[V]$ to $f \in \mathcal{O}_V(V)$, then φ is a K-algebra isomorphism.

$$\varphi : K[V] \cong \mathcal{O}_V(V)$$
$$(\varphi : f \longrightarrow f \quad) .$$

Proof. (We follow the proof of Springer [1, Theorem 1.4.5].) It is clear that φ is a well-defined injective K-algebra homomorphism.

Let $f \in \mathcal{O}_V(V)$. Let v be an arbitrary element of V , then there exists an open neighbourhood U_v of v and $g_v, h_v \in K[V]$ such that $g_v(y) \neq 0$ and

$$f(y) = \frac{h_v(y)}{g_v(y)}$$

for all $y \in U_v$. From Remark 2.9 we have $U_v = V_{f_1} \cup \ldots \cup V_{f_n}$ for some finite elements $f_1, f_2, \ldots, f_n \in K[V]$. Hence for some $1 \leq i \leq n$ we have $v \in V_{f_i}$ and we may assume that $U_v = V_{f_i}$. We write a_v for f_i . Since $V_{a_v} \subset V_{g_v}$, we have

$$V_{g_v}^c \subset V_{a_v}^c .$$

Thus we have $\mathscr{V}(K[V]g_v) \subset \mathscr{V}(K[V]a_v)$. From the Hilbert's Nullstellensatz we have

$$\mathscr{I}(\mathscr{V}(K[V]a_v)) = \sqrt{K[V]a_v} \subset \mathscr{I}(\mathscr{V}(K[V]g_v)) = \sqrt{K[V]g_v} .$$

Hence there exists a natural number m such that

$$a_v^m = g_v g_v'$$

for some $g_v' \in K[V]$. Thus on V_{a_v} we have

$$f = \frac{h_v g_v'}{a_v^m} .$$

Since $V_{a_v} = V_{a_v^m}$, we may further assume that on V_{a_v}

$$f = \frac{h_v}{a_v} .$$

Since $V = \bigcup_{v \in V} V_{a_v}$, we have $V^c = \bigcap_{v \in V} V_{a_v}^c = \emptyset$, which implies $\mathscr{V}(\sum_{v \in V} K[V]a_v) = \emptyset$. Therefore,

$$K[V] = \sqrt{\sum_{v \in V} K[V]a_v} \quad \text{and} \quad 1 \in \sum_{v \in V} K[V]a_v .$$

Hence there exists a finite subset $\{a_1,\ldots,a_t\}$ of $\{a_v \mid v \in V\}$ such that

$$1 \in \sum_{i=1}^{t} K[V]a_i .$$

It is clear that $V = V_{a_1} \cup \ldots \cup V_{a_t}$, because

$$\mathscr{V}(\sum_{i=1}^{t} K[V]a_i) = \bigcap_{i=1}^{t} V_{a_i}^c = \emptyset .$$

Since $\frac{h_i}{a_i} = f = \frac{h_j}{a_j}$ on $V_{a_i} \cap V_{a_j} = V_{a_i a_j}$ $(1 \leq i,j \leq t)$, we have

$$(h_i a_j - a_i h_j)a_i a_j = 0 \quad \text{on} \quad V = V_{a_i a_j} \cup V_{a_i a_j}^c .$$

Since $V_{a_i^2} = V_{a_i}$ and $V = \bigcup_{i=1}^{t} V_{a_i^2}$, there exists $\{b_1,\ldots,b_t\} \subset K[V]$ such that

$$b_1 a_1^2 + b_2 a_2^2 + \ldots + b_t a_t^2 = 1 .$$

Now let $p = \sum_{i=1}^{t} b_i h_i a_i \in K[V]$ and we shall show that $\varphi(p) = f$.

Let $x \in V_{a_j}$, then we have

$$a_j^2(x)p(x) = a_j^2(x)(b_1(x)h_1(x)a_1(x) + \ldots + b_t(x)h_t(x)a_t(x)) .$$

Since $a_j^2 h_i a_i = a_i^2 h_j a_j$ $(1 \leq i,j \leq t)$, we have

$$\begin{aligned} a_j^2(x)p(x) &= b_1(x)a_1^2(x)h_j(x)a_j(x) + \ldots + b_t(x)a_t^2(x)h_j(x)a_j(x) \\ &= (b_1(x)a_1^2(x) + \ldots + b_1(x)a_t^2(x))h_j(x)a_j(x) \\ &= h_j(x)a_j(x) \\ &= a_j^2(x)f(x) \quad \text{for any } 1 \leq j \leq t . \end{aligned}$$

Thus on $V = \bigcup_{j=1}^{t} V_{a_j}$, we have $p = f$. Q.E.D.

(9.8) Corollary. Let $(V,K[V]) \in \mathcal{A}(K)$ and $f \in K[V]-(0)$, then

(1) $\mathcal{O}_{V_f}(O) = \mathcal{O}_V(O)$ for any open set O in V_f (see Exercise 8 on P.13), and

(2) $\mathcal{O}_V(V_f) = K[V]_f$.

Proof. (1) Let $g \in \mathcal{O}_V(O)$, then for each $x \in O$, there exists an open neighbourhood U_x $(\subset O)$ of x in V such that

$$g: U_x \longrightarrow K$$
$$(g: y \longrightarrow \frac{a(y)}{s(y)})$$

for some $a,s \in K[V]$. Since U_x is also open in $(V_f, K[V]_f)$ and $a/1, s/1 \in K[V]_f$, we have $g \in \mathcal{O}_{V_f}(O)$.

Conversely let $g \in \mathcal{O}_{V_f}(O)$, then also from the definition for each $x \in O$ there exists an open neighbourhood U_x of x in V_f such that

$$g: U_x \longrightarrow K$$
$$(g: y \longrightarrow \frac{\frac{a}{f^m}(y)}{\frac{s}{f^n}(y)})$$

for some $\frac{a}{f^m}, \frac{s}{f^n} \in K[V]_f$, where $m,n \in \mathbb{N}$. Since U_x is also open in V and

$$g(y) = \frac{(af^n)(y)}{(sf^m)(y)}$$

for any $y \in U_x \subset O$, we have $g \in \mathcal{O}_V(O)$. Hence $\mathcal{O}_{V_f}(O) = \mathcal{O}_V(O)$.

(2) is clear from the theorem. Q.E.D.

<u>(9.9) Proposition</u>. Let V,W be affine varieties over K with coordinate rings $K[V]$ and $K[W]$ respectively. Let $\varphi: V \to W$ be a map of V into W , then φ is a morphism of affine varieties defined as in Definition 1.3 if and only if φ is a morphism of sheaves of functions of $(V,\mathcal{O}_V)$ into $(W,\mathcal{O}_W)$.

<u>Proof</u>. We first assume that φ is a morphism of affine varieties, then it is clear that φ is continuous (see Proposition 1.8). Let O be an open set of W and $f \in \mathcal{O}_W(O)$, then we have a map $f \circ (\varphi|_{\varphi^{-1}(O)})$ of $\varphi^{-1}(O)$ into K . Let $x \in \varphi^{-1}(O)$. Since $\varphi(x) \in O$ and $f \in \mathcal{O}_W(O)$, there exists an open neighbourhood $U_{\varphi(x)}$ in O and $a,s \in K[W]$ such that $s(y) \neq 0$ and

$$f(y) = \frac{a(y)}{s(y)}$$

for all $y \in U_{\varphi(x)}$. Hence $f\circ\varphi(z) = \frac{a\circ\varphi(z)}{s\circ\varphi(z)}$ for all $z \in \varphi^{-1}(U_{\varphi(x)})$. Since $a\circ\varphi, s\circ\varphi \in K[V]$ and $\varphi^{-1}(U_{\varphi(x)})$ is an open neighbourhood of x in $\varphi^{-1}(O)$, we have

$$f\circ(\varphi|_{\varphi^{-1}(O)}) \in \mathcal{O}_V(\varphi^{-1}(O))$$

for any $f \in \mathcal{O}_W(O)$. Hence φ is a morphism of ringed spaces.

Now assume φ is a morphism of ringed spaces $(V,\mathcal{O}_V)$ into $(W,\mathcal{O}_W)$. Since $f\circ\varphi \in \mathcal{O}_V(\varphi^{-1}(W))$ for any $f \in \mathcal{O}_W(W)$ and

$\mathcal{O}_V(\varphi^{-1}(W)) = \mathcal{O}_V(V) = K[V]$ and $\mathcal{O}_W(W) = K[W]$ from Theorem 9.7, it is clear that φ is a morphism of affine varieties. Q.E.D.

<u>(9.10) Lemma</u>. Let $(V,K[V]) \in \mathcal{A}(K)$ and O be an open set of V. Let $f \in \mathcal{O}_V(O)$, then

$$O_f = \{v \in O \mid f(v) \neq 0\}$$

is open in V.

<u>Proof</u>. Since $f \in \mathcal{O}_V(O)$, there exist an open neighbourhood U_v $(\subset O)$ of v for each $v \in O$ and $g,h \in K[V]$ such that $g(y) \neq 0$ and

$$f(y) = \frac{h(y)}{g(y)}$$

for all $y \in U_v$. Since $\{y \in U_v \mid f(y) \neq 0\} = U_v \cap V_h \cap V_g$ is open in V and $O = \bigcup_{v \in O} U_v$,

$$O_f = \bigcup_{v \in O} \{y \in U_v \mid f(y) \neq 0\}$$

is also open in V. Q.E.D.

<u>Exercise 32</u>. Let $(V,K[V]), (K,K[X]) \in \mathcal{A}(K)$. Let O be an open subset of V and $f \in \mathcal{O}_V(O)$. Show that $f:O \to K$ is a continuous map.

10. Varieties

Before defining a prevariety we prove the following proposition.

(10.1) Proposition. Let (V,A) be an affine variety over K, then any open covering of V has a finite subcovering.

Proof. Let $\{O_\lambda \mid \lambda \in \Lambda\}$ be an open covering of V. Since $O_\lambda = V_{f_{\lambda_1}} \cup \dots \cup V_{f_{\lambda_{l(\lambda)}}}$ for some finite elements $f_{\lambda_1}, \dots, f_{\lambda_{l(\lambda)}} \in A$ for each $\lambda \in \Lambda$ (see Remark 2.9), we have

$$V = \bigcup_{\lambda\in\Lambda} O_\lambda = \bigcup_{\lambda\in\Lambda} (V_{f_{\lambda_1}} \cup \dots \cup V_{f_{\lambda_{l(\lambda)}}}) .$$

Hence
$$V^c = \bigcap_{\lambda\in\Lambda} (V^c_{f_{\lambda_1}} \cap \dots \cap V^c_{f_{\lambda_{l(\lambda)}}})$$

$$= \mathcal{V}(\sum_{\lambda\in\Lambda} (Af_{\lambda_1} + \dots + Af_{\lambda_{l(\lambda)}})) = \emptyset .$$

From the Hilbert's Nullstellensatz we have

$$\sqrt{\sum_{\lambda\in\Lambda} (Af_{\lambda_1} + \dots + Af_{\lambda_{l(\lambda)}})} = A .$$

Therefore there exist finite elements $f_1, \dots, f_l \in \{f_{\lambda_1}, \dots, f_{\lambda_{l(\lambda)}} \mid \lambda \in \Lambda\}$ such that

$$1 \in Af_1 + \dots + Af_l .$$

Hence $\mathcal{V}(Af_1 + \dots + Af_l) = V^c_{f_1} \cap \dots \cap V^c_{f_l} = \emptyset$. Since $V = V_{f_1} \cup \dots \cup V_{f_l}$ and each V_{f_i} is contained in some $O_{\lambda(i)}$ $(\lambda(i) \in \Lambda)$, $\{O_\lambda \mid \lambda \in \Lambda\}$ has a finite subcovering $\{O_{\lambda(1)}, \dots, O_{\lambda(l)}\}$ of V.

Q.E.D.

(10.2) Definition. A topological space V is called <u>quasi-compact</u> if any open covering of V has a finite subcovering.

Exercise 33. Let V be a quasi-compact topological space, then any closed subspace is also quasi-compact.

(10.3) Definition. A ringed space $(X,\mathcal{S}_X)$ over K is said to be a prevariety over K if X is a finite union of open subsets $U_1,U_2,\ldots,U_m$ such that $(U_i,\mathcal{S}_X(U_i)) \in \mathcal{A}(K)$ and $\mathcal{S}_X(O) = \mathcal{O}_{U_i}(O)$ for any open set O in U_i where $i = 1,2,\ldots,m$ and $\mathcal{O}_{U_i}$ is the canonical sheaf of functions on the affine variety U_i. We call U_i and affine open set of X.

More generally, a non-empty open subset U of X is called an affine open set of X if $(U,\mathcal{S}_X(U)) \in \mathcal{A}(K)$ and $\mathcal{S}_X(O) = \mathcal{O}_U(O)$ for any open set O in U where $\mathcal{O}_U$ is a canonical sheaf of functions on the affine variety $(U,\mathcal{S}_X(U))$.

It is clear that a subset O of X is open in X if and only if $O \cap U_i$ is open in each U_i $(1 \leq i \leq m)$, and a subset F of X is closed in X if and only if $F \cap U_i$ is closed in each U_i $(1 \leq i \leq m)$.

Now let $O_1 \subset O_2 \subset \ldots \subset O_n \subset \ldots$ be an ascending chain of open sets in X. Since each U_i is an affine variety, there exists $n_o \in \mathbb{N}$ such that

$$O_l \cap U_i = O_{n_o} \cap U_i$$

for any $l \geq n_o$ and $1 \leq i \leq m$. Thus

$$O_l = X \cap O_l = \bigcup_{i=1}^{m} (O_l \cap U_i) = \bigcup_{i=1}^{m} (O_{n_o} \cap U_i) = O_{n_o}$$

for any $l \geq n_o$. Hence X is a Noetherian space.

Further let $\{O_\lambda \mid \lambda \in \Lambda\}$ be an open covering of X, i.e., $X = \bigcup_{\lambda\in\Lambda} O_\lambda$. Since each U_i is quasi-compact and $U_i = \bigcup_{\lambda\in\Lambda} (O_\lambda \cap U_i)$, U_i is covered by finite open sets

$O_{\lambda_1}, \ldots, O_{\lambda_t}$. Hence X is quasi-compact, because $X = \bigcup_{i=1}^{m} U_i$.

Exercise 34. Let $(X, \mathcal{S}_X)$ be a prevariety over K and let $\mathcal{S}'_X$ be a sheaf of functions over K on X such that $(U_i, \mathcal{S}'_X(U_i)) \in \mathcal{A}(K)$ and $\mathcal{S}'_X(O) = \mathcal{O}_{U_i}(O)$ for any open set O in U_i , where $i = 1,2,\ldots,m$. Show that $\mathcal{S}_X = \mathcal{S}'_X$.

Exercise 35. Let $(X, \mathcal{S}_X)$ be a prevariety over K . Then a subset C of X is constructible if and only if $C \cap U_i$ is constructible in each $(U_i, K[U_i])$.

(10.4) Proposition. Let $(X, \mathcal{S}_X)$ be a prevariety over K with a finite affine open covering $\{U_i \mid i = 1,2,\ldots,m\}$. Then:

(1) A non-empty closed subset F of X is a prevariety with a finite affine open covering $\{U_i \cap F \mid i = 1,2,\ldots,m\}$ and a sheaf of functions $\mathcal{S}_F$ over K such that

$$\mathcal{S}_F(O) = \mathcal{O}_{U_i \cap F}(O)$$

for any open set O in F which is contained in $U_i \cap F$ $(1 \leq i \leq m)$, where $\mathcal{O}_{U_i \cap F}$ is a canonical sheaf of functions on the affine variety $U_i \cap F$.

(2) A non-empty open set U of X is a prevariety with a sheaf of functions $\mathcal{S}_U$ over K such that

$$\mathcal{S}_U(O) = \mathcal{S}_X(O)$$

for any open set O in U .

We call $(F, \mathcal{S}_F)$ a closed subprevariety of $(X, \mathcal{S}_X)$ and $(U, \mathcal{S}_U)$ an open subprevariety of $(X, \mathcal{S}_X)$.

Proof. (1) Since $X = \bigcup_{i=1}^{m} U_i$, we have a finite cover of open sets $\{U_i \cap F \mid 1 \leq i \leq m\}$ of F . Since each $U_i \cap F$ is closed in U_i , $U_i \cap F$ is a closed affine subvariety of U_i with a canonical sheaf of functions $\mathcal{O}_{U_i \cap F}$ $(1 \leq i \leq m)$. We write U_i' for $U_i \cap F$.

Now let U be an open set of F (of course in the relative topology), then we define $\mathcal{G}_F(U)$ to be the set of all K-valued functions f of U into K such that

$$f|_{U \cap U_i'} \in \mathcal{O}_{U_i'}(U \cap U_i')$$

for all $1 \leq i \leq m$. It is clear that $\mathcal{G}_F$ is a sheaf of functions over K . Let O be an open set of F and U be an open set of X such that $U \subset U_i$ for some $1 \leq i \leq m$ and $O = F \cap U$. Assume that $f \in \mathcal{G}_F(O)$, then from the definition we have

$$f|_{O \cap U_i'} \in \mathcal{O}_{U_i'}(U \cap U_i') .$$

Since $O \cap U_i' = O$, we have $f \in \mathcal{O}_{U_i'}(O)$. Conversely assume that $f \in \mathcal{O}_{U_i'}(O)$. We shall show that

$$f|_{O \cap U_j'} \in \mathcal{O}_{U_j'}(O \cap U_j')$$

for any $1 \leq j \leq m$.

Let $v \in O \cap U_j'$, then there exists an open neighbourhood U_v of v in U (U_v is open in X) and $a, s \in K[U_i]$ such that $s(y) \neq 0$ and

$$f(y) = \frac{a(y)}{s(y)}$$

for all $y \in U_v \cap F$, because $K[U_i'] = \{g|_{U_i'} \mid g \in K[U_i]\}$. Since

$$\mathcal{O}_{U_i}(U_i \cap U_j) = \mathcal{G}_X(U_i \cap U_j) = \mathcal{O}_{U_j}(U_i \cap U_j) ,$$

we have $a|_{U_i \cap U_j}$, $s|_{U_i \cap U_j} \in \mathcal{O}_{U_j}(U_i \cap U_j)$. On $U_v \cap U_j'$ $(\subset U_i \cap U_j \cap F)$ we have

$$f(y) = \frac{a(y)}{s(y)}$$

for all $y \in U_v \cap U_j'$. Since $a|_{U_i \cap U_j}$, $s|_{U_i \cap U_j} \in \mathcal{O}_{U_j}(U_i \cap U_j)$ and

$v \in U_i \cap U_j$, there exists an open neighbourhood U'_v of v in $U_i \cap U_j$ (U'_v is open in X) and $a_1, a_2, s_1, s_2 \in K[U_j]$ such that $a_2(y) \neq 0$, $s_2(y) \neq 0$ and

$$a(y) = \frac{a_1(y)}{a_2(y)} , \quad s(y) = \frac{s_1(y)}{s_2(y)}$$

for all $y \in U'_v$. Hence on $U_v \cap U'_v \cap F$ $(\subset U_j \cap F)$, we have

$$f(y) = \frac{\frac{a_1}{a_2}(y)}{\frac{s_1}{s_2}(y)}$$

for all $y \in U_v \cap U'_v \cap F$. Since $f(y) = \frac{(a_1 s_2)(y)}{(a_2 s_1)(y)}$, $(a_2 s_1)(y) \neq 0$ for all $y \in U_v \cap U'_v \cap F$ $(\subset U'_j)$ and $a_1 s_2|_{U'_j}$, $a_2 s_1|_{U'_j} \in K[U'_j]$, we have

$$f|_{O \cap U'_j} \in \mathcal{O}_{U'_j}(O \cap U'_j) .$$

Therefore $f \in \mathcal{S}_F(O)$, and we have $\mathcal{S}_F(O) = \mathcal{O}_{U'_i}(O)$.

(2) Let $\mathcal{S}_U(O) = \mathcal{S}_X(O)$ for any open set O in U , then it is clear that $\mathcal{S}_U$ is a sheaf of functions on U over K .

Since $U \cap U_i$ is open in an affine open set U_i $(1 \leq i \leq m)$, each $U \cap U_i$ is the union of a finite number of principal open sets in U_i . Hence U is a finite union of principal open sets $U_{1,f_1}, \ldots, U_{1,f_r}, U_{2,g_1}, \ldots, U_{2,g_s}, \ldots, U_{m,h_1}, \ldots, U_{m,h_t}$, where $f_1, \ldots, f_r \in K[U_1]$, $g_1, \ldots, g_s \in K[U_2]$, ..., $h_1, \ldots, h_t \in K[U_m]$. Since principal open sets are affine varieties, we only have to show that

$$\mathcal{S}_U(O) = \mathcal{O}_{U_{1,f}}(O)$$

for any open set O in $U_{1,f}$ where $f \in \{f_1, \ldots, f_r\}$. Since $\mathcal{O}_{U_{1,f}}(O) = \mathcal{O}_{U_1}(O)$ from Corollary 9.8, we have

$$\mathcal{O}_{U_{1,f}}(O) = \mathcal{O}_{U_1}(O) = \mathcal{S}_X(O) = \mathcal{S}_U(O) .$$

Hence $(U,\mathcal{O}_U)$ is a prevariety over K with a finite affine open covering

$$\{U_{1,f_1},\ldots,U_{1,f_r},\ U_{2,g_1},\ldots,U_{2,g_s},\ldots,U_{m,h_1},\ldots,U_{m,h_t}\} .$$

Q.E.D.

<u>Exercise 36</u>. Let $(X,\mathcal{O}_X)$ be a prevariety over K with a finite affine open covering $\{U_i \mid i = 1,2,\ldots,m\}$ and $(F,\mathcal{O}_F)$ be a closed subprevariety of $(X,\mathcal{O}_X)$. Let $\{O_j \mid 1 \leq j \leq l\}$ be another affine open covering of $(X,\mathcal{O}_X)$, then for an open set U of F we can also define $\mathcal{O}'_F(U)$ to be the set of all K-valued functions f of U into K such that

$$f|_{U\cap O_j\cap F} \in \mathcal{O}_{O_j\cap F}(U \cap O_j \cap F)$$

for all $1 \leq i \leq l$. Show that $\mathcal{O}'_F(U) = \mathcal{O}_F(U)$.

<u>Exercise 37</u>. Let $(X,\mathcal{O}_X)$ be a prevariety over K and $(F,\mathcal{O}_F)$ and $(U,\mathcal{O}_U)$ be open and closed subprevarieties respectively. Show that $\mathcal{O}_F = \mathcal{O}_U$ if $F = U$.

<u>(10.5) Corollary</u>. Let $(X,\mathcal{O}_X)$ be a prevariety over K and $(U,\mathcal{O}_U)$ be an open subprevariety defined as in (2). Let O be an open set in U , then O is an affine open set in $(U,\mathcal{O}_U)$ if and only if O is an affine open set in $(X,\mathcal{O}_X)$.

<u>Exercise 38</u>. Prove Corollary 10.5.

<u>(10.6) Definition</u>. Let $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ be prevarieties over K . We call a morphism of ringed spaces φ of $(X,\mathcal{O}_X)$ into $(Y,\mathcal{O}_Y)$ a <u>morphism of prevarieties</u>, i.e., φ is a continuous map of X into Y such that for any open set O in Y and function $f \in \mathcal{O}_Y(O)$ $f\circ(\varphi|_{\varphi^{-1}(O)})$ belongs to $\mathcal{O}_X(\varphi^{-1}(O))$.

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & & \\ \cup & & \cup & & \\ \varphi^{-1}(O) & \xrightarrow{\varphi} & O & \xrightarrow{f} & K\ , \end{array}$$

$f \in \mathcal{O}_Y(O)$.

<u>Exercise 39</u>. Let $\varphi:(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be an isomorphism of prevarieties over K, i.e., isomorphism of ringed spaces, and O be an affine open set of X. Show that $\varphi(O)$ is an affine open set of Y and

$$\varphi|_O:(O,\mathcal{O}_X(O)) \cong (\varphi(O),\mathcal{O}_Y(\varphi(O))$$

as affine varieties.

<u>(10.7) Proposition</u>. Let $(X,\mathcal{O}_X)$, $(Y,\mathcal{O}_Y)$ be prevarieties over K and φ be a map of X into Y. Suppose there is a finite covering of Y by affine open sets $\{W_\tau \mid 1 \leq \tau \leq t\}$ and a covering of X by open sets $\{V_\tau \mid 1 \leq \tau \leq t\}$ such that

(1) $\varphi(V_\tau) \subset W_\tau$ and

(2) $f\circ(\varphi|_{V_\tau}) \in \mathcal{O}_X(V_\tau)$ whenever $f \in \mathcal{O}_Y(W_\tau)$, for all $1 \leq \tau \leq t$.

Then φ is a morphism of prevarieties of $(X,\mathcal{O}_X)$ into $(Y,\mathcal{O}_Y)$.

<u>Proof</u>. Let $\{U_i \mid 1 \leq i \leq m\}$ be a finite cover of affine open sets of X as in Definition 10.3. Since $V_\tau \cap U_i$ is a union of finite principal open sets

$$U_{i,g_1},\ldots,U_{i,g_l} \quad \text{in } U_i \quad (g_1,\ldots,g_l \in K[U_i])$$

and each $(U_{i,g_k}, \mathcal{O}_X(U_{i,g_k}))$ is an affine open set in X $(1 \leq k \leq l)$ from Corollary 10.5, expanding the suffixes set $\{\tau \mid 1 \leq \tau \leq t\}$, we can assume that $\{V_\tau \mid 1 \leq \tau \leq t\}$ are also affine open sets. Thus $\varphi|_{V_\tau}:V_\tau \to W_\tau$ is a morphism of affine varieties from the assumption.

Hence $\varphi|_{V_\tau}$ is continuous. Since $X = \bigcup_{\tau=1}^{t} V_\tau$ and $Y = \bigcup_{\tau=1}^{t} W_\tau$, φ is also continuous on X .

Now let W_o be an open set in Y and $V_o = \varphi^{-1}(W_o)$. Let $f \in \mathcal{S}_Y(W_o)$. since $\varphi|_{V_\tau}$ is a morphism of affine varieties of V_τ into W_τ and

$$f|_{W_o \cap W_\tau} \in \mathcal{S}_Y(W_o \cap W_\tau) = \mathcal{O}_{W_\tau}(W_o \cap W_\tau) ,$$

we have $f\circ(\varphi|_{V_\tau \cap \varphi^{-1}(W_o \cap W_\tau)}) \in \mathcal{O}_{V_\tau}(V_\tau \cap \varphi^{-1}(W_o \cap W_\tau))$ from Proposition 9.9. Hence

$$\begin{aligned} f\circ(\varphi|_{V_\tau \cap \varphi^{-1}(W_o \cap W_\tau)}) &\in \mathcal{O}_{V_\tau}(V_\tau \cap \varphi^{-1}(W_o \cap W_\tau)) \\ &= \mathcal{S}_X(V_\tau \cap \varphi^{-1}(W_o \cap W_\tau)) \end{aligned}$$

for any $1 \leq \tau \leq t$. Since $V_o = \varphi^{-1}(W_o) = \varphi^{-1}(\bigcup_{\tau=1}^{t} (W_o \cap W_\tau)) = \bigcup_{\tau=1}^{t} \varphi^{-1}(W_o \cap W_\tau)$, we have $f\circ(\varphi|_{V_\tau \cap V_o}) \in \mathcal{S}_X(V_\tau \cap V_o)$. Hence $f\circ(\varphi|_{V_o}) \in \mathcal{S}_X(V_o)$, because $X = \bigcup_{\tau=1}^{t} V_\tau$. Q.E.D.

<u>Exercise 40</u>. Let Ψ be a morphism of prevarieties $(X,\mathcal{S}_X)$ into $(Y,\mathcal{S}_Y)$ over K . Let $\overline{\Psi(X)}$ be the closure of $\Psi(X)$ in Y . Show that

$$\begin{aligned} \psi : X &\longrightarrow \overline{\Psi(X)} \\ (\psi : x &\longrightarrow \Psi(x)) \end{aligned}$$

is a morphism of prevarieties of $(X,\mathcal{S}_X)$ into $(\overline{\Psi(X)},\mathcal{S}_{\overline{\Psi(X)}})$.

Now we shall show how to construct the product of prevarieties. Let $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ be prevarieties over K with finite affine open coverings

$$\{U_i \mid 1 \leq i \leq l\} \quad \text{and} \quad \{O_j \mid 1 \leq j \leq m\}$$

respectively. We first examine the structure of $X\times Y$. Since $X = \bigcup_{i=1}^{l} U_i$ and $Y = \bigcup_{j=1}^{m} O_j$, we have

$$X\times Y = \bigcup_{\substack{i=1,2,\ldots,l \\ j=1,2,\ldots,m}} U_i \times O_j ,$$

where each $U_i \times O_j$ is an affine variety. Hence it is natural to define the following topology on $X\times Y$:

A subset O is open in $X\times Y$ if and only if $O \cap (U_i \times O_j)$ is open in the Zariski topology of the affine variety $U_i \times O_j$ for any $1 \leq i \leq l$ and $1 \leq j \leq m$.

It is clear that this is a well-defined topology on $X\times Y$.

Exercise 41. In the above topological space $X\times Y$.
(1) Each $U_i \times O_j$ is open in $X\times Y$.
(2) A subset O in $U_i \times O_j$ is open in $U_i \times O_j$ if and only if O is open in $X\times Y$.

Next let O be an open set of $X\times Y$, then we define $\mathcal{S}_{X\times Y}(O)$ to be the set of all K-valued functions f of O into K such that

$$f|_{O\cap(U_i\times O_j)} \in \mathcal{O}_{U_i\times O_j}(O \cap (U_i \times O_j))$$

for all $1 \leq i \leq l$, $1 \leq j \leq m$. It is clear that $\mathcal{S}_{X\times Y}$ is a sheaf of functions over K on $X\times Y$.

(10.8) Theorem. Let $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ be prevarieties over K with finite affine open coverings

$$\{U_i \mid 1 \leq i \leq l\} \quad \text{and} \quad \{O_j \mid 1 \leq j \leq m\}$$

respectively. Let $\mathcal{S}_{X\times Y}$ be a sheaf of functions over K on $X\times Y$ defined as before. Then:

(1) $(X\times Y, \mathcal{S}_{X\times Y})$ is a prevariety over K with finite affine open covering $\{U_i \times O_j \mid 1 \leq i \leq l,\ 1 \leq j \leq m\}$.
(2) Let $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ be projections, $(\pi_1 : (x,y) \longrightarrow x)$ $(\pi_2 : (x,y) \longrightarrow y)$ then π_1 and π_2 are morphisms of prevarieties and are open maps.

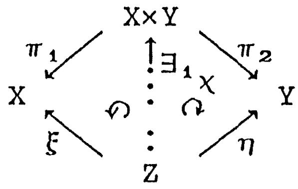
(3) For any prevariety Z over K and morphisms of prevarieties

$$\xi : Z \longrightarrow X \quad \text{and} \quad \eta : Z \longrightarrow Y$$

there exists a unique morphism of prevarieties

$$\chi : Z \longrightarrow X \times Y$$

which makes the following diagramm commutative:



.

<u>Proof</u>. (1) It is enough to show that $\mathcal{G}_{X\times Y}(O) = \mathcal{O}_{U_1\times O_1}(O)$ for any open set O in $U_1 \times O_1$. It is clear that $\mathcal{G}_{X\times Y}(O) \subset \mathcal{O}_{U_1\times O_1}(O)$ from the definition of $\mathcal{G}_{X\times Y}$.

Now let $f \in \mathcal{O}_{U_1\times O_1}(O)$ and we shall show that

$$f|_{O\cap(U_i\times O_j)} \in \mathcal{O}_{U_i\times O_j}(O \cap (U_i \times O_j))$$

for any i,j. Since $f|_{O\cap(U_i\times O_j)} \in \mathcal{O}_{U_1\times O_1}(O \cap (U_i \times O_j))$, for any point $v = (v_1,v_2)$ of $O \cap (U_i \times O_j)$ there exists an open neighbourhood U_v of v in $O \cap (U_i \times O_j)$ and $g,h \in K[U_1] \otimes K[O_1]$ such that $g(y) \neq 0$ and

$$f(y) = \frac{h(y)}{g(y)}$$

for all $y \in U_v$. Let $h = \Sigma\, \alpha \otimes \beta$ and $g = \Sigma\, \gamma \otimes \delta$ where $\alpha,\gamma \in K[U_1]$ and $\beta,\delta \in K[O_1]$. Since $\alpha|_{U_1\cap U_i}$, $\gamma|_{U_1\cap U_i} \in \mathcal{O}_{U_i}(U_1 \cap U_i)$ and $\beta|_{O_1\cap O_j}$, $\delta|_{O_1\cap O_j} \in \mathcal{O}_{O_j}(O_1 \cap O_j)$ and

$$O \cap (U_i \times O_j) \subset (U_1 \times O_1) \cap (U_i \times O_j) = (U_1 \cap U_i) \times (O_1 \cap O_j) \,,$$

there exist an open neighbourhood U_{i,v_1} of v_1 in $U_1 \cap U_i$ and O_{j,v_2} of v_2 in $O_1 \cap O_j$ and $\alpha_1,\alpha_2 \in K[U_i]$, $\beta_1,\beta_2 \in K[O_j]$ such that $\alpha_2(x) \neq 0$, $\beta_2(x') \neq 0$ and

$$\alpha(x) = \frac{\alpha_1(x)}{\alpha_2(x)} \quad \text{for all} \quad x \in U_{i,v_1} \quad \text{and}$$

$$\beta(x') = \frac{\beta_1(x')}{\beta_2(x')} \quad \text{for all} \quad x' \in O_{j,v_2} \,.$$

Similarly there exist $\gamma_1,\gamma_2 \in K[U_i]$, $\delta_1,\delta_2 \in K[O_j]$ such that

$$\gamma = \frac{\gamma_1}{\gamma_2}$$

on some open neighbourhood of v_1 in $U_1 \cap U_i$ and

$$\delta = \frac{\delta_1}{\delta_2}$$

on some open neighbourhood of v_2 in $O_1 \cap O_j$.

Since $U_{i,v_1} \times O_{j,v_2}$ is an open neighbourhood of $v = (v_1,v_2)$ in $U_i \times O_j$ (see Proposition 3.4), there exists an open neighbourhood O_v of v in $U_i \times O_j$ such that

$$f(y) = \frac{h(y)}{g(y)} = \frac{(\Sigma\alpha\otimes\beta)(y)}{(\Sigma\gamma\otimes\delta)(y)} = \frac{\Sigma\alpha(y_1)\cdot\beta(y_2)}{\Sigma\gamma(y_1)\cdot\delta(y_2)}$$

$$= \frac{\Sigma\ \frac{\alpha_1(y_1)}{\alpha_2(y_1)}\cdot\frac{\beta_1(y_2)}{\beta_2(y_2)}}{\Sigma\ \frac{\gamma_1(y_1)}{\gamma_2(y_1)}\cdot\frac{\delta_1(y_2)}{\delta_2(y_2)}} = \frac{\Sigma\ \frac{\alpha_1\otimes\beta_1(y)}{\alpha_2\otimes\beta_2(y)}}{\Sigma\ \frac{\gamma_1\otimes\delta_1(y)}{\gamma_2\otimes\delta_2(y)}}$$

for all $y = (y_1,y_2) \in O_v$. Hence $f|_{O\cap(U_i\times O_j)} \in \mathcal{O}_{U_i\times O_j}(O \cap (U_i \times O_j))$ for any i,j . Thus $\mathcal{S}_{X\times Y}(O) = \mathcal{O}_{U_1\times O_1}(O)$ for any open set O in $U_1 \times O_1$.

(2) It is enough to prove that $\pi_1 : X\times Y \longrightarrow X$ $(\pi_1 : (x,y) \longrightarrow x)$ is an open morphism. Let $\{U_i \times O_j \mid 1 \leq i \leq l,\ 1 \leq j \leq m\}$ and $\{U_i \mid 1 \leq i \leq l\}$ be affine open coverings of $X\times Y$ and X respectively as above. Since $\pi_1(U_i \times O_j) = U_i$ and

$$f\circ(\pi_1 \mid U_i \times O_j) \in \mathcal{O}_{U_i\times O_j}(U_i \times O_j) = \mathcal{S}_{X\times Y}(U_i \times O_j)$$

for any $f \in \mathcal{O}_{U_i}(U_i) = \mathcal{S}_X(U_i)$ from Proposition 3.3, π_1 is a morphism of prevarieties from Proposition 10.7.

Now let O be an open set in $X\times Y$, i.e., $O \cap (U_i \times O_j)$ is open in $U_i \times O_j$ for any i and j . Hence $\pi_1(O \cap (U_i \times O_j))$ is open in U_i from Proposition 3.3. Thus

$$\bigcup_{\substack{i=1,\ldots,l\\ j=1,\ldots,m}} \pi_1(O \cap (U_i \times O_j)) = \pi_1\Big(\bigcup_{\substack{i=1,\ldots,l\\ j=1,\ldots,m}} (O \cap (U_i \times O_j))\Big) =$$

$$= \pi_1\Big(O \cap \bigcup_{\substack{i=1,\ldots,l\\ j=1,\ldots,m}} (U_i \times O_j)\Big) = \pi_1(O \cap (X\times Y)) = \pi_1(O)$$

is open in X .

(3) Let Z be a prevariety over K and $\xi:Z \to X$ and $\eta:Z \to Y$ be given two morphisms of prevarieties. Define

$$\chi:Z \to X\times Y$$

to be a map of Z into $X\times Y$ which takes $x \in Z$ to $(\xi(x), \eta(x)) \in X\times Y$. Since $X = \bigcup_{i=1}^{l} U_i$ and $Y = \bigcup_{j=1}^{m} O_j$, we have

$$Z = \xi^{-1}\Big(\bigcup_{i=1}^{l} U_i\Big) = \bigcup_{i=1}^{l} \xi^{-1}(U_i) = \eta^{-1}\Big(\bigcup_{j=1}^{m} O_j\Big) = \bigcup_{j=1}^{m} \eta^{-1}(O_j) .$$

Thus $Z = (\bigcup_i \xi^{-1}(U_i)) \cap (\bigcup_j \eta^{-1}(O_j)) = \bigcup_{i,j} (\xi^{-1}(U_i) \cap \eta^{-1}(O_j))$.

Notice $\chi(\xi^{-1}(U_i) \cap \eta^{-1}(O_j)) \subset U_i \times O_j$. Since $\xi:Z \to X$ and $\eta:Z \to Y$ are morphisms of prevarieties, we have

$$f\circ(\xi|_{\xi^{-1}(U_i) \cap \eta^{-1}(O_j)}) \in \mathcal{O}_Z(\xi^{-1}(U_i) \cap \eta^{-1}(O_j))$$

for any $f \in \mathcal{O}_X(U_i)$ and

$$g\circ(\eta|_{\xi^{-1}(U_i) \cap \eta^{-1}(O_j)}) \in \mathcal{O}_Z(\xi^{-1}(U_i) \cap \eta^{-1}(O_j))$$

for any $g \in \mathcal{O}_Y(O_j)$, because $f\circ(\xi|_{\xi^{-1}(U_i)}) \in \mathcal{O}_Z(\xi^{-1}(U_i))$ and $g\circ(\eta|_{\eta^{-1}(O_j)}) \in \mathcal{O}_Z(\eta^{-1}(O_j))$. Since $f \otimes g \in \mathcal{O}_{U_i\times O_j}(U_i \times O_j) = \mathcal{O}_{X\times Y}(U_i \times O_j)$, we have

$$(f \otimes g)\circ(\chi|_{\xi^{-1}(U_i) \cap \eta^{-1}(O_j)}) =$$

$$= f\circ(\xi|_{\xi^{-1}(U_i) \cap \eta^{-1}(O_j)})\cdot g\circ(\eta|_{\xi^{-1}(U_i) \cap \eta^{-1}(O_j)})$$

$$\in \mathcal{O}_Z(\xi^{-1}(U_i) \cap \eta^{-1}(O_j))$$

for any $f \in \mathcal{O}_X(U_i)$ and $g \in \mathcal{O}_Y(O_j)$. Hence χ is a morphism of prevarieties from Proposition 10.7. The uniqueness follows from the definition of χ . Q.E.D.

We call $(X\times Y, \mathcal{S}_{X\times Y})$ the <u>product</u> of prevarieties $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ over K .

<u>Exercise 42</u>. Let $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ be prevarieties over K . Let F and F' be closed subsets of X and Y respectively and O and O' be open subsets of X and Y respectively. Prove the following remarks.

(1) $F\times F'$ is closed in $(X\times Y, \mathcal{S}_{X\times Y})$.

(2) Let $F\times F'$ be the closed subprevariety of $(X\times Y, \mathcal{S}_{X\times Y})$ and $F\dot{\times}F'$ be the product of closed subprevarieties F $(\subset (X,\mathcal{S}_X))$ and F' $(\subset (Y,\mathcal{S}_Y))$ respectively. Then we have
$$\mathcal{S}_{F\times F'} = \mathcal{S}_{F\dot{\times}F'} .$$

(3) $O\times O'$ is open in $(X\times Y, \mathcal{S}_{X\times Y})$.

(4) Let $O\times O'$ be the open subprevariety of $(X\times Y, \mathcal{S}_{X\times Y})$ and $O\dot{\times}O'$ be the product of open subprevarieties O $(\subset (X,\mathcal{S}_X))$ and O' $(\subset (Y,\mathcal{S}_Y))$ respectively. Then we have $\mathcal{S}_{O\times O'} = \mathcal{S}_{O\dot{\times}O'}$.

<u>Exercise 43</u>. Let $(X,\mathcal{S}_X)$, $(Y,\mathcal{S}_Y)$ and $(Z,\mathcal{S}_Z)$ be prevarieties over K . Show that the maps
$$\begin{array}{ccc} (X\times Y)\times Z \longrightarrow X\times(Y\times Z) & \text{and} & X\times Y \longrightarrow Y\times X \\ ((x,y),z) \rightarrow (x,(y,z)) & & (x,y) \rightarrow (y,x) \end{array}$$
are isomorphisms of prevarieties.

<u>Exercise 44</u>. Let $(X,\mathcal{S}_X)$, $(Y,\mathcal{S}_Y)$ be irreducible prevarieties over K . Show that $(X\times Y, \mathcal{S}_{X\times Y})$ is also irreducible.

<u>(10.9) Definition</u>. Let $(X,\mathcal{S}_X)$ be a prevariety over K . We call X a <u>variety</u> if the diagonal
$$\Delta(X) = \{(x,x) \mid x \in X\}$$
is closed in $(X\times X, \mathcal{S}_{X\times X})$. We call a morphism of prevarieties of Y into Z a <u>morphism of varieties</u> when Y and Z are varieties over K .

<u>(10.10) Remark</u>. Let $(X,\mathcal{S}_X)$ be prevariety over K . Then $(X,\mathcal{S}_X)$ is a variety over K if and only if for any prevariety $(Y,\mathcal{S}_Y)$ over K and morphisms of prevarieties φ and ψ of Y into X

$$\{y \in Y \mid \varphi(y) = \psi(y)\}$$

is closed in Y .

<u>Proof</u>. Assume that $(X,\mathcal{S}_X)$ is a prevariety and $\{y \in Y \mid \varphi(y) = \psi(y)\}$ is closed for any prevariety Y and morphisms φ and ψ . Since

$$\begin{aligned} \Delta(X) &= \{(x,x) \mid x \in X\} \\ &= \{y \in X\times X \mid \pi_1(y) = \pi_2(y)\} , \end{aligned}$$

where $\pi_1,\pi_2 : X\times X \to X$ are projections, $\Delta(X)$ is closed in $X\times X$. Hence $(X,\mathcal{S}_X)$ is a variety over K .

Conversely if $\Delta(X)$ is closed in $X\times X$, then

$$(\varphi\times\psi)^{-1}(\Delta(X)) = \{y \in Y \mid \varphi(y) = \psi(y)\}$$

is also closed in Y , because the map

$$\begin{aligned} \varphi\times\psi : Y &\longrightarrow X\times X \\ (\varphi\times\psi : y &\longrightarrow (\varphi(y),\psi(y))) \end{aligned}$$

is a morphism of prevarieties from Theorem 10.8. Q.E.D.

<u>(10.11) Examples</u>.

(1) Let $(V, \mathcal{O}_V(V)) \in \mathcal{A}(K)$ be an affine variety, then $(V,\mathcal{O}_V)$ is a variety.

(2) Let $(X,\mathcal{S}_X)$ be a variety over K and $(F,\mathcal{S}_F)$ be a closed subprevariety of X and $(U,\mathcal{S}_U)$ be an open subprevariety of X , then $(F,\mathcal{S}_F)$ and $(U,\mathcal{S}_U)$ are varieties over K . We call $(F,\mathcal{S}_F)$ a <u>closed subvariety</u> of X and $(U,\mathcal{S}_U)$ and <u>open subvariety</u> of X .

(3) Let $(X,\mathcal{S}_X)$ and $(Y,\mathcal{S}_Y)$ be varieties over K , then $(X\times Y, \mathcal{S}_{X\times Y})$ is also a variety over K .

<u>Proof.</u> (1) It is clear that $(V,\mathcal{O}_V)$ is a prevariety with affine open covering $\{V\}$. Let χ be a map of V into $V\times V$ which takes $v\in V$ to $(v,v)\in V\times V$, then from Proposition 2.5

$$\chi(V)=\Delta(V)$$

is closed in $V\times V$. Hence an affine variety is a variety.

(2) Let $\{U_i \mid i=1,2,\ldots,m\}$ be a finite affine open covering of $(X,\mathcal{S}_X)$. Since $(F,\mathcal{S}_F)$ is a prevariety with finite affine open covering $\{U_i\cap F \mid i=1,2,\ldots,m\}$, we only have to show that

$$\begin{aligned}&\Delta(F)\cap\{(U_i\cap F)\times(U_j\cap F)\}\\ &=\Delta(F)\cap(U_i\times U_j)\cap(F\times F)\\ &=\Delta(X)\cap(U_i\times U_j)\cap(F\times F)\end{aligned}$$

is closed in $(U_i\cap F)\times(U_j\cap F)$ for all i,j. Since $(U_i\cap F)\times(U_j\cap F)$ is closed in $U_i\times U_j$ (see Proposition 3.4),

$$\begin{aligned}&\Delta(F)\cap\{(U_i\cap F)\times(U_j\cap F)\}\\ &=\{\Delta(X)\cap(U_i\times U_j)\}\cap\{(U_i\cap F)\times(U_j\cap F)\}\end{aligned}$$

is closed in $(U_i\cap F)\times(U_j\cap F)$ (see Exercise 5 on P.9).

Now since $(U,\mathcal{S}_U)$ is a prevariety with finite affine open covering $C=\{U_{1,f_1},\ldots,U_{1,f_r},U_{2,g_1},\ldots,U_{2,g_s},\ldots,U_{m,h_1},\ldots,U_{m,h_t}\}$ (see the proof of Proposition 10.4), it is enough to show that $\Delta(U)\cap(U_{i,f}\times U_{j,g})$ is closed in $U_{i,f}\times U_{j,g}$ for all $U_{i,f},U_{j,g}\in C$. Since

$$\Delta(U)\cap(U_{i,f}\times U_{j,g})=\Delta(X)\cap(U_{i,f}\times U_{j,g})=\Delta(X)\cap(U_i\times U_j)_{f\otimes g}$$

and $\Delta(X)\cap(U_i\times U_j)$ is closed in $U_i\times U_j$, $\Delta(U)\cap(U_{i,f}\times U_{j,g})$ is closed in $(U_i\times U_j)_{f\otimes g}=U_{i,f}\times U_{j,g}$ (see Exercise 8 on P.13).

(3) Let $(Z,\mathcal{S}_Z)$ be a prevariety over K and φ,ψ be morphisms of prevarieties Z into $X\times Y$. Since

$$\begin{aligned} &\{z \in Z \mid \varphi(z) = \psi(z)\} \\ &= \{z \in Z \mid \pi_1 \circ \varphi(z) = \pi_1 \circ \psi(z) \text{ and } \pi_2 \circ \varphi(z) = \pi_2 \circ \psi(z)\} \\ &= \{z \in Z \mid \pi_1 \circ \varphi(z) = \pi_1 \circ \psi(z)\} \cap \{z \in Z \mid \pi_2 \circ \varphi(z) = \pi_2 \circ \psi(z)\} \end{aligned}$$

is closed in Z , $(X \times Y, \mathcal{O}_{X \times Y})$ is also a variety over K from Remark 10.10. Q.E.D.

11. Projective varieties

In this section we introduce the notion of projective varieties and show that they are varieties defined as in Definition 10.9.

A projective n-space $\mathbb{P}^n$ over K is defined to be the set of equivalence classes of

$$K^{n+1} - \{(\underbrace{0,0,\ldots,0}_{n+1})\}$$

relative to the equivalence relation

$$(x_o,x_1,\ldots,x_n) \sim (y_o,y_1,\ldots,y_n)$$

if and only if there exists $a \in K-\{0\}$ such that

$$a(x_o,x_1,\ldots,x_n) = (y_o,y_1,\ldots,y_n) .$$

We write $[x_o,x_1,\ldots,x_n]$ for the equivalence class in $\mathbb{P}^n$ which contains $(x_o,x_1,\ldots,x_n) \in K^{n+1} - \{(0,0,\ldots,0)\}$.

Assume that V is an $n+1$-dimensional vector space over K, then we can identify the set of all 1-dimensional subspace of V with $\mathbb{P}^n$.

$$\{Kv \mid v \in V-\{0\}\} \xleftrightarrow{1:1} \mathbb{P}^n$$

$$K(x_ov_o+x_1v_1+\ldots+x_nv_n) \longleftrightarrow [x_o,x_1,\ldots,x_n] ,$$

where $(v_o,v_1,\ldots,v_n)$ is a K-basis of V . We write $P(V)$ for the set of all 1-dimensional subspace of V .

Now let $\mathbb{P}^n_i = \{[x_o,x_1,\ldots,x_n] \in \mathbb{P}^n \mid x_i \neq 0\}$, where $i = 0,1,\ldots,n$, then we can identify $\mathbb{P}^n_i$ with the affine n-space K^n by the following bijection φ_i .

$$\varphi_i:\mathbb{P}^n_i \longrightarrow K^n$$

$$(\varphi_i:[x_o,x_1,\ldots,x_n] \longrightarrow (\frac{x_o}{x_i},\ldots,\frac{\check{x}_i}{x_i},\ldots,\frac{x_n}{x_i})) ,$$

where the $\frac{\check{x}_i}{x_i}$ is omitted. By the above identification we define $K[\mathbb{P}^n_i]$ to be

$$K[\mathbb{P}^n_i] = K[\frac{X_o}{X_i}, \ldots, \check{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}]$$
$$\subset K(X_o, X_1, \ldots, X_n) \ .$$

<u>Justification</u>. Let φ^*_i be the comorphism of φ_i , then we have

$$\varphi^*_i(K[X_1, \ldots, X_n]) = K[X_1 \circ \varphi_i, \ldots, X_n \circ \varphi_i] \ .$$

Let ρ be a map of $K[X_1 \circ \varphi_i, \ldots, X_n \circ \varphi_i]$ into $K(X_o, X_1, \ldots, X_n)$, the quotient field of $K[X_o, X_1, \ldots, X_n]$, such that

$$\rho : K[X_1 \circ \varphi_i, \ldots, X_n \circ \varphi_i] \longrightarrow K(X_o, X_1, \ldots, X_n)$$
$$(\rho : X_j \circ \varphi_i \longrightarrow \frac{X_{j-1}}{X_i}) \qquad (j = 1, \ldots, i)$$

and $\rho(X_j \circ \varphi_i) = \frac{X_j}{X_i}$ $(j = i+1, \ldots, n)$, then ρ is an injective K-algebra homomorphism. Hence we have

$$K[\mathbb{P}^n_i] = K[\frac{X_o}{X_i}, \ldots, \check{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}]$$

and

$$\frac{X_o}{X_i} : [x_o, x_1, \ldots, x_n] \longrightarrow \frac{x_o}{x_i} , \quad \frac{X_1}{X_i} : [x_o, x_1, \ldots, x_n] \longrightarrow \frac{x_1}{x_i} , \ldots ,$$
$$\frac{X_n}{X_i} : [x_o, x_1, \ldots, x_n] \longrightarrow \frac{x_n}{x_i} \ .$$

<u>(11.1) Lemma</u>. Let $\mathbb{P}^n$ be a projective n-space over K , then $\mathbb{P}^n_i \cap \mathbb{P}^n_j$ is a principal open set in $\mathbb{P}^n_i$, where $0 \leq i, j \leq n$.

<u>Proof</u>. $(\mathbb{P}^n_i \cap \mathbb{P}^n_j) = (\mathbb{P}^n_i)_{\frac{X_j}{X_i}}$. Q.E.D.

Since $\mathbb{P}^n = \bigcup_{i=0}^{n} \mathbb{P}^n_i$ and each $(\mathbb{P}^n_i, K[\mathbb{P}^n_i]) \in \mathcal{A}(K)$, it is natural to define the following topology on $\mathbb{P}^n$:

A subset O of $\mathbb{P}^n$ is open in $\mathbb{P}^n$ if and only if $O \cap \mathbb{P}^n_i$ is open in the Zariski topology of the affine varieties $\{\mathbb{P}^n_i \mid 0 \leq i \leq n\}$.

<u>(11.2) Lemma</u>. Let $\mathbb{P}^n_i \in \{\mathbb{P}^n_o, \mathbb{P}^n_1, \ldots, \mathbb{P}^n_n\}$ and O be a subset of $\mathbb{P}^n_i$, then

O is open in $(\mathbb{P}^n_i, K[\mathbb{P}^n_i])$ if and only if

O is open in $\mathbb{P}^n$.

<u>Proof</u>. It is enough to show that $O \cap \mathbb{P}^n_j$ is open in $(\mathbb{P}^n_j, K[\mathbb{P}^n_j])$ for any $0 \leq j \leq n$ when O is an open subset of $(\mathbb{P}^n_i, K[\mathbb{P}^n_i])$.

Assume that O is open in $(\mathbb{P}^n_i, K[\mathbb{P}^n_i])$, then $\varphi_i(O)$ is open in K^n , i.e., $\varphi_i(O)$ is a finite union of principal open sets $\{(K^n)_{f_1}, \ldots, (K^n)_{f_t}\}$ of K^n . Let $f \in \{f_1, \ldots, f_t\}$, then

$$\varphi_i^{-1}((K^n)_f) = \{[x_o, \ldots, x_n] \in \mathbb{P}^n_i \mid f(\frac{x_o}{x_i}, \ldots, \overset{\vee}{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}) \neq 0\} .$$

Assume

$$f(\frac{X_o}{X_i}, \ldots, \overset{\vee}{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i}) = c_{m_o, \ldots, \overset{i}{\vee}, \ldots, m_n} (\frac{X_o}{X_i})^{m_o} \ldots \overset{\vee}{(\frac{X_i}{X_i})^{m_i}} \ldots (\frac{X_n}{X_i})^{m_n} + \ldots$$

and put

$$F(X_o, \ldots, X_i, \ldots, X_n) = X_i^{m_o + \ldots + \overset{\vee}{m_i} + \ldots + m_n} f(\frac{X_o}{X_i}, \ldots, \overset{\vee}{\frac{X_i}{X_i}}, \ldots, \frac{X_n}{X_i})$$

$$= c_{m_o, \ldots, \overset{i}{\vee}, \ldots, m_n} X_o^{m_o} \ldots \overset{\vee}{X_i} \ldots X_n^{m_n} + \ldots$$

$$\ldots + c_{m'_o, \ldots, \overset{i}{\vee}, \ldots, m'_n} X_i^{(m_o + \ldots, \overset{i}{\vee}, \ldots + m_n) - (m'_o + \ldots, \overset{i}{\vee}, \ldots + m'_n)} X_o^{m'_o} \ldots \overset{\vee}{X_i} \ldots X_n^{m'_n} + \ldots$$

Thus we have

$$\varphi_i^{-1}((K^n)_f) = \{[x_o, \ldots, x_n] \in \mathbb{P}^n_i \mid F(x_o, \ldots, x_i, \ldots, x_n) \neq 0\} .$$

Hence

$$\varphi_i^{-1}((K^n)_f) \cap \mathbb{P}_j^n$$

$$= \{[x_o,\ldots,x_n] \in \mathbb{P}^n \mid x_i \neq 0,\ x_j \neq 0,\ F(x_o,\ldots,x_i,\ldots,x_n) \neq 0\}$$

$$= \{[x_o,\ldots,x_n] \in \mathbb{P}_j^n \mid x_i \neq 0,\ F(x_o,\ldots,x_i,\ldots,x_n) \neq 0\}$$

$$= \{[x_o,\ldots,x_n] \in \mathbb{P}_j^n \mid \frac{x_i}{x_j} \neq 0,\ \frac{F(x_o,\ldots,x_n)}{x_j^{m_o+\ldots+m_n}} \neq 0\}$$

is open in $\mathbb{P}_j^n$. Since

$$O = \varphi_i^{-1}((K^n)_{f_1} \cup\ldots\cup (K^n)_{f_t}) = \varphi_i^{-1}((K^n)_{f_1}) \cup\ldots\cup \varphi_i^{-1}((K^n)_{f_t}) ,$$

$O \cap \mathbb{P}_j^n$ is open in $\mathbb{P}_j^n$. Q.E.D.

Next let O be an open set of $\mathbb{P}^n$, then we define $\mathcal{G}_{\mathbb{P}^n}(O)$ to be the set of all K-valued functions f of O into K such that

$$f|_{O\cap\mathbb{P}_i^n} \in \mathcal{O}_{\mathbb{P}_i^n}(O \cap \mathbb{P}_i^n)$$

for all $i = 0,1,2,\ldots,n$. It is clear that $\mathcal{G}_{\mathbb{P}^n}$ is a sheaf of functions over K on $\mathbb{P}^n$.

<u>(11.3) Lemma</u>. $(\mathbb{P}^n,\mathcal{G}_{\mathbb{P}^n})$ is a prevariety over K with a finite affine open covering $\mathbb{P}_o^n,\mathbb{P}_1^n,\ldots,\mathbb{P}_n^n$.

<u>Proof</u>. It is enough to show that

$$\mathcal{G}_{\mathbb{P}^n}(O) = \mathcal{O}_{\mathbb{P}_o^n}(O)$$

for any open set O in $\mathbb{P}_o^n$. It is clear that

$$\mathcal{G}_{\mathbb{P}^n}(O) \subset \mathcal{O}_{\mathbb{P}_o^n}(O)$$

from the definition of $\mathcal{G}_{\mathbb{P}^n}$. Let $f \in \mathcal{O}_{\mathbb{P}_o^n}(O)$ and we shall show that

$$f|_{O\cap\mathbb{P}_i^n} \in \mathcal{O}_{\mathbb{P}_i^n}(O \cap \mathbb{P}_i^n)$$

for all i . Since $O \cap \mathbb{P}_i^n$ is open in $\mathbb{P}_o^n$ and $f|_{O\cap\mathbb{P}_i^n} \in \mathcal{O}_{\mathbb{P}_o^n}(O \cap \mathbb{P}_i^n)$, for any $v \in O \cap \mathbb{P}_i^n$ there exists an open neighbourhood U_v of v in $O \cap \mathbb{P}_i^n$ and $g,h \in K[\mathbb{P}_o^n]$ such that $g(y) \neq 0$ and

$$f(y) = \frac{h(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o})(y)}{g(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o})(y)}$$

for all $y = [y_o, y_1, \ldots, y_n] \in U_v$. Notice

$X_o^m h(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o})$, $X_o^m g(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o}) \in K[X_o, X_1, \ldots, X_n]$ for certain large numbers $m \in \mathbb{N}$ and

$$f(y) = \frac{h(\frac{\overset{\vee}{y_o}}{y_o}, \frac{y_1}{y_o}, \ldots, \frac{y_n}{y_o})}{g(\frac{\overset{\vee}{y_o}}{y_o}, \frac{y_1}{y_o}, \ldots, \frac{y_n}{y_o})} = \frac{y_o^m h(\frac{\overset{\vee}{y_o}}{y_o}, \frac{y_1}{y_o}, \ldots, \frac{y_n}{y_o})}{y_o^m g(\frac{\overset{\vee}{y_o}}{y_o}, \frac{y_1}{y_o}, \ldots, \frac{y_n}{y_o})} .$$

Since $y \in U_v \subset \mathbb{P}_i^n$, we have

$$f(y) = \frac{y_o^m h(\frac{\overset{\vee}{y_o}}{y_o}, \frac{y_1}{y_o}, \ldots, \frac{y_n}{y_o})/y_i^l}{y_o^m g(\frac{\overset{\vee}{y_o}}{y_o}, \frac{y_1}{y_o}, \ldots, \frac{y_n}{y_o})/y_i^l} = \frac{X_o^m h(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o})/X_i^l}{X_o^m g(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o})/X_i^l}(y) ,$$

where $l = \deg X_o^m h(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o}) = \deg X_o^m g(\frac{\overset{\vee}{X_o}}{X_o}, \frac{X_1}{X_o}, \ldots, \frac{X_n}{X_o})$. Hence $f|_{O\cap\mathbb{P}_i^n} \in \mathcal{O}_{\mathbb{P}_i^n}(O \cap \mathbb{P}_i^n)$, because $X_o^m h/X_i^l$ and X_o^{mg}/X_i^l belong to $K[\mathbb{P}_i^n]$. Q.E.D.

__(11.4) Proposition__. $(\mathbb{P}^n, \mathcal{G}_{\mathbb{P}^n})$ is a variety over K with finite affine open coverings $\mathbb{P}_o^n, \mathbb{P}_1^n, \ldots, \mathbb{P}_n^n$.

__Proof__. It is enough to show that

$$\Delta(\mathbb{P}^n) \cap (\mathbb{P}_i^n \times \mathbb{P}_j^n)$$

is closed in $\mathbb{P}_i^n \times \mathbb{P}_j^n$ for any $0 \le i,j \le n$. Since

$$\Delta(\mathbb{P}^n) \cap (\mathbb{P}_i^n \times \mathbb{P}_j^n) = \{(x,x) \mid x \in \mathbb{P}_i^n \cap \mathbb{P}_j^n\}$$

and
$$(\frac{X_l}{X_i} \otimes 1 - \frac{X_j}{X_i} \otimes \frac{X_l}{X_j})(x,y) = 0$$

for all $l = 0,1,\ldots,n$ if and only if

$$(x,y) \in \Delta(\mathbb{P}^n) \cap (\mathbb{P}^n_i \times \mathbb{P}^n_j) \ ,$$

$\Delta(\mathbb{P}^n) \cap (\mathbb{P}^n_i \times \mathbb{P}^n_j)$ is closed in $\mathbb{P}^n_i \times \mathbb{P}^n_j$. Q.E.D.

Exercise 45. Let $A = (a_{ij}) \in GL(n+1,K)$ and φ be a map of $\mathbb{P}^n$ into $\mathbb{P}^n$ which takes $[x_o,x_1,\ldots,x_n] \in \mathbb{P}^n$ to $[(x_o,\ldots,x_n)A] \in \mathbb{P}^n$. Show that φ is an isomorphism of varieties of $\mathbb{P}^n$ onto $\mathbb{P}^n$.

(11.5) Definition. We call a closed subvariety of $(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n})$ a projective variety over K and an open subvariety of a projective variety a quasi-projective variety over K .

(11.6) Lemma. Let $V_i = \{[x_o,x_1,\ldots,x_n] \in \mathbb{P}^n \mid x_i = 0\}$, where $i = 0,1,\ldots,n$, then V_i is a closed subvariety of $\mathbb{P}^n$ and isomorphic to $\mathbb{P}^{n-1}$.

Proof. Since $V_i = \mathbb{P}^n - \mathbb{P}^n_i$, it is clear that V_i is a closed subvariety of $\mathbb{P}^n$ with a finite affine open covering $\{V_i \cap \mathbb{P}^n_j \mid j = 0,1,\ldots,n\}$. Let φ be a map of $\mathbb{P}^{n-1}$ into V_i which takes $[x_o,\ldots,x_{n-1}] \in \mathbb{P}^{n-1}$ to $[x_o,x_1,\ldots,x_{i-1},0,x_i,\ldots,x_{n-1}] \in V_i$.

$$\varphi: \mathbb{P}^{n-1} \longrightarrow V_i$$

$$(\varphi:[x_o,\ldots,x_{n-1}] \longrightarrow [x_o,\ldots,x_{i-1},0,x_i,\ldots,x_{n-1}]$$

It is clear that φ is well-defined and bijective. Since

$$\varphi(\mathbb{P}^{n-1}_j) = V_i \cap \mathbb{P}^n_j \qquad (0 \leq j \leq i-1) \ ,$$

$$\varphi(\mathbb{P}^{n-1}_j) = V_i \cap \mathbb{P}^n_{j+1} \qquad (i \leq j \leq n-1)$$

and $\varphi|_{\mathbb{P}^{n-1}_j}$ is an isomorphism of affine varieties of $\mathbb{P}^{n-1}_j$ onto $V_i \cap \mathbb{P}^n_j$ or $V_i \cap \mathbb{P}^n_{j+1}$ for each $0 \leq j \leq n-1$, φ is an isomorphism of varieties from Proposition 10.7. Q.E.D.

<u>(11.7) Proposition</u>. Let $\mathbb{P}^n$ and $\mathbb{P}^m$ be projective n- and m-spaces over K respectively. Let φ be a map of $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{(n+1)(m+1)-1}$ which takes $([x_o,\dots,x_n],[y_o,\dots,y_m]) \in \mathbb{P}^n \times \mathbb{P}^m$ to $[(x_oy_o,\dots,x_oy_m,x_1y_o,\dots,x_1y_m,\dots,x_ny_o,\dots,x_ny_m] \in \mathbb{P}^{(n+1)(m+1)-1}$. Then $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$ is a closed subvariety of $\mathbb{P}^{mn+m+n}$ and φ is an isomorphism of varieties of $\mathbb{P}^n \times \mathbb{P}^m$ onto $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$.

<u>Proof</u>. Let $q = (n+1)(m+1)-1$. Clearly φ is a well-defined injective map. Let $\mathbb{P}^q_{ij} = \mathbb{P}^q_{i(m+1)+j}$ where $0 \leq i \leq n$ and $0 \leq j \leq m$, then we have

$$\varphi(\mathbb{P}^n_i \times \mathbb{P}^m_j) \subset \mathbb{P}^q_{ij}$$

for any i,j. Since $(\varphi|_{\mathbb{P}^n_i \times \mathbb{P}^m_j})^*(K[\mathbb{P}^q_{ij}]) = K[\mathbb{P}^n_i] \otimes K[\mathbb{P}^m_j]$, $\varphi(\mathbb{P}^n_i \times \mathbb{P}^m_j)$ is closed in $\mathbb{P}^q_{ij}$ and the map

$$\varphi|_{\mathbb{P}^n_i \times \mathbb{P}^m_j} : \mathbb{P}^n_i \times \mathbb{P}^m_j \longrightarrow \varphi(\mathbb{P}^n_i \times \mathbb{P}^m_j)$$

is an isomorphism of affine varieties from Proposition 2.5. Since $\varphi(\mathbb{P}^n \times \mathbb{P}^m) \cap \mathbb{P}^q_{ij} = \varphi(\mathbb{P}^n_i \times \mathbb{P}^m_j)$ is closed in $\mathbb{P}^q_{ij}$ for any i,j, $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$ is closed in $\mathbb{P}^q$. Thus $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$ is a closed subvariety of $\mathbb{P}^q$.

It is clear that φ is an isomorphism of varieties of $\mathbb{P}^n \times \mathbb{P}^m$ onto $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$, because

$$\varphi|_{\mathbb{P}^n_i \times \mathbb{P}^m_j} : \mathbb{P}^n_i \times \mathbb{P}^m_j \longrightarrow \varphi(\mathbb{P}^n_i \times \mathbb{P}^m_j) = \varphi(\mathbb{P}^n \times \mathbb{P}^m) \cap \mathbb{P}^q_{ij}$$

is an isomorphism of affine varieties (see Proposition 10.7). Q.E.D.

<u>(11.8) Corollary</u>. Products of projective varieties are projective varieties.

<u>Proof</u>. Let X and Y be closed subvarieties of $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively. Since $(X\times Y) \cap (\mathbb{P}^n_i \times \mathbb{P}^m_j) = (X \cap \mathbb{P}^n_i) \times (Y \cap \mathbb{P}^m_j)$ is closed in $\mathbb{P}^n_i \times \mathbb{P}^m_j$ (see Proposition 3.4) for any i,j, $X\times Y$ is closed in $\mathbb{P}^n \times \mathbb{P}^m$. Since $\varphi(X \times Y)$ is closed in $\varphi(\mathbb{P}^n \times \mathbb{P}^m)$ and $\varphi|_{X\times Y}$ is

an isomorphism of varieties of $X\times Y$ onto $\varphi(X\times Y)$, $X\times Y$ is a projective variety. Q.E.D.

Now we introduce Grassmann varieties and flag varieties over K, which are examples of projective varieties. First we review the idea of <u>alternating products</u>. Let R be a commutative ring with unity element 1 and E and F be R-modules. We call an R-multilinear map of E^r $(= \underbrace{E\times\ldots\times E}_{r})$ into F alternating if $f(x_1,x_2,\ldots,x_r) = 0$ for any $(x_1,\ldots,x_r) \in E^r$ which satisfies $x_i = x_j$ for some $i \neq j$ $(1 \leq i,j \leq r)$. We use a_r to denote the submodule of $T^r(E)$ $(= \underbrace{E \otimes\ldots\otimes E}_{r}$, the tensor product over $R)$ generated by all elements of the following form

$$x_1 \otimes\ldots\otimes x_r\ , \quad \text{where } x_i = x_j \text{ for some } i \neq j\ .$$

We define $\Lambda^r E = T^r(E)/a_r$. Then we have a canonical alternating R-multilinear map τ of E^r into $\Lambda^r E$ which takes $(x_1,\ldots,x_r) \in E^r$ to $x_1 \otimes\ldots\otimes x_r + a_r \in \Lambda^r E$. It can be easily shown that for any R-multilinear alternating map f of E^r into an arbitrary R-module F there exists a unique linear map f^* of $\Lambda^r E$ into F which makes the following diagram commutative.

$$\begin{array}{ccc} & & \Lambda^r E \\ & \overset{\tau}{\nearrow} & \\ E^r & \circlearrowright & \Big\downarrow \exists_1 f^* \\ & \underset{f}{\searrow} & \\ & & F \end{array} \quad .$$

We write $x_1\wedge\ldots\wedge x_r$ for $\tau(x_1,\ldots,x_r)$ where $(x_1,\ldots,x_r) \in E^r$.

<u>(11.9) Proposition</u>. Let E be a free R-module with a basis $\{e_1,\ldots,e_n\}$ over R. Then $\Lambda^r E = \{0\}$ if $r > n$. If $1 \leq r \leq n$, $\Lambda^r E$ is a free R-module with an R-basis

$$\{e_{i_1}\wedge\ldots\wedge e_{i_r} \mid i_1<\ldots<i_r\}\ .$$

For a proof of the above proposition see Lang [1, Proposition 9.1 on P.590].

(11.10) Proposition. Let V be an n-dimensional vector space over K and $\mathcal{G}_d(V)$ be the set of all d-dimensional subspaces of V $(1 \leq d \leq n)$. Let ψ be a map of $\mathcal{G}_d(V)$ into $P(\Lambda^d V)$ which takes $D \in \mathcal{G}_d(V)$ to $\Lambda^d D \in P(\Lambda^d V)$. Then

(1) ψ is well-defined and injective and

(2) $\psi(\mathcal{G}_d(V))$ is closed in the projective variety $P(\Lambda^d V)$.

Exercise 46. Verify that the variety structure of $P(\Lambda^d V)$ is uniquely defined up to isomorphism (see Exercise 45 on P.127).

For the proof of the above proposition we need the following two lemmas.

(11.11) Lemma. Let V be an n-dimensional vector space over a field k and D, D' be two d-dimensional k-subspaces of V $(1 \leq d \leq n)$. Assume that $\Lambda^d D = \Lambda^d D'$, then $D = D'$.

Proof. Since $\Lambda^d D = \Lambda^d D'$, we have $D \cap D' \neq \{0\}$ from Proposition 11.9. Let $\{v_1, v_2, \ldots, v_d\}$ be a k-basis of D such that

$$D \cap D' = kv_r \oplus \ldots \oplus kv_d$$

for certain $1 \leq r \leq d$, then D' has a k-basis of the form $\{v_r, \ldots, v_d, v_{d+1}, \ldots, v_{d+r-1}\}$. Since

$$\dim_k(D+D') = \dim_k D + \dim_k D' - \dim_k(D \cap D'),$$

$\{v_1, \ldots, v_d, v_{d+1}, \ldots, v_{d+r-1}\}$ forms a k-basis of $D+D'$. Since $\Lambda^d D = kv_1 \wedge v_2 \wedge \ldots \wedge v_d = \Lambda^d D' = kv_r \wedge \ldots \wedge v_d \wedge \ldots \wedge v_{d+r-1}$, we have $D = D'$ from Proposition 11.9. Q.E.D.

(11.12) Lemma. Let (U,A), (V,B) be affine varieties over K and φ be a morphism of affine varieties of U into V, then

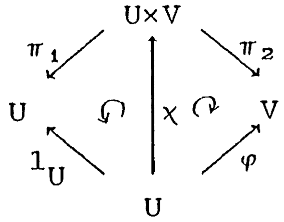
$$\{(u, \varphi(u)) \mid u \in U\}$$

is closed in $(U \times V, A \otimes B)$.

Proof. Let χ be a map of U into $U \times V$ such that

$$\begin{array}{l} \chi : U \longrightarrow U\times V \\ (\chi : u \longrightarrow (u,\varphi(u))) \ , \end{array}$$

then from Proposition 3.3, χ is a morphism of affine varieties.



Since $\chi^*(A \otimes B) = A$, $\chi(U) = \{(u,\varphi(u)) \mid u \in U\}$ is closed in $U\times V$ from Proposition 2.5.
Q.E.D.

<u>Proof of Proposition 11.10</u>. (1) is clear from Lemma 11.11.
(2) Let $\{v_1,v_2,\dots,v_n\}$ be a fixed K-basis of V , then $\Lambda^d V$ has a K-basis $\{v_{i_1}\wedge\dots\wedge v_{i_d} \mid i_1 <\dots< i_d\}$. Let U be an affine open set of $P(\Lambda^d V)$ which consists of points of $P(\Lambda^d V)$ whose homogeneous coordinate relative to $v_1\wedge v_2\wedge\dots\wedge v_d$ is non-zero. It is enough to show that $\psi(\mathcal{G}_d(V)) \cap U$ is closed in U . Let $D_o = Kv_1+\dots+Kv_d \in \mathcal{G}_d(V)$ and π be a projection of V onto D_o which takes

$$c_1v_1+\dots+c_dv_d+c_{d+1}v_{d+1}+\dots+c_nv_n \quad (c_1,\dots,c_n \in K)$$

to $c_1v_1+\dots+c_dv_d$. Now let $D \in \mathcal{G}_d(V)$ and $\{v_1',\dots,v_d'\}$ be a K-basis of D , then we have

$$v_i' = \sum_{j=1}^{n} a_{ij}v_j ,$$

where $1 \leq i \leq d$ and $a_{ij} \in K$. Since

$$\begin{aligned} v_1'\wedge\dots\wedge v_d' &= ((\sum_{j=1}^{d} a_{1j}v_j)+(\sum_{j=d+1}^{n} a_{1j}v_j)) \wedge\dots \\ &\quad \dots\wedge ((\sum_{j=1}^{d} a_{dj}v_j)+(\sum_{j=d+1}^{n} a_{dj}v_j)) \\ &= (\sum_{j=1}^{d} a_{1j}v_j) \wedge\dots\wedge (\sum_{j=1}^{n} a_{dj}v_j) +\dots \\ &= \det \begin{bmatrix} a_{11}\cdots\cdots a_{1d} \\ a_{21} \qquad a_{2d} \\ \vdots \qquad\quad \vdots \\ a_{d1}\cdots\cdots a_{dd} \end{bmatrix} v_1\wedge\dots\wedge v_d+\dots , \end{aligned}$$

$\psi(D) \in U$ if and only if $\det \begin{bmatrix} a_{11} \cdots\cdots a_{1d} \\ a_{21} \qquad a_{2d} \\ \vdots \qquad \vdots \\ a_{d1} \cdots\cdots a_{dd} \end{bmatrix} \neq 0$, i.e.,

$\pi(D) = D_o$.

Assume that $\psi(D) \in U$, i.e., $\pi(D) = D_o$. Since

$$(\pi|D)^{-1}(v_i) = v_i + \sum_{j=d+1}^{n} c_{ij}v_j$$

$(i = 1,2,\ldots,d;\ c_{ij} \in K)$ form a K-basis of D , $\psi(D)$ is generated by

$$(\pi|D)^{-1}(v_1)\wedge\ldots\wedge(\pi|D)^{-1}(v_d)$$

$$= v_1\wedge\ldots\wedge v_d + \sum_{i=1}^{d}(v_1\wedge\ldots\wedge\ (\sum_{j=d+1}^{n} c_{ij}v_j)\wedge\ldots\wedge v_d)+L ,$$

where L is a K-linear combination of the terms $v_{i_1}\wedge\ldots\wedge v_{i_d}$ such that

$$|\{v_{i_1},\ldots,v_{i_d}\} \cap \{v_1,\ldots,v_d\}| \leq d-2 .$$

Thus there exists a one-to-one correspondence between $\psi(\mathscr{G}_d(V)) \cap U$ and affine coordinates

$$(\underbrace{\ldots\ c_{ij}\ldots}_{d(n-d)},\ldots\ f_k(\ldots\ c_{ij}\ldots)\ldots) ,$$

where c_{ij}'s are arbitrary $d(n-d)$ scalars, and f_k's are polynomial functions on $K^{d(n-d)}$ independent of $\psi(D)$. Hence $\psi(\mathscr{G}_d(V)) \cap U$ is closed in U from Lemma 11.12. Q.E.D.

We call these projective varieties $\mathscr{G}_d(V)$ <u>Grassman varieties</u> over K .

Now let V be an n-dimensional vector space over K as before. We define $\mathscr{F}(V)$ to be the set of all sequences of K-subspaces $(0,V_1,\ldots,V_n)$ of V such that

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq\cdots\subsetneq V_n = V$$

(i.e. $\dim_K V_{i+1}/V_i = 1$, $i = 0,1,\ldots,n-1$) where O is the O-dimensional subspace.

<u>(11.13) Proposition</u>. Let $\mathcal{F}(V)$ be as above and φ be a map of $\mathcal{F}(V)$ into $\mathcal{G}_1(V) \times \mathcal{G}_2(V) \times\ldots\times \mathcal{G}_n(V)$ which takes $(O,V_1,\ldots,V_n) \in \mathcal{F}(V)$ to $(V_1,V_2,\ldots,V_n) \in \mathcal{G}_1(V) \times\ldots\times \mathcal{G}_n(V)$. Then

(1) φ is injective and

(2) $\varphi(\mathcal{F}(V))$ is closed in the projective variety $\mathcal{G}_1(V) \times \mathcal{G}_2(V) \times\ldots\times \mathcal{G}_n(V)$.

<u>Proof</u>. (2) We have the following embeddings

$$\mathcal{F}(V) \xrightarrow{\varphi} \mathcal{G}_1(V) \times\ldots\times \mathcal{G}_n(V) \hookrightarrow P(V) \times P(\Lambda^2 V) \times\ldots\times P(\Lambda^n)$$

$$O \subset V_1 \subset V_2 \subset\ldots\subset V_n \longrightarrow (V_1,V_2,\ldots,V_n) \longrightarrow (V_1,\Lambda^2 V_2,\ldots,\Lambda^n V_n) .$$

Let $(v_1,v_2,\ldots,v_n)$ be a fixed K-basis of V . U^d denotes the affine open set of $P(\Lambda^d V)$ which consists of points of $P(\Lambda^d V)$ whose homogeneous coordinate relative to $v_1 \wedge v_2 \wedge\ldots\wedge v_d$ is non-zero, where $d = 1,2,\ldots,n$. It is enough to prove that

$$\varphi(\mathcal{F}(V)) \cap (U^1 \times U^2 \times\ldots\times U^n)$$

is closed in $U^1 \times U^2 \times\ldots\times U^n$.

Now suppose that $D^{(d)}$ be an element of $\mathcal{G}_d(V)$ such that $\Lambda^d D^{(d)} \in U^d$ $(1 \leq d \leq n)$, then from the proof of Proposition 11.10 $D^{(d)}$ has the K-basis as follows

$$(v_i + \sum_{j=d+1}^{n} c_{ij}^d v_j \mid i = 1,2,\ldots,d) \quad (c_{ij}^d \in K) .$$

Assume that $D^{(d)} \subset D^{(d+1)}$ $(d = 1,2,\ldots,n-1)$. Since $(v_i + \sum_{j=d+1}^{n} c_{ij}^d v_j \mid i = 1,2,\ldots,d)$ $(c_{ij}^d \in K)$ is contained in the K-subspace of V generated by

$$(v_i + \sum_{j=d+2}^{n} c_{ij}^{d+1} v_j \mid i = 1,2,\ldots,d+1) \quad (c_{ij}^{d+1} \in K) ,$$

$v_i + \sum_{j=d+1}^{n} c_{ij}^d v_j$ is a K-linear combination of $v_i + \sum_{j=d+2}^{n} c_{ij}^{d+1} v_j$

and $v_{d+1} + \sum_{j=d+2}^{n} c_{d+1,j}^{d+1} v_j$, where $i = 1,2,\ldots,d$. Thus we have

$$(*) \quad \sum_{j=d+1}^{n} c_{ij}^{d} v_j = \sum_{j=d+2}^{n} c_{ij}^{d+1} v_j + c_{i,d+1}^{d}\left(v_{d+1} + \sum_{j=d+2}^{n} c_{d+1,j}^{d+1} v_j\right)$$

for $1 \leq i \leq d$. Conversely, $D^{(d)} \subset D^{(d+1)}$ if (*) holds for $1 \leq i \leq d$. Since (*) can be written as certain polynomial condition on $\{c_{ij}^{d}\}$ which is independent of $\{c_{ij}^{d}\}$, $\varphi(\mathcal{F}(V)) \cap (U^1 \times U^2 \times \ldots \times U^n)$ is closed in $U^1 \times U^2 \times \ldots \times U^n$.

Q.E.D.

We call these projective varieties $\mathcal{F}(V)$ <u>flag varieties</u> over K .

12. Complete varieties

Finally we define complete varieties and show that projective varieties are complete. The idea of completeness of varieties is very important in the structure theory of algebraic groups.

(12.1) Definition. A variety V over K is said to be complete if the projection map

$$\pi_2 : V\times W \to W$$

is a closed map (i.e. π_2 maps a closed subset of $V\times W$ to a closed subset of W) for any variety W over K .

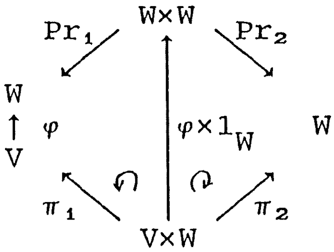
(12.2) Proposition.

(1) A closed subvariety of a complete variety is complete.
(2) Let V and W be varieties over K and $\varphi : V \to W$ be a morphism of varieties. If V is complete, then $\varphi(V)$ is closed in W and is a complete variety.
(3) Let V be a complete affine variety over K then V is a finite set.
(4) If V and W are complete varieties over K , then $V\times W$ is complete.

Proof. (1) and (4) are straightforward.
(2) Let $\varphi : V \to W$ be a given morphism of a complete variety V into a variety W . We define $\varphi\times 1_W$ to be a map of $V\times W$ into $W\times W$ such that

$$\varphi\times 1_W : V\times W \longrightarrow W\times W$$
$$(\varphi\times 1_W : (v,w) \to (\varphi(v),w)) \ .$$

It is clear that $\varphi\times 1_W$ is a morphism of varieties which makes the following diagram commutative.



where $Pr_1 : W\times W \longrightarrow W$ and $Pr_2 : W\times W \longrightarrow W$. Hence
$(Pr_1 : (x,y) \longrightarrow x)$ $(Pr_2 : (x,y) \longrightarrow y)$
$(\varphi\times 1_W)^{-1}(\{(w,w) \mid w \in W\}) = \{(v,\varphi(v)) \mid v \in V\}$ is closed in $V\times W$.
Since V is complete,

$$\pi_2(\{(v,\varphi(v)) \mid v \in V\}) = \varphi(V)$$

is closed in W . Hence $\varphi(V)$ is a closed subvariety.

Now let W' be an arbitrary variety over K and $\pi_2' : \varphi(V) \times W' \to W'$ be the projection. Let S be a closed subset of $\varphi(V) \times W'$. Since

$$\pi_2'(S) = \pi_2((\varphi\times 1_{W'})^{-1}(S)) ,$$

where π_2 is the projection of $V\times W'$ onto W' , $\pi_2'(S)$ is closed in W' . Hence $\varphi(V)$ is complete.

(3) Let $(V,K[V])$ be a complete affine variety over K and $(K,K[X])$ be the affine 1-space. Let $f \in K[V]$. Since the closed set

$$S = \{(v,x) \in V\times K \mid (f \otimes X - 1 \otimes 1)(v,x) = 0\}$$

does not contain any point of $V\times\{0\}$ and V is complete, $\pi_2(S)$ is a proper closed subset in K . Hence $\pi_2(S)$ is a finite set. Thus $f(V)$ is also a finite set for any $f \in K[V]$. Therefore, V is a finite set. Q.E.D.

<u>(12.3) Corollary</u>. Let φ be a morphism of an irreducible complete variety V over K into an affine variety W over K , then φ is a constant.

<u>Exercise 47</u>. Prove Proposition 12.2.(1) and (4).

<u>(12.4) Theorem</u>. A projective variety over K is complete.

<u>Proof</u> (see Humphreys [2, §6]). From Proposition 12.2 we only have to prove that $\mathbb{P}^n$ is complete, i.e., the projection map

$$\pi_2 : \mathbb{P}^n \times W \to W$$

is a closed map for any variety W over K. Let $\{O_j \mid 1 \leq j \leq m\}$ be a finite cover of affine open sets of W. Assume that F is a closed subset of $\mathbb{P}^n \times W$. Since

$$\pi_2(F) \cap O_j = \pi_2(F \cap (\mathbb{P}^n \times O_j))$$

and $\pi_2(\mathbb{P}^n \times O_j) = O_j$ for any j, it is enough to show that

$$\pi_2 \mid \mathbb{P}^n \times O_j$$

is a closed map for any j. Thus we only have to show that

$$\pi_2 : \mathbb{P}^n \times V \to V$$

is a closed map for any affine variety V over K. Let $V_1, V_2, \ldots, V_l$ be the irreducible components of V and F be a closed subset of $\mathbb{P}^n \times V$. Since $\pi_2((\mathbb{P}^n \times V_i) \cap F) = \pi_2(F) \cap V_i$ for any i, we can also assume that V is irreducible.

Now let $U_i = \mathbb{P}^n_i \times V$ $(i = 0,1,\ldots,n)$ be the affine open covering of $\mathbb{P}^n \times V$. Then we have

$$K[U_i] = K[X_o/X_i, \ldots, \check{X}_i/X_i, \ldots, X_n/X_i] \otimes K[V] .$$

Since $(\pi_2|_{U_i})^* : K[V] \longrightarrow K[X_o/X_i, \ldots, X_n/X_i] \otimes K[V]$ is injective and

$$((\pi_2|_{U_i})^* : \; f \longrightarrow 1 \otimes f)$$

$$\zeta = \sum_t (X_o/X_i)^{t_o} \cdots (X_o/X_i)^{t_n} \otimes g_t = 0$$

in $K[U_i]$ if and only if $1 \otimes g_t = 0$ for all t, $K[U_i]$ can be considered as a polynomial ring $K[V][X_o/X_i, \ldots, X_n/X_i]$ in n-variables $\{X_o/X_i, \ldots, X_n/X_i\}$ over $K[V]$.

Let Z be a closed subset of $\mathbb{P}^n \times V$ and v be any point of $V - \pi_2(Z)$. In order to show that $\pi_2(Z)$ is closed in V it is enough to find an element f in $K[V] - \mathcal{I}_V(\{v\})$ such that $f(\pi_2(Z)) = \{0\}$, i.e., $v \in V_f \subset V - \pi_2(Z)$. Since $f \circ \pi_2|_Z = 0$ for such an f, we have

$$f\circ\pi_2|_{U_i} \in \mathscr{I}_{U_i}(Z\cap U_i)$$

for all i. Now let I_m be the set of all homogeneous polynomials $f(X_o,X_1,\dots,X_n) \in K[V][X_o,\dots,X_n]$ of degree m such that

$$f(X_o/X_i,\dots,X_n/X_i) \in \mathscr{I}_{U_i}(Z\cap U_i)$$

for all $0 \le i \le n$. Let $I = \sum_{m=0}^{\infty} I_m$, then I is a homogeneous ideal of $K[V][X_o,\dots,X_n] = \sum_{m=0}^{\infty} H_m$, where H_m is the set of all homogeneous polynomials of degree m. Let

$$f \in \mathscr{I}_{U_i}(Z\cap U_i) \quad (\subset K[V][X_o/X_i,\dots,X_n/X_i]) ,$$

where $0 \le i \le n$, then

$$X_i^m f \in H_m \quad (\subset K[V][X_o,\dots,X_n])$$

for all $m > \deg f$. Since $\dfrac{X_i^m f}{X_j^m} \in K[V][X_o/X_j,\dots,X_n/X_j]$ and

$\left.\dfrac{X_i^m f}{X_j^m}\right|_{Z\cap U_j\cap U_i} = 0$ and $\left.\dfrac{X_i^m f}{X_j^m}\right|_{(Z\cap U_j)-U_i} = 0$ for all $0 \le j \le n$, $X_i^m f \in I_m$ for any $m > \deg f$. Since $Z\cap U_i$ and $\mathbb{P}_i^n \times \{v\}$ are disjoint closed subsets of U_i, we have

$$\mathscr{V}_{U_i}(\mathscr{I}_{U_i}(Z\cap U_i) + \mathscr{I}_{U_i}(\mathbb{P}_i^n \times \{v\}))$$

$$= \mathscr{V}_{U_i}(\mathscr{I}_{U_i}(Z\cap U_i)) \cap \mathscr{V}_{U_i}(\mathscr{I}_{U_i}(\mathbb{P}_i^n \times \{v\}))$$

$$= (Z\cap U_i) \cap (\mathbb{P}_i^n \times \{v\}) = \emptyset .$$

Hence $\mathscr{I}_{U_i}(Z\cap U_i) + \mathscr{I}_{U_i}(\mathbb{P}_i^n \times \{v\}) = K[V][X_o/X_i,\dots,X_n/X_i]$ for any $v \in V - \pi_2(Z)$, because

$$\sqrt{\mathscr{I}_{U_i}(Z\cap U_i) + \mathscr{I}_{U_i}(\mathbb{P}_i^n \times \{v\})} = \mathscr{I}_{U_i}(\emptyset) = K[V][X_o/X_i,\dots,X_n/X_i]$$

from the Hilbert's Nullstellensatz. Since

$$\mathscr{I}_{U_i}(\mathbb{P}_i^n \times \{v\}) = \mathscr{I}_V(\{v\})[X_o/X_i,\dots,X_n/X_i] ,$$

there exist $f_i \in \mathcal{I}_{U_i}(Z \cap U_i)$, $c_{ij} \in \mathcal{I}_V((v))$ and $g_{ij} \in K[X_o/X_i,\ldots,X_n/X_i]$ such that

$$1 = f_i + \sum_j c_{ij}g_{ij}$$

for each $0 \leq i \leq n$. Since $X_i^m f_i \in I_m$ for any $m > \deg f_i$, there exists a natural number $N_i > 0$ for each $0 \leq i \leq n$ such that

$$X_i^m = X_i^m f_i + X_i^m \sum_j c_{ij}g_{ij} \in I_m + \mathcal{I}_V((v))\ H_m$$

for any $m > N_i$. Therefore, there exists a natural number $N > 0$ such that for all $m > N$

$$X_i^m \in I_m + \mathcal{I}_V((v))\ H_m ,$$

where $0 \leq i \leq n$.

Now assume that $m > (n+1)N$ and $\deg X_o^{m_o} X_1^{m_1}\ldots X_n^{m_n} = m$, i.e. $m_o+m_1+\ldots+m_n = m$, then there exists m_i such that $m_i > N$. Since $X_i^{m_i} \in I_{m_i} + \mathcal{I}_V((v))\ H_{m_i}$, $X_o^{m_o}\ldots X_n^{m_n} \in I_m + \mathcal{I}_V((v))\ H_m$. Hence

$$H_m = I_m + \mathcal{I}_V((v))\ H_m$$

for all $m > (n+1)N$. Thus we have a finitely generated $K[V]$-module H_m/I_m which satisfies the condition

$$\mathcal{I}_V((v))(H_m/I_m) = H_m/I_m \quad (m > (n+1)N) .$$

Hence from the Nakayama's Lemma there exists $f \in K[V] - \mathcal{I}_V((v))$ such that $f\cdot H_m/I_m = 0$, i.e.,

$$fH_m \subset I_m \quad (m > (n+1)N) .$$

Since $X_j^m f \in I_m$ $(m > (n+1)N)$ where $0 \leq j \leq n$, $(X_j^m f)(X_o/X_i,\ldots,X_n/X_i) \in \mathcal{I}_{U_i}(Z \cap U_i)$ for each $0 \leq i \leq n$. Hence for all $0 \leq j \leq n$ we have

$$f\cdot(X_j/X_j)^m \in \mathcal{I}_{U_j}(Z \cap U_j) ,$$

which implies $(1 \otimes f)(Z \cap U_j) = f\circ(\pi_2|_{U_j})(Z \cap U_j) = 0$ for all $0 \leq j \leq n$. Therefore $f(\pi_2(Z)) = 0$ as desired. Q.E.D.

13. Dimension and tangent spaces in general

(13.1) Proposition. Let $(X,\mathcal{O}_X)$ be an irreducible prevariety over K and U,V be affine open sets of X, then U and V are irreducible and

$$K(U) \cong K(V)$$

as fields, where $K(U)$ and $K(V)$ are the quotient fields of the coordinate rings of U and V respectively.

Proof. Since X is irreducible, we have $\bar{U} = \bar{V} = X$ and U,V are irreducible from Proposition 5.4. Since

$$\mathcal{O}_X(U \cap V) = \mathcal{O}_U(U \cap V) = \mathcal{O}_V(U \cap V)$$

from the definition of affine open set, we have

$$\bigcap_{x \in U \cap V} K[U]_{\mathcal{I}_U(x)} \cong \bigcap_{x \in U \cap V} K[V]_{\mathcal{I}_V(x)}$$

from Proposition 9.6. Hence $K(U)$ is isomorphic to $K(V)$ as field.

Q.E.D.

(13.2) Definition. Let $(X,\mathcal{O}_X)$ be an irreducible prevariety over K and U be an affine open set of X. We write $K(X)$ for $K(U)$ and call it the function field of $(X,\mathcal{O}_X)$. We define the dimension of $(X,\mathcal{O}_X)$ to be $\dim U$. In case $(X,\mathcal{O}_X)$ has more than one irreducible components

$$X = X_1 \cup \ldots \cup X_l \ ,$$

we define dim X to be the maximum of $\{\dim X_k \mid 1 \leq k \leq l\}$.

Exercise 48. (1) Let $(X,\mathcal{O}_X)$ be a prevariety over K. Then $\dim X = 0$ if and only if X is a non-empty finite set.

(2) Let $(X,\mathcal{O}_X)$ be an irreducible prevariety over K and Y be a proper non-empty closed irreducible subset of X, then we have $\dim Y < \dim X$.

<u>(13.3) Proposition</u>. Let $(X, \mathcal{O}_X)$ be a prevariety over K. Let $x \in X$, then we have

$$K[U]_{\mathcal{I}_U(x)} \cong K[V]_{\mathcal{I}_V(x)}$$

as K-algebras for any affine open sets U and V of X which contain x.

<u>Proof</u>. Let $\hat{\mathcal{O}}_x = \{(U_x, f) \mid U_x$ is an open neighbourhood of x in U and $f \in \mathcal{O}_U(U_x)\}$ and $\hat{\mathcal{O}}'_x = \{(V_x, f) \mid V_x$ is an open neighbourhood of x in V and $f \in \mathcal{O}_V(V_x)\}$. From Proposition 9.5 it is enough to prove that $\hat{\mathcal{O}}_x/\sim$ and $\hat{\mathcal{O}}'_x/\sim$ are isomorphic as K-algebras.

Let ζ be a map of $\hat{\mathcal{O}}_x/\sim$ into $\hat{\mathcal{O}}'_x/\sim$ which takes $\overline{(U_x, f)}$ to $\overline{(V \cap U_x, f|_{V \cap U_x})}$, then it is clear that ζ is well-defined and also a K-algebra isomorphism. Q.E.D.

We write $\mathcal{O}_x$ for the above local ring $K[U]_{\mathcal{I}_U(x)}$ at x.

Now let $(V, \mathcal{O}_V)$ be an affine variety over K and v be any fixed point of V. Since $\mathcal{I}_V(v)$ is a maximal ideal of $K[V]$ and $K[V]/\mathcal{I}_V(v) \cong K$, we can consider $\mathcal{I}_V(v)/(\mathcal{I}_V(v))^2$ to be a K-vector space from Lemma 7.12. Let γ be a tangent vector to V at v, i.e., γ is a K-linear map of $K[V]$ into K such that

$$\gamma(ab) = a(v)\gamma(b) + \gamma(a)b(v)$$

for any $a, b \in K[V]$, then $\gamma((\mathcal{I}_V(v)^2) = 0$ and we can define a K-linear map $\varphi(\gamma)$ of $\mathcal{I}_V(v)/\mathcal{I}_V(v)^2$ into K as follows.

$$\varphi(\gamma) : \mathcal{I}_V(v)/\mathcal{I}_V(v)^2 \longrightarrow K$$
$$(\varphi(\gamma) : f + \mathcal{I}_V(v)^2 \longrightarrow \gamma(f)\ .$$

<u>(13.4) Lemma</u>. The map $\varphi : T(V)_v \to \mathrm{Hom}_K(\mathcal{I}_V(v)/\mathcal{I}_V(v)^2, K)$ is a bijective K-linear map.

Proof. It is clear that φ is well-defined and K-linear. Let $\lambda \in \mathrm{Hom}_K(\mathscr{I}_V(v)/\mathscr{I}_V(v)^2, K)$. We define $\psi(\lambda)$ to be the tangent vector to V at v which takes $f \in K[V]$ to

$$\lambda(f-f(v) + \mathscr{I}_V(v)^2) \ .$$

It can be easily checked that $\psi(\lambda)$ is well-defined and the map

$$\psi : \mathrm{Hom}_K(\mathscr{I}_V(v)/\mathscr{I}_V(v)^2, K) \to T(V)_v$$

is K-linear. Since $\varphi(\psi(\lambda))(f+\mathscr{I}_V(v)^2) = \psi(\lambda)(f) = \lambda(f-f(v) + \mathscr{I}_V(v)^2) = \lambda(f + \mathscr{I}_V(v)^2)$, where $f \in \mathscr{I}_V(v)$ and $\psi(\varphi(\gamma))(g) = \varphi(\gamma)(g-g(v) + \mathscr{I}_V(v)^2) = \gamma(g)$, where $\gamma \in T(V)_v$ and $g \in K[V]$, φ is a bijective K-linear map. Q.E.D.

Now let $(X, \mathscr{O}_X)$ be a prevariety over K and x be any fixed point of X . Assume that U is an affine open set of X containing x . Since

$$T(U)_x \cong \mathrm{Hom}(\mathscr{I}_U(x)/\mathscr{I}_U(x)^2, K) \cong \mathrm{Hom}_K(\mathscr{M}_x/\mathscr{M}_x^2, K)$$

as K-linear spaces from Lemmas 13.4 and 7.12, where $\mathscr{M}_x$ is the unique maximal ideal of $\mathscr{O}_x$, it is reasonable to define the tangent space $T(X)_x$ of X at x to be

$$\mathrm{Hom}_K(\mathscr{M}_x/\mathscr{M}_x^2, K) \ .$$

(13.5) Definition. Let $(X, \mathscr{O}_X)$ be a prevariety over K and $x \in X$. Let $\mathscr{M}_x$ be the unique maximal ideal of the local ring $\mathscr{O}_x$ at x . Then we define the tangent space $T(X)_x$ of X at x to be

$$\mathrm{Hom}_K(\mathscr{M}_x/\mathscr{M}_x^2, K) \ .$$

(13.6) Definition. Let $(X, \mathscr{O}_X)$ be an irreducible prevariety over K . A point $x \in X$ is said to be simple or non-singular if

$$\dim_K T(X)_x = \dim X \ .$$

If all points of X are simple, we call X smooth or non-singular.

Now let $(X,\mathcal{G}_X)$ and $(Y,\mathcal{G}_Y)$ be irreducible varieties over K and $\varphi:X \to Y$ be a <u>dominant morphism</u> of varieties, i.e., φ is a morphism of varieties and $\overline{\varphi(X)} = Y$. Let O_1 and O_2 be open subsets of X and Y respectively such that $\varphi(O_1) \subset O_2$, then

$$\varphi|_{O_1}: O_1 \to O_2$$

is also a dominant morphism of varieties. Assume that O_1 and O_2 are affine open subsets, then

$$(\varphi|_{O_1})^* : K[O_2] \longrightarrow K[O_1]$$
$$((\varphi|_{O_1})^* : \quad f \longmapsto f\circ\varphi)$$

is injective and we have got the following embeddings:

$$K[O_2] \hookrightarrow K[O_1],\ K(O_2) \hookrightarrow K(O_1) \quad \text{and} \quad K(Y) \hookrightarrow K(X) .$$

<u>(13.7) Theorem</u>. Let $\varphi:X \to Y$ be a dominant morphism of irreducible varieties $(X,\mathcal{G}_X)$ into $(Y,\mathcal{G}_Y)$, i.e., $\varphi(X)$ is dense in Y . Let $r = \dim X - \dim Y$. Let W be a closed irreducible subset of Y and Z be an irreducible component of $\varphi^{-1}(W)$ such that $\overline{\varphi(Z)} = W$, then

$$\dim Z \geq \dim W + r .$$

<u>Proof</u> (see Humphreys [2, Theorem 4.1]). We first prove the Theorem in case Y is affine. Let $s = \operatorname{codim} W$, i.e., $s = \dim Y - \dim W$, and assume that $s \geq 1$. Then W is an irreducible component of $\mathcal{V}_Y(K[Y]f_1+\ldots+K[Y]f_s)$ for some $f_1,\ldots,f_s \in K[Y]$ from Corollary 7.9.5. Let $g_i = f_i\circ\varphi \in \mathcal{G}_X(X)$ $(1 \leq i < s)$ and

$$\mathcal{V}_X(g_1,\ldots,g_s) = \{x \in X \mid g(x) = 0 \quad \text{for } \forall g \in \mathcal{G}_X(X)g_1+\ldots+\mathcal{G}_X(X)g_s\} .$$

Since $\mathcal{V}_X(g_1,\ldots,g_s) \supset Z$ and $\mathcal{V}_X(g_1,\ldots,g_s)$ is closed in X , Z is contained in some irreducible component Z_o of $\mathcal{V}_X(g_1,\ldots,g_s)$. Since $W = \overline{\varphi(Z)} \subset \overline{\varphi(Z_o)} \subset \mathcal{V}_Y(f_1,\ldots,f_s)$ and W is an irreducible component of $\mathcal{V}_Y(f_1,\ldots,f_s)$, we have $W = \overline{\varphi(Z)} = \overline{\varphi(Z_o)}$. Hence $Z_o \subset \varphi^{-1}(W)$. Since Z is an irreducible component of $\varphi^{-1}(W)$ and $Z \subset Z_o$, we have $Z = Z_o$. Therefore, Z is an irreducible component of $\mathcal{V}_X(g_1,\ldots,g_s)$.

Now let O be an element of finite affine open covering of X such that $O \cap Z \neq \emptyset$. Since $\dim Z = \dim_O (Z \cap O)$ and $Z \cap O \subset \mathscr{V}_O(g_1|_O, \ldots, g_s|_O)$ and $Z \cap O$ is an irreducible component of

$$\mathscr{V}_O(g_1|_O, \ldots, g_s|_O) = \mathscr{V}_X(g_1, \ldots, g_s) \cap O$$

from Exercise 14 on P.32, we have

$$\operatorname{codim} Z = \operatorname{codim}_O (Z \cap O) \leq s$$

(see Corollary 7.9.4). Hence $\dim X - \dim Z = \operatorname{codim} Z \leq s$ $= \dim Y - \dim W$, i.e.,

$$\dim W + r \leq \dim Z \ .$$

Next we show that the general case can be resolved into the affine case. Let U be an element of affine open covering of Y such that $U \cap W \neq \emptyset$. Let $X' = \varphi^{-1}(U)$, then X' is an open subvariety of X and $\varphi|_{X'}: X' \to U$ is a dominant morphism of irreducible varieties. It is clear that $W \cap U$ is irreducible and closed in U (see Exercise 14 on p.32) and $\dim W = \dim_U (W \cap U)$. Since

$$W \cap U = \overline{\varphi(Z)} \cap U \supset \varphi(Z \cap X') \ ,$$

$\overline{\varphi|_{X'}(Z \cap X')} = W \cap U$ in U . Since $Z \cap X'$ is irreducible and closed in X' , we only have to show that $Z \cap X'$ is an irreducible component of $(\varphi|_{X'})^{-1}(U \cap W)$ and $\dim Z = \dim_{X'}(Z \cap X')$. Since $Z \cap X' \neq \emptyset$, $Z \cap X'$ is an irreducible component of $(\varphi|_{X'})^{-1}(U \cap W)$ from Exercise 14 on p.32 Let V be an element of affine open covering of X such that $Z \cap X' \cap V \neq \emptyset$. Then there exists a principal open set V_f in V $(f \in K[V])$ such that $X' \cap V \supset V_f$ and $V_f \cap Z \neq \emptyset$ from Remark 2.9. Since V_f is an affine open set of X (see Corollary 10.5), we have

$$\dim Z = \dim_{V_f}(Z \cap V_f) = \dim_{X'}(Z \cap X') \ .$$

Q.E.D.

<u>(13.8) Corollary</u>. Let $\varphi: X \to Y$ and r be as in the theorem, then $\dim \varphi^{-1}(y) \geq r$ for any $y \in \varphi(X)$.

<u>(13.9) Lemma</u>. Let X, X' and Y be irreducible varieties over K and $\psi : X \to X'$, $\varphi' : X' \to Y$ be dominant morphisms. Assume that $\dim X = \dim X' = \dim Y$. Let U and U' be non-empty open sets in X and X' respectively. Let $\varphi = \varphi' \circ \psi$, $U_1 = U \cap \psi^{-1}(U')$, $D = X - U_1$, $U_1' = U' \cap \varphi'^{-1}(Y - \overline{\varphi(D)})$ and $U_o = U_1 \cap \psi^{-1}(U_1')$, then U_o is a non-empty open set in X such that $U_o \subset U$, $\psi(U_o) \subset U'$ and $\varphi^{-1}(\varphi(u)) \subset U$ for any $u \in U_o$.

$$
\begin{array}{lllll}
\varphi : X & \xrightarrow{\quad\psi\quad} & X' & \xrightarrow{\quad\varphi'\quad} & Y \\
\quad\;\; \cup & & \cup & & \cup \\
\quad\;\; U & & U' & & Y - \overline{\varphi(D)} \\
\quad\;\; \cup & & \cup & & \\
\quad\;\; U_1 = U \cap \psi^{-1}(U') & & U_1' = U' \cap \varphi'^{-1}(Y - \overline{\varphi(D)}) & & \\
\quad\;\; \cup & & & & \\
\quad\;\; U_o = U_1 \cap \psi^{-1}(U_1') & & & & \\
(\varphi : u & \xrightarrow{\quad\psi\quad} & \psi(u) & \xrightarrow{\quad\varphi'\quad} & \varphi(u))
\end{array}
$$

<u>Proof</u>. Since ψ is dominant, we have $\psi^{-1}(U') \neq \emptyset$. Hence $\dim D < \dim X$ (see Exercise 48 on p.140). Therefore, $\dim \overline{\varphi(D)} \leq \dim D < \dim Y$ (see Exercise 40 on p.113) and we have $Y - \overline{\varphi(D)} \neq \emptyset$. Since φ' is dominant, we have $\varphi'^{-1}(Y - \overline{\varphi(D)}) \neq \emptyset$. Hence $U_1' \neq \emptyset$. Since ψ is dominant, $\psi^{-1}(U_1') \neq \emptyset$. Hence $U_o \neq \emptyset$.

Let $u \in U_o$ and $x \in \varphi^{-1}(\varphi(u))$. Assume that $x \notin U$, then $x \in D$, which implies $\varphi(x) = \varphi(u) \in \overline{\varphi(D)}$. Since $\psi(U_o) \subset U_1'$ and $\varphi(U_o) = \varphi' \circ \psi(U_o) \subset \varphi'(U_1') \subset Y - \overline{\varphi(D)}$, we have got a contradiction. It is clear that $\psi(U_o) \subset U'$. Q.E.D.

<u>(13.10) Lemma</u>. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be irreducible varieties over K, and $\varphi : X \to Y$ be a dominant morphism of varieties. If $K(Y) = K(X)$, then there exists a non-empty open subset U in X such that $\varphi(U)$ is open in Y and φ induces an isomorphism of varieties of U onto $\varphi(U)$.

<u>Proof</u> (see Springer [1, Lemma 4.1.2]). Let $O_1 \subset X$ and $O_2 \subset Y$ be affine open subsets such that $\varphi(O_1) \subset O_2$. Then since

$$\varphi|_{O_1} : O_1 \to O_2$$

is a dominant morphism and $K(O_1) = K(X) = K(Y) = K(O_2)$, we can assume that X and Y are affine varieties.

We shall write f^* for $f \circ \varphi$ where $f \in K[Y]$. Since $\varphi^*: K[Y] \hookrightarrow K[X]$ and $K[X] \subset K(X) = K(Y)$, we can take
$(\varphi^*: f \longrightarrow f^* = f \circ \varphi)$

$f_1, \ldots, f_r \in K(Y)$ such that

$$K[X] = K[Y][f_1, \ldots, f_r] .$$

Let $f^* \in K[Y]-(0)$ $(\subset K[X])$ such that $f^* f_i \in K[Y]$ for each $1 \leq i \leq r$, then the map

$$K[Y]_f \longrightarrow K[X]_{f^*}$$
$$\left(\frac{a}{f^n} \longrightarrow \frac{a^*}{(f^*)^n} \right)$$

is a K-algebra isomorphism. Hence

$$\varphi|_{X_{f^*}} : X_{f^*} \longrightarrow Y_f$$

is the desired isomorphism from Lemma 2.4. Q.E.D.

<u>(13.11) Lemma</u> (see Springer [1, Lemma 4.1.3]). Let $(X, K[X])$ and $(Y, K[Y])$ be irreducible affine varieties over K and $\varphi: X \to Y$ be a dominant morphism. Assume that there exists a in $K[X]$ such that $K[X] = K[Y][a]$. If a is transcendental over $K(Y)$, then

(1) φ is an open map, i.e., φ maps open sets onto open sets;
(2) for any irreducible closed subset W of Y , $\varphi^{-1}(W)$ is irreducible and closed in X and
$$\dim \varphi^{-1}(W) = \dim W + 1 .$$

<u>Proof</u>. Since a is transcendental over $K(Y)$,

$$K[Y] \otimes_K K[a] \cong K[Y][a] = K[X]$$
$$(x \otimes y \longrightarrow xy)$$

as K-algebras. Hence we can consider X to be $Y \times K$ and φ to be the projection $\pi_1: Y \times K \to Y$. The assertion follows from Proposition 3.3, Proposition 5.7 and Corollary 7.9.2. Q.E.D.

(13.12) Lemma (see Springer [1, Lemma 4.1.4]). Let $(X,K[X])$ and $(Y,K[Y])$ be irreducible affine varieties over K and $\varphi: X \to Y$ be a dominant morphism. Assume that there exists a in $K[X]$ such that $K[X] = K[Y][a]$. If a is separably algebraic over $K(Y)$, then X contains a non-empty open set U such that

(1) $\varphi|_U$ is an open map and morphism of varieties of U into Y ;

(2) if W is an irreducible closed subset of Y and Z is an irreducible component of $\varphi^{-1}(W)$ such that $Z \cap U \neq \emptyset$, then we have $\dim Z = \dim W$;

(3) for any $x \in U$ we have $|\varphi^{-1}(\varphi(x))| = [K(X):K(Y)]$.

Proof. Let $K[Y][T]$ be the one variable polynomial ring over $K[Y]$. Let

$$\begin{array}{c} \upsilon: K[Y][T] \longrightarrow K[X] \\ (\upsilon: F(T) \longrightarrow F(a)) \end{array}$$

be a surjective homomorphism of $K[Y][T]$ onto $K[X]$ such that $\upsilon(T) = a$, then

$$\begin{array}{c} K[Y][T] \,/\, \mathrm{Ker}\, \upsilon \cong K[X] \\ (F(T)+\mathrm{Ker}\, \upsilon \longrightarrow F(a)) \end{array}$$

as K-algebras. Let $f(T)$ be the irreducible polynomial in $K(Y)[T]$ with leading coefficient 1 such that $f(a) = 0$. Let b be a non-zero element of $K[Y]$ such that $bf(T) \in K[Y][T]$, then all the coefficients of $f(T)$ lie in $K[Y]_b$.

Since a is separably algebraic over $K(Y)$, $f(T) = 0$ has no multiple roots. Let $x_1,\dots,x_n$ be the roots of $f(T)$ in the algebraic closure of $K(Y)$. Let

$$c = \prod_{i<j} (x_i - x_j)^2$$

be the discriminant of $f(T)$. Since c is a symmetric polynomial of $x_1,\dots,x_n$ from Lang [1, Theorem 9.1 on p.204] we have $c \in K[Y]_b$.

Let $c = \frac{d}{b^t}$, where $d \in K[Y]$ and $t \geq 0$ is an integer, then we have

$$(X_b)_c = X_{bd} \quad \text{and} \quad (Y_b)_c = Y_{bd} \; .$$

Notice that $\varphi^{-1}(Y_{bd}) = X_{bd}$, $K[X_{bd}] = K[X]_{bd} = K[Y]_{bd}[a]$ $= K[Y_{bd}][a]$ and a is separably algebraic over $K(Y_{bd}) = K(Y)$. Now we shall write X' for X_{bd} and Y' for Y_{bd}, then $a \in K[X']$ and all the assumptions of the Lemma hold for $\varphi|_{X'}: X' \to Y'$ and further we have the following additional properties:

a) Ker $\upsilon' = (K[Y'][T])f(T)$, where $\upsilon': K[Y'][T] \longrightarrow K[X']$; $(\upsilon': F(T) \longrightarrow f(a))$

b) assume that $f(T) = \sum_{i=0}^{n} f_i T^i$, $(f_o, f_1, \dots, f_{n-1}, f_n = 1 \in K[Y]_b \subset K[Y]_{bd} = K[Y'])$, then for all $y \in Y'$ $(= (Y_b)_c)$

$$f(y)(T) = \sum_{i=0}^{n} f_i(y) T^i$$

has distinct roots.

We shall show that the Lemma holds for $\varphi|_{X'}: X' \to Y'$ on $U = X'$. We shall write $\varphi' = \varphi|_{X'}$.

(1) Corresponding to the sequence of K-algebra homomorphisms

$$(\varphi')^*: K[Y'] \hookrightarrow K[Y'][T] \xrightarrow{\upsilon'} K[X'] , \qquad F(T) \longrightarrow F(a)$$

there exists a sequence of morphisms

$$\varphi': X' \xrightarrow{\rho} Y' \times K \xrightarrow{\pi_1} Y' , \qquad (y,t) \longrightarrow y$$

such that the map $\rho_0: X' \longrightarrow \rho(X')$ $(\rho_0: x \longrightarrow \rho(x))$ is an isomorphism of varieties (see Proposition 2.5). From Proposition 2.5 we have

$$\rho(X') = \mathcal{V}(\text{Ker } \upsilon') = \{(y,t) \in Y' \times K \mid f(y)(t) = 0\} .$$

Let O be the principal open set in $\rho(X')$ defined by $g \in K[Y'][T]$, i.e.,

$$O = \{(y,t) \in \rho(X') \mid g(y)(t) \neq 0\} .$$

Let $g(T) = q(T)f(T) + r(T)$, where $q(T), r(T) \in K[Y'][T]$ and $\deg r(T) < \deg f(T)$. Let

$$r(T) = \sum_{i=0}^{n-1} r_i T^i ,$$

where $n = \deg f(T)$, then we have $\pi_1(O) \subset \bigcup_{i=0}^{n-1} Y'_{r_i}$, where Y'_{r_i} is the principal open set in Y' defined by $r_i \in K[Y']$.

Conversely let $y \in Y'_{r_i}$ for some $0 \le i \le n-1$. Since $y \in Y'$, $f(y)(T) = \sum_{i=0}^{n} f_i(y)T^i$ has n distinct roots $t_1, t_2, \ldots, t_n$. Since

$$g(y)(t_j) = \sum_{i=0}^{n-1} r_i(y)t_j^i$$

and $r(y)(T) = 0$ has at most $n-1$ distinct roots, there exists t_j $(1 \le j \le n)$ such that $g(y)(t_j) \ne 0$, i.e., $(y,t_j) \in O$. Hence

$$\pi_1(O) = \bigcup_{i=0}^{n-1} Y'_{r_i}$$

and φ' is an open map.

(2) Let W be an irreducible closed subset of Y' , then

$$(\varphi|_{X'})^{-1}(W) = \rho^{-1}(\pi_1^{-1}(W)) = \rho^{-1}((W\times K) \cap \rho(X'))$$

$$= \rho^{-1}(\{(y,t) \in W\times K \mid f(y)(t) = 0\}) .$$

Since $W\times K$ is an irreducible closed subvariety of $Y'\times K$ with coordinate ring

$$K[W\times K] = \{\alpha|_{W\times K} \mid \alpha \in K[Y][T]\} ,$$

all the irreducible components of $\{(y,t) \in W\times K \mid f(y)(t) = 0\}$ are of dimension $\dim(W\times K)-1 = \dim W$ from Theorem 7.8. Hence all the irreducible components of $(\varphi|_{X'})^{-1}(W)$ have the same dimensions, $\dim W$.

(3) Now let $W = \{\varphi(u)\}$ for some $u \in X'$, then from (2) we have

$$\varphi^{-1}(\varphi(u)) = \varphi^{-1}(W) = (\varphi|_{X'})^{-1}(W)$$

$$= \rho^{-1}(\{(\varphi(u),t) \mid t \in K \text{ and } f(\varphi(u))(t) = 0\} ,$$

because $\varphi^{-1}(Y') = X'$. Hence $|\varphi^{-1}(\varphi(u))|$ is the number of distinct roots of $f(\varphi(u))(T) = 0$. Thus we have $|\varphi^{-1}(\varphi(u))| = [K(X'):K(Y')]$ $= [K(X):K(Y)]$. It is clear that the assertion of the Lemma holds for $\varphi: X \to Y$ on $U = X'$. Q.E.D.

<u>(13.13) Lemma</u> (see Springer [1, Lemma 4.1.5]). Let $(X,K[X])$ and $(Y,K[Y])$ be irreducible affine varieties over K and $\varphi: X \to Y$ be a dominant morphism. Assume that there exists a in $K[X]$ such that

$K[X] = K[Y][a]$. If the characteristic of K is $p > 0$ and $a^p \in K(Y)$, then X contains a non-empty open set U such that:

(1) $\varphi|_U$ is an open map and morphism of varieties of U into Y and the map $\begin{matrix} U \longrightarrow \varphi(U) \\ (u \longrightarrow \varphi(u)) \end{matrix}$ is a homeomorphism;

(2) for any irreducible closed subset W of Y there exists at most one irreducible component Z of $\varphi^{-1}(W)$ such that $Z \cap U \neq \emptyset$. For such a Z we have

$$\dim Z = \dim W .$$

<u>Proof</u>. Let $a^p = b = \frac{d}{c}$ for some $c,d \in K[Y]$ such that $c \neq 0$. Let $X' = X_c$ and $Y' = Y_c$, then $\varphi^{-1}(Y') = X'$ and $b \in K[Y]_c = K[Y']$. Notice that $K[X'] = K[X]_c = K[Y]_c[a] = K[Y'][a]$ and all the assumptions of the Lemma hold for $\varphi|_{X'}:X' \to Y'$. We shall show that the Lemma holds on $U = X'$. We shall write $\varphi' = \varphi|_{X'}$.

(1) Corresponding to the sequence of K-algebra homomorphisms

$$(\varphi')^*:K[Y'] \hookrightarrow K[Y'][T] \xrightarrow{\upsilon'} K[X'] , \qquad F(T) \longrightarrow F(a)$$

there exists a sequence of morphisms

$$\varphi':X' \xrightarrow{\rho} Y'\times K \xrightarrow{\pi_1} Y' \qquad ((y,t) \longrightarrow y)$$

such that the map $\begin{matrix} \rho_0:X' \longrightarrow \rho(X') \\ (\rho_0:x \longrightarrow \rho(x)) \end{matrix}$ is an isomorphism of varieties (see Proposition 2.5). From Proposition 2.5 we have $\rho(X') = \mathcal{V}(\mathrm{Ker}\ \upsilon')$. Let $f(T)$ be the non-zero minimal polynomial in $K[Y'][T]$ such that $f(a) = 0$. Since $f(T) \mid T^p-a^p$ in $K(Y')[T]$ and for any $1 \leq m < p$

$$(T-a)^m = T^m-maT^{m-1}+\ldots+(-1)^m a^m \in K[Y'][T]$$

if and only if $a \in K[Y']$, we have

$$f(T) = T^p-a^p \quad \text{or} \quad T-a .$$

Hence
$$\rho(X') = \{(y,t) \in Y'\times K \mid f(y)(t) = 0\}$$
$$= \{(y,t) \in Y'\times K \mid t = b(y)^{\frac{1}{p}}\} .$$

Therefore φ' is bijective. Let O be the principal open subset in $\rho(X')$ defined by $g \in K[Y'][T]$, i.e.,

$$O = \{(y, b(y)^{\frac{1}{p}}) \in Y' \times K \mid g(y)(b(y)^{\frac{1}{p}}) \neq 0\} .$$

Let $g(y)(T) = \sum_{i=0}^{s} g_i(y)T^i$, then we have $g(y)(b(y)^{\frac{1}{p}}) \neq 0$ if and only if

$$\{g(y)(b(y)^{\frac{1}{p}})\}^p = \sum_{i=0}^{s} g_i^p(y)b(y)^i \neq 0 .$$

Since $\sum_{i=0}^{s} g_i^p b^i \in K[Y']$, $\pi_1(O)$ is open in Y'. Hence φ' is an open map.

(2) Let W be an irreducible closed subset of Y. We can assume that $U \cap \varphi^{-1}(W) \neq \emptyset$. Since $W \cap \varphi(U)$ is irreducible and

$$(\varphi|_U)^{-1}(W \cap \varphi(U)) = \varphi^{-1}(W) \cap U ,$$

$U \cap \varphi^{-1}(W)$ is also irreducible. Since $\varphi|_U$ is a homeomorphism of U onto $\varphi(U)$, we have

$$\dim W = \dim(W \cap \varphi(U)) = \dim(\varphi^{-1}(W) \cap U)$$

from Corollary 7.9.1.

Now let Z be an irreducible component of $\varphi^{-1}(W)$ such that $Z \cap U \neq \emptyset$, then $Z \cap U$ is an irreducible component of $\varphi^{-1}(W) \cap U$ from Exercise 14 on p.32. Hence $Z \cap U = \varphi^{-1}(W) \cap U$. Since $Z = \overline{\varphi^{-1}(W) \cap U}$, we have at most one such Z. Q.E.D.

<u>(13.14) Theorem</u> (see Springer [1, Theorem 4.1.6]). Let $\varphi: X \to Y$ be a dominant morphism of irreducible varieties $(X, \mathscr{S}_X)$ into $(Y, \mathscr{S}_Y)$ and let $r = \dim X - \dim Y$. Then X has a non-empty open set U such that:

(1) $\varphi|_U$ is an open map and morphism of varieties of U into Y;

(2) if W is an irreducible closed subset of Y and Z is an irreducible component of $\varphi^{-1}(W)$ such that $Z \cap U \neq \emptyset$, then we have $\dim Z = \dim W + r$;

(3) if $K(X)$ is algebraic over $K(Y)$, i.e., $r = 0$, then for all $x \in U$ we have $|\varphi^{-1}(\varphi(x))| = [K(X)_s : K(Y)]$, where

$K(X)_s = \{\alpha \in K(X) \mid \alpha$ is separably algebraic over $K(Y)\}$.

<u>Proof</u>. Let O be an affine open set of Y and V be an affine open set of X contained in $\varphi^{-1}(O)$, then

$$\varphi|_V : V \longrightarrow O$$
$$(\varphi|_V : x \to \varphi(x))$$

is also a dominant morphism of affine varieties.

Assume that $K(X)$ is algebraic over $K(Y)$, i.e., $\dim X = \dim Y$. Let $D = \varphi^{-1}(O) - V$, then $\dim D < \dim \varphi^{-1}(O)$ (see Exercise 48.2 on p.140). Hence

$$\dim \overline{\varphi|_{\varphi^{-1}(O)}(D)} \leqq \dim D < \dim \varphi^{-1}(O) = \dim O ,$$

where $\overline{\varphi|_{\varphi^{-1}(O)}(D)}$ is the closure of $\varphi|_{\varphi^{-1}(O)}(D)$ in O (see Exercise 40 on p.113). Let O' be an affine open set in $O - \overline{\varphi(D)}$, then $\varphi^{-1}(O') \subset V$ and

$$(\varphi|_V)^{-1}(u) = \varphi^{-1}(u) \quad \text{for all} \quad u \in O' .$$

Suppose that the theorem holds for V, O and $\varphi|_V : V \to O$. Let U be a non-empty open set of V which satisfies the conditions (1), (2) and (3). Then U satisfies (1) and (2) as an open subset of X. Further if $r = 0$, then we take a new non-empty open subset $U' = U \cap \varphi^{-1}(O')$ of V instead of U. Clearly, (1), (2) and (3) hold for U' and $\varphi|_V : V \to O$, and thus U' satisfies (1), (2) and (3) as an open subset of X.

Hence it is enough to prove the theorem in case X and Y are affine varieties.

(1) and (2). Let $\varphi = \varphi' \circ \psi$ be a factorization of φ by dominant morphisms of irreducible affine varieties

$$\psi : X \to X' \quad \text{and} \quad \varphi' : X' \to Y .$$

Let U and U' be open subsets of X and X' respectively which satisfy (1) and (2) for ψ and φ' respectively. Let

$U_o = U \cap \psi^{-1}(U')$, then U_o satisfies (1) and (2) for $\varphi: X \to Y$.

$$\begin{array}{ccccc} \varphi: X & \xrightarrow{\psi} & X' & \xrightarrow{\varphi'} & Y \\ \cup & & \cup & & \\ U & & U' & & \\ \cup & & & & \\ U_0 = U \cap \psi^{-1}(U') & . & & & \end{array}$$

Since $K[X]$ is finitely generated over $K[Y]$, i.e.,

$$K[X] = K[Y][a_1, \ldots, a_r]$$

for some $a_1, \ldots, a_r \in K[X]$, we have a sequence of finitely generated K-algebras with trivial nilradicals:

$$K[Y] \subset K[Y][a_1] \subset K[Y][a_1, a_2] \subset \ldots \subset K[X] .$$

In case $\text{ch } K = p > 0$ and a_i is algebraic over the quotient field of $K[Y][a_1, a_2, \ldots, a_{i-1}]$ for some $1 \leq i \leq r$ there exists an integer $\mu \geq 0$ such that $a_i^{p^\mu}$ is separably algebraic over the quotient field of $K[Y][a_1, a_2, \ldots, a_{i-1}]$ (see Lang [1, Proposition 4.3 on p.283]), then we can refine the sequence by adding the new terms:

$$K[Y][a_1, \ldots, a_{i-1}] \subset K[Y][a_1, \ldots, a_{i-1}, a_i^{p^\mu}] \subset K[Y][a_1, \ldots, a_{i-1}, a_i^{p^{\mu-1}}]$$
$$\subset \ldots \subset K[Y][a_1, \ldots, a_{i-1}, a_i^p] \subset K[Y][a_1, \ldots, a_{i-1}, a_i] .$$

Thus from Theorem 6.10 and Lemma 8.3 we get a seuqnce of dominant morphisms of irreducible affine varieties each step of which satisfies the condition of one of the Lemmas 13.11, 13.12 and 13.13. Hence (1) and (2) hold for affine varieties.

(3) First we assume that $\text{ch } K = p > 0$. Let $K[X] = K[Y][a_1, \ldots, a_r]$ for some $a_1, \ldots, a_r \in K[X]$, then from Lang [1, Proposition 4.3 on p.283] there exist integers $\mu_i \geq 0$ for each $1 \leq i \leq r$ such that $a_i^{p^{\mu_i}}$ is separably algebraic over $K(Y)$. Thus we have got a sequence of finitely generated K-algebras with trivial nilradicals:

$$K[Y] \subset K[Y][a_1^{p^{\mu_1}}, \ldots, a_r^{p^{\mu_r}}] \subset K[X] .$$

From Lang [1, Proposition 7.2 on p.292] we have

$$K(X)_s = K(Y)[a_1^{p^{\mu_1}}, \dots, a_r^{p^{\mu_r}}] .$$

From Theorem 6.10 and Lemma 8.3 we get a sequence of dominant morphisms of irreducible affine varieties corresponding to the above sequence of K-algebras:

$$\varphi : X \xrightarrow{\psi} X' \xrightarrow{\varphi'} Y ,$$

where $K[X'] = K[Y][a_1^{p^{\mu_1}}, \dots, a_r^{p^{\mu_r}}]$. Since $K[X] = K[Y][a_1, \dots, a_r]$ and $K[X'] = K[Y][a_1^{p^{\mu_1}}, \dots, a_r^{p^{\mu_r}}]$, we can embed X and X' into $Y\times K^r$. Let $K[Y\times K^r] = K[Y] \otimes_K K[X_1, \dots, X_r]$, then

$$\begin{array}{ccc} K[Y] \otimes_K K[X_1,\dots,X_r] & \cong & K[Y][X_1,\dots,X_r] \\ (\ a\otimes b & \longrightarrow & ab\ ) \end{array}$$

as K-algebras and we have the following commutative diagrams:

$$\begin{array}{ccccc} K[Y][a_1,\dots,a_r] & \supset & K[Y][a_1^{p^{\mu_1}},\dots,a_r^{p^{\mu_r}}] & & \\ a_i & & a_i^{p^{\mu_i}} & & \\ \uparrow & \circlearrowright & \uparrow & \circlearrowright & K[Y] \\ X_i & & X_i & & \\ K[Y][X_1,\dots,X_r] & \leftarrow & K[Y][X_1,\dots,X_r] & & \\ X_i^{p^{\mu_i}} & \longleftarrow & X_i & & \end{array}$$

$$\begin{array}{ccccc} X & \xrightarrow{\psi} & X' & & \\ \cap & & \cap & \searrow^{\varphi'} & \\ \downarrow & \circlearrowright & \downarrow & \circlearrowright & Y \\ Y\times K^r & \xrightarrow{\mu} & Y\times K^r & \nearrow & \\ & & & & y \\ (y,x_1,\dots,x_r) & \longrightarrow & (y,x_1^{p^{\mu_1}},\dots,x_r^{p^{\mu_r}}) & \nearrow & \end{array}$$

Hence ψ is injective, because μ is injective.

Now we decompose φ' into a sequence of dominant morphisms each step on which satisfies the condition of Lemma 13.12.

$$\varphi':X' \xrightarrow{\zeta_1} Y_1 \xrightarrow{\zeta_2} Y_2 \longrightarrow \cdots \longrightarrow Y_{r-1} \xrightarrow{\zeta_r} Y$$

$$\varphi'^{*}:K[Y] \overset{\zeta_r^*}{\hookrightarrow} K[Y][a_1^{p^{\mu_1}}] \overset{\zeta_{r-1}^*}{\hookrightarrow} K[Y][a_1^{p^{\mu_1}}, a_2^{p^{\mu_2}}] \hookrightarrow \cdots \overset{\zeta_1^*}{\hookrightarrow} K[X'] .$$

From Lemma 13.9 and Lemma 13.12 we can find a non-empty open set U_o in X' on which (1), (2) and (3) hold for

$$\varphi':X' \to Y .$$

Let U' be a non-empty open set of X on which (1) and (2) hold for $\psi:X \to X'$. Since ψ is injective (1), (2) and (3) hold for $\varphi:X \to Y$ on $U = U' \cap \psi^{-1}(U_o)$.

Similarly we can show the characteristic 0 case. Q.E.D.

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