Quotient Vectors and Algebraification of Integral Vector Laws by Them.

Satio OKADA, Syōkiti HOSINA
(Dep. of Elect. Eng., Faculty of Eng.)

§ 0. Introduction.

Since A. Einstein, tensor analysis (abbreviation : tan) became rapidly practical and popular. But vector analysis by direct notation (abbr.: van), especially 3-dimensionsl vector analysis (abbr.: 3van) is still usefull for classical physics. Therein, differential vector laws (abbr.: van^{+1}) is reduced to vector algebra by W. R. Hamilton's operator ∇, and van^{+1} is represnted as ∇-algebra for example in the J. A. Stratton's Electromagnetic Theory (1941) etc. But integral vectorial laws (abbr.: van^{-1}) which use by J. W. Gibbs proposed Pot, New, Max, Lap, (GV 205~59, for "GV" see references at the end) and Helmholtz's operator $H(v) = \text{Hel}\ v$ (GV 258~9) were not algebraified in spite of their algebraic properties. We succeeded the algebraification of van^{-1} by reducing it to division vector algebra (abbr.: val^{-1}) which is proposed by K. Hisazue by his book (HV). But about integro-differential laws (abbr. : van$^{\pm}$) such as Gauss, Stokes, Green etc. we intend to proceed quite another way, and its results are published in other places (OE 1~4, OP etc.), so here we don't touch them. Also extension of our result to 4-and n-dimension gives interesting problems but we don't touch them too. For projection of vector to subspace X^m in X^n we treated it in another paper (OS) by index notation. This corresponds to an extention of a part of this work.

We add 2-dimesional case in appendix. The above result is tabulated as follows :

$$\left.\begin{array}{ccc} \text{val} & : & \text{val}^{+1} \xrightarrow{A^{-1}} \text{val}^{-1} \\ \uparrow\nabla & & \uparrow\nabla \quad \nabla^{-1} \quad \uparrow\nabla \\ \text{van} & : & \text{van}^{+1} \xrightarrow[\text{van}^{\pm}]{} \text{van}^{-1} \end{array}\right\} \rightarrow \text{tan}$$

By above formal algebraification, the correspondence between network theory (OS) and field theoy (e. g. SE.) became quite clear as we would show in the forthcoming paper (OA) in another issue .

The contents of this paper : ——

in § 1, we arrange and describe shortly the neccessary concept from abstract algebra,

in § 2, we study algebraic construction of vector algebra especially→val→val.$^{-1}$

in § 3, we introduce inverse vector $\mathbf{A}^{-1}$ about inner product and general exponential $\mathbf{A}^k$ necessary for van^{-1}.

in § 4, we proceed to general value of inverse vector and its application, we owe these result· mainly to J. A. Schouten (SDG, I, 21. problem 2.14 and 2.15).

§5, describes inverse operation for outer product,

§6, is an arrangement of the algebraic properties of the operator ∇,

in §7, we reduce van^{-1} to val^{-1}.

To see only algebraification of integral vectorial laws, only §3, and §7 suffice. By this investigation it became clear that inverse operation is useful in vector calculus, especially we became able to reduce van^{-1} to val^{-1}.

Also by representaion of decomposition of an any vector to parallel and normal component to another vector by inverse vector, we knew this decomposition correspond to the decomposition of any vector to divergent and rotational components by Helmholtz.

§ 1. Algebraic preliminaries (WCG, BLT)

Art. 0. Order For the following sets:

set of natural number $M^{na} \ni 1, 2, 3 \cdots\cdots$,

set of integer $M^{i} \ni 0, 1, -1, 2, -2, \cdots\cdots$,

set of rational number $M^{ra} \ni 0, 1, \frac{1}{2} \cdots\cdots$,

set of real number $M^{re} \ni 0, 1, e, \pi, \sqrt{2}, \sqrt[3]{2} \cdots\cdots$,

(this doesn't show countability)

set of polynominal of x with coefficients of above sets $M^{x} \ni, o, a, ax+b, \cdots\cdots$,

set of linear operator $M^{p} \ni \cdots p^{3} = \frac{d^{3}}{dt^{3}}, p^{2} = \frac{d^{2}}{dt^{2}}, p = \frac{d}{dt}, p^{0} = 1$,

$p^{-1} = \int dt, \ p^{-2} = \iint dt^{2}, \cdots\cdots$,

we consider following four axioms of order (we read $\geqq$ "include" and we say $\geqq$ "inclusion relation") (BLT).

P1 reflexive : For all A, A$\geqq$A.

P2 transitive : If A$\geqq$B, B$\geqq$C, then A$\geqq$C.

We call a set "quasi-ordered" which astisfies P1, P2.

P3 antisymmetry : If A$\geqq$B and B$\geqq$A, then A=B.

A set which satisfies these three so-called "equivalence relations" P1, P2, P3, "partially ordered", "semi-ordered" or simply "ordered set". Further if a set has P4 chain-property or linearity : For any A and B,

A$\geqq$B or B$\geqq$A.

then we call this "linearly ordered set" (BLT, 5, 9), "simply ordered set" or "chain".

Art. 1 Lattice.

We call a part of set "subset". A subset above various ordered sets become the same subsets respectively.

For a subset X in ordered set P, if there exist upper bound A$\geqq x$ for every $x \ni$X, we call the element "A" upper bound. A "least upper bound" is an upper bound contained in every other upper bound. Dually we can define "greatest lower bound."

If any 2 elements A, B of an ordered set have unique l. u. b. anb g. l. b. we call this a "lattice," and we write l. u. b. $A \smile B$ and call "join" and, write g. l. b. $A \frown B$ and call "meet".

Art. 2 Combination

For mere convention we call M^{ra} or M^{re}1-dimensional set M^1 hereafter, and we write

$M^1 \ni o, 1, a, b, c, \cdots\cdots\cdots\cdots$.

Further

complex number : $M^c \ni 0, 1, i, u, v, w, z, \cdots\cdots\cdots\cdots,$

2-dimensional set : $M^2 \ni \mathbf{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{A}, \mathbf{B}, \cdots\cdots\cdots\cdots,$

3-dimensional set : $M^3 \ni \mathbf{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{A}, \mathbf{B}, \cdots\cdots\cdots\cdots,$

n-dimensional set : $M^n \ni \mathbf{O}, \mathbf{e}_1, \cdots\cdots \mathbf{e}_n, \mathbf{A}, \mathbf{B}, \cdots\cdots\cdots\cdots,$

n-dimensional operator vector set : $M^\nabla \ni \cdots\cdots \nabla^{-4}, \nabla^{-3}, \nabla^{-2}, \nabla^{-1}, \nabla^0 = 1, \nabla^1, \nabla^2, \cdots\cdots,$

rectangular or square matrlx : $M^M \ni o, \mathbf{E}, \mathbf{A}, \mathbf{B}, \cdots\cdots\cdots\cdots,$

$$M = M^\kappa \ni A, B, \cdots\cdots\cdots\cdots,$$

$$(\kappa = na,\ i, ra, re, p, x,\ 1, c, 2, 3, \cdots\cdots\cdots\cdots, n, \nabla, M, \cdots\cdots).$$

and if we can construct a kind of combination

$$A \ast B$$

(for example : sum, direct sum $\dot{+}$, $\oplus$, product $\times$, direct product $\cdot\times, \times\cdot, \dot{\times}$, (Kronecker) $\otimes$, scalar and vector product( ), [ ], meet $\frown$ and join $\smile$), we call this "product" (though this may represent sum) and we call this set a "system". The main objects of these axiomatic discussion are equations and congruences of hypercomplex numbers, matrices, polynomials etc.

Art. 3 Commutative

In matrix calculus, both matrix product and direct product $A \ast B \neq B \ast A$ and for vector product also

$$[AB] = -[BA] \neq [BA].$$

So if

$$A \ast B = B \ast A, \tag{1}$$

we call this "commutative system."

Art. 4 Multiplicity.

If the product of a system (not necessarily commutative) decide always uniquely, we call this "one-valued" system. Geometrical mean of two number $\sqrt{AB}$ is two valued system. Matrix product AB^{-1} in which determinant of B is zero, is "indefinite system".

Art. 5 Closed.

If the product of a one-valued system is also included in the original set, namely

$$A \ast B = C \in M, \tag{2}$$

we call this system "closed" about this combination $\ast$ in H. Weyl's (W C G. 1) meaning or simply "algebraic system", and write $(M, \{\ast\})$ if necessary.

If a closed system is

Art. 6 associative : $(A \divideontimes B) \divideontimes C = A \divideontimes (B \divideontimes C)$, (3)

we call this "semigroup". Further a semigroup is

Art. 7 divisible, namely for any two element A, B there exists X and Y which satisfy

$$A \divideontimes X = B \quad (4), \qquad Y \divideontimes A = B. \quad (5)$$

wo call X and Y "right and left quotient" and write $A \backslash B$ and B/A respectively. and we call to decide X, Y from A and B "division:". So

$$A \divideontimes (A \backslash B) = B \quad (6), \qquad (B/A) \divideontimes A = B. \quad (7)$$

$(A/B) \divideontimes A$ or $A \divideontimes (B/A)$ is not always equal to B.

If both quotients of a semigroup determine uniquely, we call this a "group". A group is commutative if (1) holds, or we call it "Abelian group". In this case $A \backslash B = B/A$, so we can write this $\frac{B}{A}$. If division is possible only under certain condition, we call this semigruop "semi-divisble". Vector is semi-divisible for $(M^{s} \cdot \{[\ \ ]\})$.

Art. 8 Unit element.

For semi-group, if there exist X and Y for

$$A \divideontimes X = A \quad (8), \qquad Y \divideontimes A = A. \quad (9)$$

which are got by substituting A for B in (4) and (5), we call X and Y "right and left unit element", and we use particular letter $\divideontimes E$ and $E \divideontimes$ for $A \backslash A$, A/A. and if both coincides we call it "unit element", and write it E. Namely

$$A \divideontimes E = A \quad (10), \qquad E \divideontimes A = A. \quad (11)$$

Art, 9 Regularity

If a semigroup has a unit element by (8) and (9), and there exist X and Y which satisfy

$$A \divideontimes X = E \quad (12), \qquad Y \divideontimes A = E, \quad (13)$$

we call $A \backslash E$ and E/A "right and left inverse element" and write them $\divideontimes A^{-1}$ and $A^{-1} \divideontimes$, if both coincide, we call it "inverse element" and write A^{-1}. Namely

$$A \divideontimes A^{-1} = E \quad (14), \qquad A^{-1} \divideontimes A = E. \quad (15)$$

If a semi-group has a unit element, we call an element which has an inverse elment "regular". If there exist unit element in a semigroup and every element is regular, this becomes group . Namely Art, 7 is equivalent to Art. 8 and 9. We can define natural number M^{na} as a semigroup about addition 1 as a "generating element", and each number acquires personality by decimal or other nomenclature. If we add this zero and negative, and accomplish to M^{i}, this becomes an Abelian group about addition, as inverse (subtraction) become possible and unique for all elements. And rational number M^{ra} except zero is Abelian group about multiplication from M^{na} as generating elements. If inverse element exist, we can write $A \backslash B = A^{-1}B$, $B \backslash A = BA^{-1}$ and even in this case, the experience of matrix calculus shows $A^{-1}B \neq BA^{-1}$ therefore $A \backslash B \neq B/A$.

An element which satisfy

Art. 10 Idempotent : $X \divideontimes X = X$, (16)

we call it "idempotent". In $(M^1\{+\})$ idempotent element is zero only, but in $(M^1\{\dot{\times}\})$, O and I are both idempotent.

Art. 11 Distributive.

For a set, if adopting addition + as $\divideontimes$, we get Abelian group. we call it "additive group", then we call the unit element "zero element" specially for an additive group. Namely

$$A+O=A \quad (17), \qquad O+A=A. \quad (18)$$

For an additive group, we introduce another combination $\divideontimes$, and if this is both

left distributive : $A \divideontimes (B+C) = A \divideontimes B + A \divideontimes C$, (19)

right distrbutive : $(A+B) \divideontimes C = A \divideontimes C + B \divideontimes C$, (20)

we call it "distributive system". From (19)

$$A \divideontimes O = O \quad (19'), \quad \text{similarly} \quad O \divideontimes A = O. \quad (20')$$

If a distributive system is associative and about $\divideontimes$ namely semigroup, (namely exclude division) we call it "ring". $(M^i.\{+,\times\})$ is a commutative ring.

Art. 12 Lie-ring, If in a distributive system, every element is

Art. 13 nilpotent : $A \divideontimes A = O$, (21)

Art, 14 Jacobi-property , $A \divideontimes (B \divideontimes C) + B \divideontimes (C \divideontimes A) + C \divideontimes (A \divideontimes B) = O$, (22)

we call it "Lie-ring" or "Lie-algebra" (WCG. 260). This is

Art. 15 anticommutative : $A \divideontimes B = -B \divideontimes A$. (23)

Art. 16 non-associative :

$$(A \divideontimes B) \divideontimes C = A \divideontimes (B \divideontimes C) + (A \divideontimes C) \divideontimes B = A \divideontimes (B \divideontimes C) + B \divideontimes (C \divideontimes A) \quad (24)$$

Art: 17 Null-divisor . we call $\divideontimes N = A \backslash O$ namely which satisfy

$$A \neq O \ , \ A \divideontimes N = O \quad (25)$$

"right null-divisor". Similarly we call $N \divideontimes = O/B$

$$B \neq O, \qquad N \divideontimes B = O \quad (26)$$

"left null-divisor", and we use special letter N for them.

If this distributive system is associative, $\divideontimes N \divideontimes Q$ which is multiplied from right any element Q to a right null-divisor $\divideontimes N$ of A is also null-divisor, because

$$A \divideontimes (N \divideontimes Q) = (A \divideontimes N) \divideontimes Q = O \divideontimes Q = O. \quad (27)$$

This experience is useful also for non-associative system such as Lie-ring. Every element of Lie-ring, has itself as null-divsor by nilpotent property (21). If a ring has unit element about $\divideontimes$, null-divisor is zero only, and it is called "integral domain". If every element of integral domain is regular (Art. 9), it is called a "field". Field is defined also as a distributive system which construct a group about $\divideontimes$ except zero. The minimun number which satisfy

Art. 18 Index The minimum number which satisfies

$$A^p = A \divideontimes A \divideontimes \cdots\cdots \divideontimes A = O \quad (28)$$

is called "index" p. If there is none such finite p, we call it zero-index. Positive and negative rational number is a minimum field which is generated from number "I" and is zero-index. Every zero-index field contains this rational field as a subfield. Algebraic real and complex number construct both commutative field in $\{+.\times\}$.

One can construct various closed set, distributive system, Lie-ring etc. from these fields. For example, from matrix A and B, we get Lie-ring by

$$AB-BA=A\divideontimes B. \tag{29}$$

In this meaning we can say every ring to be Lie-ring and also inversely one can show that every Lie-ring can be realised by proper ring.

Art. 19 multiplicity of quotient element.

If there exist quotient which fulfils (4) and (5) in a distributive system having null-divisor beside zero namely in non-integral domain, K be a solution of (4) $\divideontimes N=A\backslash O$
be right null-divisor, then from left distributive law (19).

$$A\divideontimes(K+N)=A\divideontimes K+A\divideontimes N=A\divideontimes K.$$

So, if $X=K$. (30) then $X=K+N$ (31)

is also solution. Further if it is associatve, by (27)

$$X=K+N\divideontimes Q \tag{32}$$

is a solution too. This experience is useful also for non-associative Lie-ring.

§ 2 Algebraic consideration of 3 val (SGA)

Now we consider algebraic construction of 3-dim. vector algebra 3val. in order to utilize experience and successes of abstract algebra. 1-dimensional set such as M^{ra} or M^{re} construct a kind of commutative field K. 3-dimensional vector set M^3 which consist of these 1-dim. set is called a set on the operator M^1 "additive vector group" or "linear additive group" generated from $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. There are also standpoints of Loewy's mixed group and Brandt's quasi-group.

Here we consider axiomatic (logical) construction of 3-val without sticking to these.

We start from 1-dim. set M^1 which is commutative field. We consider M^3 as a linear additive group generatad by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, on the M^1, then we start from a "compound set"

$$M=(M^1,M^3)\quad M^1\ni o,\ 1,\ a,\ b,\ \cdots\cdots;M^3,\ni \mathbf{O},\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{A},\ \mathbf{B},\ \cdots\cdots. \tag{33}$$

M is for example M^{ra}, algebraic field, M^{re} etc. To introduce operators M^p, M^V, we necessitate M^{re} at least. We can also start from a ring etc. As M^3 is additive group.

Art. 20 Additive properties :

Additvely 1. commutative $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$, (34)

2. associative $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$, (34)

3. if $\mathbf{A}+\mathbf{B}=\mathbf{A}+\mathbf{C}$, then $\mathbf{B}=\mathbf{C}$. (36)

If necessary, we write ordinary product of M^1 as follows :

$$\divideontimes_1=\divideontimes(M^1, M^1\rightarrow M^1) \qquad (37)$$

Next, there exist already numerical product of M^1 and M^3 as M^3 is a linear additive group, but we consider this product as a kind of product : ——

$$\divideontimes_2=\divideontimes(M^1, M^3\rightarrow M^3), \qquad (38)$$

$$M^1\divideontimes_2 M^3=M^3. \qquad (39)$$

For example, $3\divideontimes_2\mathbf{A}=3\mathbf{A}=3\{4,5,7\}=\{12,15,21\}=\mathbf{B}$.

About this product $\divideontimes_2$,

Art. 21 Compound properties :

compound commutativity : $a\mathbf{A}=\mathbf{A}a$ (40)

especially $1\divideontimes_2\mathbf{A}=\mathbf{A}$ (41), $0\divideontimes_2\mathbf{A}=\mathbf{0}$; (42)

$(-1)\divideontimes_2\mathbf{A}=-\mathbf{A}$ (43), $\mathbf{A}-\mathbf{A}=\mathbf{0}$; (44)

compound first distributive property :

$$a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}, \qquad (45)$$

componnd second distributive property :

$$(a+b)\mathbf{A}=a\mathbf{A}+b\mathbf{A}, \qquad (46)$$

compound associativity :

$$a(b\mathbf{A})=(ab)\mathbf{A} \quad (47), \qquad (\mathbf{A}a)b=\mathbf{A}(ab) \quad (48), \qquad (a\mathbf{A})b=a(\mathbf{A}b). \quad (49)$$

So M^3 satisfies axioms of vector space or linear space (34—6), (41), (45—7),

Art, 22 Vector lattice ,

Further, we introduce "order" by the following rule : ——

if $A_\lambda \geqq B_\lambda$ $(\lambda=1,2,3)$, then $\mathbf{A}\geqq\mathbf{B}$. (50)

So there does not exist order between any two vector, namely chain property P4 in Art. 0 does not satisfied, e. g. surely $\mathbf{A}=\{1,2,5\}\geqq B=\{0,1,4\}$ but between $\mathbf{A}=\{1.2.5\}$ and $\mathbf{C}=\{2,1,3\}$ we cannot give order, therefore this is semi-ordered. About this order :

VLI : There is join of $\mathbf{A}$ and $\mathbf{0}$. for $A=\{-1,-5,3\}$ this join is $\{0,0,3\}$.

VL2 ; if $\mathbf{A}\geqq\mathbf{B}$, then $\mathbf{A}+\mathbf{C}\geqq\mathbf{B}+\mathbf{C}$,

VL3 ; if $\mathbf{A}\geqq\mathbf{B}$, $a>0$, then $a\mathbf{A}\geqq a\mathbf{B}$.

So in M^3, we can introduce "vector lattice" because axioms of vector lattice VL1~3 are satisfied. But the concept of vector lattice began from limit calculation in functional space and has not relations to our algebraic properties in spite of its algebaic definition, therefore we stop this topic here.

Art. 23 Linear relations.

From above description, we can define linearly independence, dependence, dimension etc, we omit them.

Art. 24 Inverse operation of $\divideontimes_2$

If $\mathbf{A}$ and $\mathbf{B}$ is collinear (parallel), we call x which fit

$$\mathbf{A}\,x=\mathbf{B} \qquad (51)$$

"numerical quotient", and if we write the magnitude of vector **A** A as defined below from inner product, then (51) has a unique solution

$$x = A^{-1} B \tag{52}$$

Next, $$a\mathbf{X} = \mathbf{B} \tag{53}$$

is satisfied by $\mathbf{X} = a^{-1}\mathbf{B}$. (54)

The operation to determine x from (51) is possible only for the case $\mathbf{A} \| \mathbf{B}$, so this is semi-divisible, in spite of the fact that the solution (52) has no conatraint such as $\mathbf{A} \| \mathbf{B}$, but the answer is in M^1, so close in the meaning of Art. 5 (53) is divisible and close. For $\divideontimes_2$ there exists no null-divisor except o.

Art. 25 Axiom of inner product.

If we introduce a third operation $\divideontimes_3$

$$\divideontimes_3 = \divideontimes\{M^3, M^3 \to M^1\} \tag{55}$$

which is called "inner product" ( ) by:

$$\mathbf{A}, \mathbf{B} \in M^3, \quad (\mathbf{A}, \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = C \in M^1, \tag{56}$$

this construct commutative distributive system, namely

inner commutativity: $(\mathbf{AB}) = (\mathbf{BA})$, (57)

inner distributive p.: ——

left distributive p.: $(\mathbf{A}, \mathbf{B} + \mathbf{C}) = (\mathbf{A}, \mathbf{B}) + (\mathbf{A}, \mathbf{C})$, (58)

right distributive p.: $(\mathbf{A} + \mathbf{B}, \mathbf{C}) = (\mathbf{A}, \mathbf{C}) + (\mathbf{B}, \mathbf{C})$. (59)

Between $\divideontimes_2$ and $\divideontimes_3$: ——

inner compound associativity: $a(\mathbf{A}, \mathbf{B}) = (a\mathbf{A}, \mathbf{B}) = (\mathbf{A}, a\mathbf{B})$. (60)

Art. 26 $M^3 \to M^1$ cerrespondence

If we adopt **A** as **B** in (56),

$$(\mathbf{AA}.) = A_1A_1 + A_2A_2 + A_3A_3 \tag{61}$$

we call it "norm" of vectet **A**, and we call positive root of it "magnitude" of **A** and write A, $|\mathbf{A}|$, $\|\mathbf{A}\|$ etc. For every element of M^3, two element of M^1 (norm and magnitude) correspond, but inversely for the element O in M^1 corresponds the vector **O** in M^3 (in indefinite quadratic metric space, even this correspondence is not unique e. g. in 4-dimensional relativistic world). But for every element of M^1 does not correspond the element of M^3 uniquely.

Art. 27 Inner null-divisor.

Above introduced inner product does not fulfil (2)in M^3, namely M^3 is not closed about $\divideontimes_3$., so this cannot be associative in the meaning of Art. 6, nor divisible in Art. 7. Therefore M^3 is not group about $\divideontimes_3$, even not semi- group, but the result falls in M^1, so this is closed as a compound set (33), and M^1 is commutative field, so M^1 has O and 1. Therefore we call the **X** which satisfies

$$\mathbf{A} \neq \mathbf{O}, \ (\mathbf{AX}) = O \in M^1 \tag{62}$$

"right inner null-divisor" of **A**, this is also the left one by (57), so we can call it simply "inner null-divisor" and we can write it $\mathbf{A}\backslash O$, $O/\mathbf{A}$, $\frac{O}{\mathbf{A}}$, or special letter **N**.

Besides zero vector $\mathbf{O}$, any vertical vector to $\mathbf{A}$ is also inner null-divisor, namely

$$\mathbf{X}=\mathbf{O} \text{ or } \mathbf{X}=\mathbf{A}\backslash O=O/\mathbf{A}=\frac{O}{\mathbf{A}}=\mathbf{N}\,.\ \mathbf{N}\perp\mathbf{A}\,. \tag{63}$$

So M^3 is non-integral domain about inner product. By the experience of Art. 17, if

$$\mathbf{X}=\mathbf{N}\,.\ \text{then } \mathbf{X}=k\mathbf{N}\,,\ k\in M^1:\ \text{any value.} \tag{64}$$

is also solution.

Art. 28 Inner inverse element.

Let us call $\mathbf{X}$ "inner inverse element" which satisfies

$$\mathbf{A}\neq\mathbf{O}\,,\ (\mathbf{AX})=1 \tag{65}$$

and write it $\mathbf{A}\backslash 1=1/\mathbf{A}=\frac{1}{\mathbf{A}}$ (66)

Vector parallel to $\mathbf{A}$ and of inverse magnitude to A is surely one of the solution. If $\mathbf{K}$ is one solution, from the experience of Art. 19 $\mathbf{K}+k\mathbf{N}$ is also solution,

namely if $\mathbf{X}=\mathbf{K}$, then $\mathbf{X}=\mathbf{K}+k\mathbf{N}$. (67)

If wo adopt $\mathbf{X}$ parallel to $\mathbf{A}$, then about magnitude in M^1,

$$AX=1. \tag{68}$$

So surely (66) is an extension of reciprocal number.

Art. 29 Inner quotieut.

If we call $\mathbf{X}$ which satisfies

$$\mathbf{A}\neq\mathbf{O},\ (\mathbf{AX})=b\neq 0 \tag{69}$$

"right inner quotient", this becomes also left quotient, so we can call it simply "inner quotient". So we can write it

$$\mathbf{A}\backslash b=b/\mathbf{A}=\frac{b}{\mathbf{A}}. \tag{70}$$

If $\mathbf{K}$ is a solution, also $\mathbf{K}+k\mathbf{N}$ is also solution. In case where $\mathbf{A}$ and $\mathbf{X}$ are parallel.

$$AX=b. \tag{71}$$

But in genearl case,

$$AK\cos(\mathbf{AK})=b. \tag{72}$$

If we adopt $(\mathbf{AX})=(\mathbf{AB})$ (73)

instead of (69), this represents a plane which passes through the terminal point of $\mathbf{B}$ and vertical to $\mathbf{A}$, in this case we do not need the constraint that $\mathbf{A},\mathbf{B}$ are not zero.

Art. 30 Axioms of outer product.

If we introduce forth operation in M^3

$$\divideontimes_4=\divideontimes(M^3,M^3\to M^3) \tag{74}$$

"vector product", "outer product" [ ] $\times$

$$[\mathbf{AB}]=\mathbf{A}\times\mathbf{B}=e_1(A_2B_3-A_3B_2)+e_2(A_3B_1-A_1B_3)+e_3(A_1B_2-A_2B_1)$$
$$=3!\,\mathbf{e}_{[1}A_2B_{3]}, \tag{75}$$

this fulfils Art.5 and is close. Between $\divideontimes_2$ and $\divideontimes_4$, holds outer compound associativity

$$a[\mathbf{AB}]=[a\mathbf{A},\mathbf{B}]=[\mathbf{A},a\mathbf{B}]. \tag{76}$$

$(M^3\{+,[\ \ ]\})$ constructs Lie-ring, by (21) and (24).

Art. 31 Outer null-divisor.

$$[\mathbf{AB}]=\mathbf{C} \tag{77}$$

holds only when **A** and **C** are vertical, so we cannot decide **B** for any **A** and **C**, so outer product is reversible only for vertical **A** and **C**, namely this Lie-ring is semi-divisible. Algebraic criterion that (77) is soluble about B_t is

$$(\mathbf{AC})=0, \tag{78}$$

this is nothing else than vertical condition of **A** and **C**. Now,

$$[\mathbf{AX}]=\mathbf{A} \tag{79}$$

is satisfied only by

$$\mathbf{A}=\mathbf{O}.\ \mathbf{X}=\text{any finite vector} \tag{80}$$

and for any non-zero vector **A**, **X** does not exist, so this Lie-ring is semi-divisible but this has not unit vector independent of coordinates, therefore this has also no outer inverse vector. After all, distinction between inner and outer quotient is necessary but we do not need the word "inner" for inverse vector. There is **O** which fits

$$\mathbf{A}+\mathbf{O}=\mathbf{A},\ [\mathbf{AO}]=\mathbf{O}, \tag{81}$$

so, $$\mathbf{A}\neq\mathbf{O},\quad [\mathbf{AX}]=\mathbf{O} \tag{82}$$

holds for proper **X**, we call this **X** "right outer null-divisor". this is also left one from Art. 15., we write this $\mathbf{A}\backslash\mathbf{O}$ and $\mathbf{O}/\mathbf{A}$ respectively,

$$\mathbf{X}=\mathbf{A}\backslash\mathbf{O}=-\mathbf{O}/\mathbf{A}\ \textit{or}\ \mathbf{X}=\mathbf{O}/\mathbf{A}=-\mathbf{A}/\mathbf{O}. \tag{83}$$

A is itself outer null-divisor to **A** by Art.13, so by applying experience of Art.17 (27) to $\divideontimes_2$, we get

$$\mathrm{X}=k\mathbf{A},\ k\in \mathrm{M}^1:\ \text{arbitrary}. \tag{84}$$

$k=1$: (21) nilpotent, $k=-1$: (83)

(84) is the most general form of outer null-divisor. (84) can be written also.

$$\mathbf{X}=\mathbf{K},\ \mathbf{K}\|\mathbf{A}$$

So $$[\mathbf{AB}]=0 \text{ is equivalent to } \mathbf{A}\|\mathbf{B} \tag{85}$$

$$(\mathbf{AB})=0 \text{ is equivalent to } \mathbf{A}\perp\mathbf{B} \tag{86}$$

Art. 32 Outer quotient vector.

For any vectors which have conditions:

$$\mathbf{A}\neq 0.\ \mathbf{A}\perp\mathbf{B} \text{ namely } (\mathbf{AB})=0,$$

there exist **X** for

$$[\mathbf{AX}]=\mathbf{B}. \tag{87}$$

K which is vertical to **A** and **B** and

$$\mathrm{K}=\mathrm{A}^{-1}\mathrm{B} \tag{88}$$

is a solution of (87). About the sense of **A**,**K**,**B**, it is necessary to construct right hand system in this order. Let us call such solution right outer quotient vector and write $\mathbf{A}\backslash\mathbf{B}$. From antisymmetry of $\divideontimes_4$, between **X** and **Y** which fits

$$[\mathbf{YA}]=\mathbf{B}, \tag{89}$$

there is a relation

$$\mathbf{A}\backslash\mathbf{B}=-\mathbf{B}/\mathbf{A}. \tag{90}$$

So we cannot write this $\frac{\mathbf{B}}{\mathbf{A}}$, neither $\mathbf{A}^{-1}\mathbf{B},\mathbf{B}\mathbf{A}^{-1}$ because of lack of outer inverse vector. To avoid the vertical condition of $\mathbf{A}$ and $\mathbf{B}$ in the problem, we can start from

$$[\mathbf{AX}]=[\mathbf{AC}].\ \mathbf{A}.\mathbf{C}: \text{quite free,} \tag{91}$$

because the right side is always vertical to $\mathbf{A}$. This $\mathbf{X}$ represents a straight line passing through the terminal point of $\mathbf{C}$ and parallel to $\mathbf{A}$. Using the same expression,

$$(\mathbf{PV})=0,\ \mathbf{P}\backslash O=O/\mathbf{P}=\frac{O}{\mathbf{P}}=[\mathbf{QP}] \text{ or } p[\mathbf{QP}],\ p.\ \mathbf{Q}: \text{arbitrary} \tag{92}$$

This is one of the most general form of inner null-divisor.

Inner quotient $\mathbf{A}\backslash b$ has numerical numerater $\in M^1$, on the contrary outer quotient has vectorial numerator $\in M^3$, so we don't need special notation to distinguish them. For $\divideontimes_4$, M^3 is a Lie-ring, so it has no unit nor inverse vector, this is the reason that we wrote divisibility (4), (5) in Art. 7 before Art. 8 and 9; also we wrote Art. 3 commutativity before Art. 5 closed property because inner product is commutative but not clossd.

§3 Inverse vector.

Art. 33 Inverse vector.

If we denote a unit vector of sense $\mathbf{A}$,

$$\mathbf{e}_A=\frac{\mathbf{A}}{A}=\mathbf{A}A^{-1}, \tag{93}$$

then inverse vector is from Art. 28

$$\mathbf{X}=A^{-1}\mathbf{e}_A=\frac{1}{A}\frac{\mathbf{A}}{A}=\frac{\mathbf{A}}{A^2}=\frac{\mathbf{A}}{(\mathbf{AA})} \tag{94}$$

or
$$\mathbf{X}=\mathbf{A}(\mathbf{AA})^{-1}=\mathbf{A}A^{-2}. \tag{95}$$

because
$$(\mathbf{AA})=A^2,\ [\mathbf{AA}]\equiv 0, \tag{96}$$

we can use $\mathbf{A}^2$ and not A^2 for $(\mathbf{AA})$.

If we adopt $\mathbf{A}^2$, we can write $\mathbf{A}^{-1}$ for $\mathbf{X}$ from addition rule of exponent. Now we ignore $\divideontimes_4$ for $\mathbf{A}^m$ from nilpotent property, and define

$$(\mathbf{AA})\overset{D}{=}\mathbf{A}^2, \tag{97}$$

$\overset{D}{=}$ means an equation of definition. As $\mathbf{A}^2\in M^1$, so its reciprocal exists, which we write $\mathbf{A}^{-2}$, namely

$$\mathbf{A}^{-2}\overset{D}{=}(\mathbf{A}^2)^{-1}=(\mathbf{AA})^{-1}. \tag{98}$$

From axiom of M^1,

$$\mathbf{A}^2\mathbf{A}^{-2}=\mathbf{A}^{-2}\mathbf{A}^2=1 \tag{99}$$

By this notation, (95) is written as

$$\mathbf{A}\mathbf{A}^{-2}=\mathbf{A}^{-2}\mathbf{A}\overset{D}{=}\mathbf{A}^{-1}. \tag{100}$$

That $\mathbf{A}^{-1}$ fits (65) is written

$$(\mathbf{A}\mathbf{A}^{-1})=(\mathbf{A}^{-1}\mathbf{A})=1. \tag{101}$$

This is shown by substituting (100) to $\mathbf{A}^{-1}$. similarly, by (100),

$$[\mathbf{A}\mathbf{A}^{-1}]=[\mathbf{A},\mathbf{A}\mathbf{A}^{-2}]=[\mathbf{A}\mathbf{A}^{-2},\mathbf{A}]=[\mathbf{A}^{-1}\mathbf{A}]=\mathbf{A}^{-2}[\mathbf{AA}]=0.$$

By (23),

$$[\mathbf{A}\mathbf{A}^{-1}] = -[\mathbf{A}^{-1}\mathbf{A}], \text{so } [\mathbf{A}\mathbf{A}^{-1}] = \pm[\mathbf{A}^{-1}\mathbf{A}],$$

after all,

$$[\mathbf{A}\mathbf{A}^{-1}] = [\mathbf{A}^{-1}\mathbf{A}] = 0. \tag{102}$$

Next,

$$\mathbf{A}^2\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}, \tag{103}$$

$$(\mathbf{A}^{-1}\mathbf{A}^{-1}) = (\mathbf{A}^{-2}\mathbf{A}.\mathbf{A}^{-2}\mathbf{A}) = \mathbf{A}^{-2}\mathbf{A}^{-2}(\mathbf{A},\mathbf{A}) = \mathbf{A}^{-2}, \tag{104}$$

$$[\mathbf{A}^{-1}\mathbf{A}^{-1}] = 0, \tag{105}$$

$$\mathbf{A}^0 = 1,\ \mathbf{A}^1 = \mathbf{A}. \tag{106}$$

Art. 34 Exponent of vector.

Above equations are summarized to

$$\mathbf{A}^{2p+1} = (\mathbf{A}\mathbf{A})^p\mathbf{A} \in \mathrm{M}^3 : \text{vector},\ p = -1,0 \tag{107}$$

$$\mathbf{A}^{2q} = (\mathbf{A}\mathbf{A})^q \in \mathrm{M}^1 : \text{scalar},\ q = 0,1,-1 \tag{108}$$

Adopting proper product from $\divideontimes_2, \divideontimes_3, \divideontimes_4$, we can extend p and q to any integer M^i.

Similarly adopting $\divideontimes_k$ properly,

$$\mathbf{A}^a\mathbf{A}^b = \mathbf{A}^b\mathbf{A}^a = \mathbf{A}^{a+b} \tag{109}$$

$a+b=1+1=2 : (97);\ 2-2=0 : (99);\ 1-2=-1 : (100);$

$1-1=0 : (101);\ 2-1=1 : (103);\ -1-1=-2 : (104).$

Namely these exhaust all the cases where $a.b : -2\sim2$ and $|a+b|\leqq 2$. But further we can define

$$-2-2=-4 : \mathbf{A}^{-2}\mathbf{A}^{-2} \overset{D}{=} \mathbf{A}^{-4}, \tag{110}$$

$$1-4=-3 : \mathbf{A}\mathbf{A}^{-4} = \mathbf{A}^{-4}\mathbf{A} \overset{D}{=} \mathbf{A}^{-3}, \tag{111}$$

$$2-4=-2 : \mathbf{A}^2\mathbf{A}^{-4} = \mathbf{A}^{-4}\mathbf{A}^2 \overset{D}{=} \mathbf{A}^2. \tag{112}$$

By these we exhausted all cases of a and $b : -2\sim2$

In

$$(\mathbf{A}^r)^s = \mathbf{A}^{rs}, \tag{113}$$

$r,s=0,1$ are trivial,

$r=2, s=1$ is a definition (98),

$$r=-1,\ s=2 : (\mathbf{A}^{-1})^2 = \mathbf{A}^{-2}, \text{ by } (104) \tag{114}$$

$$s=-1,\quad (\mathbf{A}^{-1})^{-1} = (\mathbf{A}^{-2}\mathbf{A})^{-1} = (\mathbf{A}^{-2})^{-1}(\mathbf{A})^{-1} = \mathbf{A}^2\mathbf{A}^{-2}\mathbf{A} = \mathbf{A} \tag{115}$$

We did not used above the property:

$$(\mathbf{A}\mathbf{B})^m = (\mathbf{A}^m\mathbf{B}^m)$$

which does not hold generally, but the property which we used above is the case of

$$p = -1$$

in

$$(a\mathbf{A})^p = a^p\mathbf{A}^p, \tag{116}$$

which can be shown from (60).

$s=-2$: from (98),

$$(\mathbf{A}^{-1})^{-2} = (\mathbf{A}^{-1}\mathbf{A}^{-1})^{-1} = \mathbf{A}^2. \tag{117}$$

Art. 35 Application of vectorial exponent.

Following equations are used in van^{-1}. To that object, we write $\mathbf{D}$ for $\mathbf{A}$ and we write down various results operated from scalar a and vector $\mathbf{A}$ by proper $\divideontimes_k (k=1,2,3,4,)$

$$a=\mathbf{D}^2\mathbf{D}^{-2}a=\mathbf{D}^{-2}\mathbf{D}^2a=\mathbf{D}^{-2}(\mathbf{D},\mathbf{D}a)=(\mathbf{D},\mathbf{D}^{-2}\mathbf{D}a)=(\mathbf{D}^{-2}\mathbf{D},\mathbf{D}a) \tag{118}$$

$$=(\mathbf{D},\mathbf{D}^{-1}a)=(\mathbf{D}^{-1},\mathbf{D}a), \tag{119}$$

$$(\mathbf{D}^2)^{-1}a=\mathbf{D}^{-2}a, \tag{120}$$

$$\mathbf{A}=\mathbf{D}^2\mathbf{D}^{-2}\mathbf{A}=\mathbf{D}^{-2}\mathbf{D}^2\mathbf{A}=\mathbf{D}^{-2}(\mathbf{D}\mathbf{D})\mathbf{A}=(\mathbf{D}\mathbf{D}^{-1})\mathbf{A}=(\mathbf{D}^{-1},\mathbf{D})\mathbf{A}, \tag{121}$$

$$(\mathbf{D}^2)^{-1}\mathbf{A}=\mathbf{D}^{-2}\mathbf{A}, \tag{122}$$

$$\mathbf{D}\mathbf{D}^{-2}a=\mathbf{D}^{-2}\mathbf{D}a=\mathbf{D}^{-1}a, \tag{123}$$

$$(\mathbf{D},\mathbf{D}^{-2}\mathbf{A})=\mathbf{D}^{-2}(\mathbf{D}\mathbf{A})=(\mathbf{D}^{-1}\mathbf{A}), \tag{124}$$

$$[\mathbf{D},\mathbf{D}^{-2}\mathbf{A}]=\mathbf{D}^{-2}[\mathbf{D}\mathbf{A}]=[\mathbf{D}^{-1}\mathbf{A}], \tag{125}$$

$$[\mathbf{D},\mathbf{D}^{-1}a]=[\mathbf{D}^{-1},\mathbf{D}a]=[\mathbf{D},\mathbf{D}^{-2}\mathbf{D}a]=\mathbf{D}^{-2}[\mathbf{D},\mathbf{D}a]=0, \tag{126}$$

$$\mathbf{D}(\mathbf{D},\mathbf{D}^{-2}\mathbf{A})=\mathbf{D}(\mathbf{D},{}^{-1}\mathbf{A})=\mathbf{D}\mathbf{D}^{-2}(\mathbf{D}\mathbf{A})=\mathbf{D}^{-1}(\mathbf{D}\mathbf{A})=\mathbf{D}^{-2}\mathbf{D}(\mathbf{D}\mathbf{A}), \tag{127}$$

$$(\mathbf{D}[\mathbf{D}^{-1}\mathbf{A}])=([\mathbf{D}\mathbf{D}^{-1}]\mathbf{A})=(\mathbf{D}\mathbf{D}^{-2},[\mathbf{D}\mathbf{A}])=(\mathbf{D}^{-1}[\mathbf{D}\mathbf{A}])=\mathbf{D}^{-2}(\mathbf{D}[\mathbf{D}\mathbf{A}])=0, \tag{128}$$

$$\Big[\mathbf{D}[\mathbf{D}^{-1}\mathbf{A}]\Big]=\Big[\mathbf{D}\mathbf{D}^{-2}[\mathbf{D}\mathbf{A}]\Big]=\Big[\mathbf{D}^{-1}[\mathbf{D}\mathbf{A}]\Big]=\mathbf{D}^{-1}(\mathbf{D}\,\mathbf{A})-(\mathbf{D}\,\mathbf{D}^{-1})\mathbf{A}, \tag{129}$$

$$=\mathbf{D}^{-1}(\mathbf{D}\mathbf{A})-\mathbf{A}=\mathbf{D}(\mathbf{D}^{-1}\mathbf{A})-\mathbf{A}, \tag{130}$$

$$=\Big[\mathbf{D}[\mathbf{D},\mathbf{D}^{-2}\mathbf{A}]\Big]=\mathbf{D}^{-2}\Big[\mathbf{D}[\mathbf{D}\mathbf{A}]\Big]=\mathbf{D}^{-2}\Big\{\mathbf{D}(\mathbf{D}\mathbf{A})-\mathbf{D}^2\mathbf{A}\Big\}. \tag{131}$$

Art. 36 Decomposition of vector.

From above results,

$$\mathbf{A}=\mathbf{D}^{-1}(\mathbf{D}\mathbf{A})-\Big[\mathbf{D}^{-1}[\mathbf{D}\mathbf{A}]\Big], \tag{132}$$

$$=\mathbf{D}(\mathbf{D}^{-1}\mathbf{A})-\Big[\mathbf{D}[\mathbf{D}^{-1}\mathbf{A}]\Big], \tag{133}$$

$$=\mathbf{D}^{-2}\Big\{\mathbf{D}(\mathbf{D}\mathbf{A})-\Big[\mathbf{D}[\mathbf{D}\mathbf{A}]\Big]\Big\}=\mathbf{D}\mathbf{D}^{-2}(\mathbf{D}\mathbf{A})-\Big[\mathbf{D}\mathbf{D}^{-2}[\mathbf{D}\mathbf{A}]\Big]. \tag{134}$$

If we write (132∼4) by using $\mathrm{D}^{-1}\mathrm{D}^{-1}$ instead of $(\mathbf{D}\mathbf{D})=\mathbf{D}^{-2}$,

$$\mathbf{A}=\frac{1}{\mathrm{D}^2}\Big\{\mathbf{D}(\mathbf{D}\mathbf{A})-\Big[\mathbf{D}[\mathbf{D}\mathbf{A}]\Big]\Big\}=\frac{\mathbf{D}}{\mathrm{D}}\Big(\frac{\mathbf{D}}{\mathrm{D}}\mathbf{A}\Big)-\Big[\frac{\mathbf{D}}{\mathrm{D}}\Big[\frac{\mathbf{D}}{\mathrm{D}}\mathbf{A}\Big]\Big]$$

$$=\mathbf{e}_D[\mathbf{e}_D\mathbf{A}]-\Big[\mathbf{e}_D[\mathbf{e}_D\mathbf{A}]\Big]. \tag{135}$$

This is nothing else than the well-known formula of decomposition of any vector $\mathbf{A}$ into parallel and vertical component to any vector $\mathbf{D}$, $\mathbf{A}_\parallel$ and $\mathbf{A}_\perp$: $\mathbf{A}=\mathbf{A}_\parallel+\mathbf{A}_\perp$ (136)

$$\mathbf{A}_\parallel \parallel \mathbf{D} \text{ namely } [\mathbf{D}\mathbf{A}_\parallel]=0, \tag{137}$$

$$\mathbf{A}_\perp \perp \mathbf{D} \;\;'' \;\; (\mathbf{D}\mathbf{A}_\perp)=0, \tag{138}$$

where

$$\mathbf{A}_\parallel=\mathbf{A}_{\parallel D}=\mathbf{e}_D(\mathbf{e}_D\mathbf{A}), \tag{139}$$

$$\mathbf{A}_\perp=\mathbf{A}_{\perp D}=-\Big[\mathbf{e}_D[\mathbf{e}_D\mathbf{A}]\Big], \tag{140}$$

also

$$\mathbf{A}_{\parallel}=\mathbf{D}^{-1}(\mathbf{D}\mathbf{A})=\mathbf{D}(\mathbf{D}^{-1}\mathbf{A})=\mathbf{D}^{-1}(\mathbf{D}\mathbf{A}_{\parallel})=\mathbf{D}(\mathbf{D}^{-1}\mathbf{A}_{\parallel})=\mathbf{A}-\mathbf{A}_{\perp}, \tag{141}$$

$$\mathbf{A}_{\perp}=-\left[\mathbf{D}^{-1}[\mathbf{D}\mathbf{A}]\right]=-\left[\mathbf{D}[\mathbf{D}^{-1}\mathbf{A}]\right]=-\left[\mathbf{D}^{-1}[\mathbf{D}\mathbf{A}_{\perp}]\right]=-\left[\mathbf{D}[\mathbf{D}^{-1}\mathbf{A}_{\perp}]\right]=\mathbf{A}-\mathbf{A}_{\parallel} \tag{142}$$

$$\mathbf{A}_{\parallel}:(\mathbf{D}\mathbf{A}_{\parallel})=(\mathbf{D}\mathbf{D}^{-1})(\mathbf{D}\mathbf{A})=(\mathbf{D}\mathbf{A}),$$
$$[\mathbf{D}\mathbf{A}_{\parallel}]=[\mathbf{D}\mathbf{D}^{-1}](\mathbf{D}\mathbf{A})=0,$$
$$\mathbf{A}_{\perp}:[\mathbf{D}\mathbf{A}_{\perp}]=-\left[\mathbf{D}[\mathbf{D}^{-1}[\mathbf{D}\mathbf{A}]]\right]=-\mathbf{D}^{-1}(\mathbf{D}[\mathbf{D}\mathbf{A}])+(\mathbf{D}\mathbf{D}^{-1})[\mathbf{D}\mathbf{A}]=[\mathbf{D}\mathbf{A}],$$
$$(\mathbf{D}\mathbf{A}_{\perp})=-(\mathbf{D}[\mathbf{D}[\mathbf{D}^{-1}\mathbf{A}]])=0, \tag{143}$$

Namely

$$(\mathbf{D}\mathbf{A}_{\parallel})=(\mathbf{D}\mathbf{A}),\ [\mathbf{D}\mathbf{A}_{\parallel}]=0,$$
$$[\mathbf{D}\mathbf{A}_{\perp}]=[\mathbf{D}\mathbf{A}],\ (\mathbf{D}\mathbf{A}_{\perp})=0 \tag{144}$$

is shown algebraically. The operators:

$$\mathbf{D}^{-1}(\mathbf{D}\cdot)=\mathbf{D}(\mathbf{D}^{-1}\cdot)=\mathbf{e}_D(\mathbf{e}_D\cdot)$$

and

$$-\left[\mathbf{D}^{-1}[\mathbf{D}\cdot]\right]=-\left[\mathbf{D}[\mathbf{D}^{-1}\cdot]\right]=-\left[\mathbf{e}_D[\mathbf{e}_D\cdot]\right]$$

ane mutually null-divisor and each is idempotent (Art.10(16)).

Art. 37 Application of vectorial exponent (continued)

$$\mathbf{D}^{-1}\mathbf{D}^2a=\mathbf{D}^2\mathbf{D}^{-1}a=\mathbf{D}a, \tag{145}$$
$$(\mathbf{D}^{-1},\mathbf{D}^2\mathbf{A})=\mathbf{D}^2(\mathbf{D}^{-1}\mathbf{A})=(\mathbf{D}\mathbf{A}), \tag{146}$$
$$[\mathbf{D}^{-1},\mathbf{D}^2\mathbf{A}]=\mathbf{D}^2[\mathbf{D}^{-1}\mathbf{A}]=[\mathbf{D}\mathbf{A}], \tag{147}$$
$$\mathbf{D}^{-2}a=(\mathbf{D}^{-1}\mathbf{D}^{-1})a=(\mathbf{D}^{-1},\mathbf{D}^{-1}a), \tag{148}$$
$$\mathbf{D}^{-2}\mathbf{A}=(\mathbf{D}^{-1},\mathbf{D}^{-1})\mathbf{A}=\mathbf{D}^{-2}\left\{\mathbf{D}(\mathbf{D}^{-1}\mathbf{A})-\left[\mathbf{D}[\mathbf{D}^{-1}\mathbf{A}]\right]\right\}=\mathbf{D}^{-1}(\mathbf{D}^{-1}\mathbf{A})-\left[\mathbf{D}^{-1}[\mathbf{D}^{-1}\mathbf{A}]\right], \tag{149}$$
$$\mathbf{D}^{-2}\mathbf{A}_{\parallel}=\mathbf{D}^{-1}(\mathbf{D}^{-1}\mathbf{A})=\mathbf{D}^{-1}(\mathbf{D}^{-1}\mathbf{A}_{\parallel}), \tag{150}$$
$$\mathbf{D}^{-2}\mathbf{A}_{\perp}=-\left[\mathbf{D}^{-1}[\mathbf{D}^{-1}\mathbf{A}]\right]=-\left[\mathbf{D}^{-1}[\mathbf{D}^{-1}\mathbf{A}_{\perp}]\right], \tag{151}$$
$$[\mathbf{D}^{-1},\mathbf{D}^{-1}a]=0, \tag{152}$$
$$\left([\mathbf{D}^{-1}\mathbf{D}^{-1}]\mathbf{A}\right)=\left(\mathbf{D}^{-1}[\mathbf{D}^{-1}\mathbf{A}]\right)=0, \tag{153}$$
$$\left[[\mathbf{D}^{-1}\mathbf{D}^{-1}]\mathbf{A}\right]=0, \tag{154}$$
$$(\mathbf{D}^{-1})^{-1}a=\mathbf{D}a,\ \left((\mathbf{D}^{-1})^{-1},\mathbf{A}\right)=(\mathbf{D}.\mathbf{A}),\ \left[(\mathbf{D}^{-1})^{-1}\mathbf{A}\right]=[\mathbf{D},\mathbf{A}], \tag{155}$$

From (110～2), also

$$(\mathbf{D}^{-2})^{-1}a=\mathbf{D}^2a,\ (\mathbf{D}^{-2})^{-1}\mathbf{A}=\mathbf{D}^2\mathbf{A}.$$
$$\mathbf{D}^{-2}\mathbf{D}^{-2}a=\mathbf{D}^{-4}a, \tag{156}$$
$$\mathbf{D}^{-2}\mathbf{D}^{-2}\mathbf{A}=\mathbf{D}^{-4}\mathbf{A}, \tag{157}$$
$$\mathbf{D}\mathbf{D}^{-4}a=\mathbf{D}^{-4}\mathbf{D}a=\mathbf{D}^{-3}a, \tag{158}$$

$$(\mathbf{D},\mathbf{D}^{-4}\mathbf{A})=\mathbf{D}^{-4}(\mathbf{DA})=(\mathbf{D}^{-3}\mathbf{A}), \tag{159}$$

$$[\mathbf{D},\mathbf{D}^{-4}\mathbf{A}]=\mathbf{D}^{-4}[\mathbf{DA}]=[\mathbf{D}^{-3}\mathbf{A}], \tag{160}$$

$$\mathbf{D}^2\mathbf{D}^{-4}a=\mathbf{D}^{-4}\mathbf{D}^2a=\mathbf{D}^{-2}a, \tag{161}$$

$$\mathbf{D}^2\mathbf{D}^{-4}\mathbf{A}=\mathbf{D}^{-4}\mathbf{D}^2\mathbf{A}=\mathbf{D}^{-2}\mathbf{A}, \tag{162}$$

$$\mathbf{D}^{-2}\mathbf{A}=\Big[\mathbf{D}[\mathbf{D},\mathbf{D}^{-4}\mathbf{A}]\Big]-\mathbf{D}(\mathbf{D},\mathbf{D}^{-4}\mathbf{A}). \tag{163}$$

As shown in later § 7, we exhausted all the formulae of van^{-1} by above formulae.

Art. 38 Representation of inner quotient by inverse vector.

The answers of (69) and (73) are:

$(\mathbf{PV})=w$, *Ans.*: $\mathbf{V}=\mathbf{P}^{-1}w=w\mathbf{P}^{-1}$,

$(\mathbf{PV})=(\mathbf{PA})$, *Ans*: $\mathbf{V}=\mathbf{P}^{-1}(\mathbf{PA})=\mathbf{P}(\mathbf{P}^{-1}\mathbf{A})$.

These are not general solution because these do not contain null-divisors. We see (132) to investigate it.

(132) : $\mathbf{A}=\mathbf{D}^{-1}(\mathbf{DA})-\Big[\mathbf{D}^{-1}[\mathbf{DA}]\Big]$.

is decomposition formula of $\mathbf{A}$ as described in Art. 36, but we can interpret this a formula to determine $\mathbf{A}$ from $(\mathbf{DA})$ and $[\mathbf{DA}]$.

If only $(\mathbf{DA})$ is given, the second term becomes null-divisor.

Namely, for

$$(\mathbf{DA})=a, \tag{166}$$

we can adopt a null-divisor such as determined by given $[\mathbf{DA}]=\mathbf{B}$

$$\mathbf{A}=\mathbf{D}^{-1}a-\Big[\mathbf{D}^{-1}[\mathbf{DA}]\Big], \tag{167}$$

$$\text{or } \mathbf{A}=\mathbf{D}^{-1}a-p\Big[\mathbf{D}^{-1}[\mathbf{DA}]\Big]. \tag{168}$$

In the second term $[\mathbf{DA}]$ is vertical to $\mathbf{D}$ and $\mathbf{A}$, whatever $\mathbf{A}$ be, but the inner null-divisor satisfies (160) if only it is vertical to $\mathbf{D}$, so $[\mathbf{DA}]$ can be substituted by a quite arbitrary vector $\mathbf{Q}$ and at the same time by Art. 17, we can adopt following representations as the answer of (166):

$$\mathbf{D}^{-1}a-[\mathbf{D}^{-1}\mathbf{Q}],\ \mathbf{D}^{-1}a-p[\mathbf{D}^{-1}\mathbf{Q}],\ \mathbf{D}^{-1}a+p[\mathbf{D}^{-1}\mathbf{Q}],\ \mathbf{D}^{-1}a-[\mathbf{DQ}],$$
$$\mathbf{D}^{-1}a-p[\mathbf{DQ}].\ \mathbf{D}^{-1}a+p[\mathbf{Q},\mathbf{D}] \quad \text{etc.} \tag{169}$$

We would call the null-divisor in (167) "Normal type" in these. Similarly if we are given only outer product $[\mathbf{DA}]$, the first term of (132) becomes outer null-divisor, so the solution of

$$[\mathbf{DA}]=\mathbf{B} \tag{170}$$

can be written down

$$-[\mathbf{D}^{-1}\mathbf{B}]+\mathbf{D}^{-1}(\mathbf{DA})$$
$$-[\mathbf{D}^{-1}\mathbf{B}]+p\mathbf{D}^{-1}(\mathbf{DA})$$
$$-[\mathbf{D}^{-1}\mathbf{B}]+p\mathbf{D} \quad \text{etc.} \tag{171}$$

We could call the null-divisor in the first form of (171) which is got directly from $(\mathbf{DA})$ Normal type". If we adopt (73) and (91) instead of (69) and (87),

$$(\mathbf{DX})=(\mathbf{DA}),\ \mathbf{X}_1=\mathbf{D}^{-1}(\mathbf{DA})-p\left[\mathbf{D}^{-1}[\mathbf{DA}]\right]$$

$$\mathbf{X}_2=\mathbf{D}^{-1}(\mathbf{DA})-p[\mathbf{DQ}] \tag{172}$$

$$[\mathbf{DX}]=[\mathbf{DA}], \tag{173}$$

$$\mathbf{X}^1=-\left[\mathbf{D}^{-1}[\mathbf{DA}]\right]+p\mathbf{D}^{-1}(\mathbf{DA}),$$

$$\mathbf{X}_2=-\left[\mathbf{D}^{-1}[\mathbf{DA}]\right]+p\mathbf{D} \tag{174}$$

distributive property of inner quotient can be got from (164):

$$\mathbf{A}\backslash(b+c)=\mathbf{A}\backslash b+\mathbf{A}\backslash c. \tag{175}$$

To utilize in van^{-1}, we would rewrite the notation of (51) and (52) in the inverse operation of ※$_2$ in Art.24,

$$\mathbf{D}a=\mathbf{A}\ (176)\ \ a=D^{-1}A=\frac{DA}{DD}=\frac{(\mathbf{DA})}{(\mathbf{DD})}=(\mathbf{D}^{-1}\mathbf{A}) \tag{177}$$

This is also distributive,

from

$$\mathbf{D}a=\mathbf{B}+\mathbf{C},\ (\mathbf{D}^{-1},\mathbf{B}+\mathbf{C})=(\mathbf{D}^{-1},\mathbf{B})+(\mathbf{D}^{-1},\mathbf{C}) \tag{178}$$

when $\mathbf{D}$ become ∇, this is known as "Law of superposition" or "linearity" in electric engineering.

§4 General inverse vector

Inner inverse element, namely $\mathbf{X}$ *satisfying (65) is not only* $\mathbf{A}^{-1}$ *but it has additional* term as a inner null-divisor. Here we search single term representation of (167～9). Let $\mathbf{r}$ be any vector which is neither zero nor vertical to $\mathbf{A}$,

namely $\mathbf{r}\neq 0$ and $\mathbf{r}\not\perp\mathbf{A}$, of in one word

$$(\mathbf{Ar})\neq 0, \tag{179}$$

Then $\mathbf{A}$-direction component of $\mathbf{r}$ is

$$(\mathbf{r}\mathbf{e}_A)=(\mathbf{r}\frac{\mathbf{A}}{A}).$$

So if we consider $\mathbf{A}$-componeht only for $\mathbf{r}$, $\mathbf{r}$ divided by this $(\mathbf{r}\mathbf{e}_A)=(\mathbf{r}\frac{\mathbf{A}}{A})$, namely

$$\frac{\mathbf{r}}{\left(\mathbf{r}\frac{\mathbf{A}}{A}\right)}=\frac{\mathbf{r}}{(\mathbf{rA})}A$$ has a length 1 in $\mathbf{A}$-direction, therefore if we divide thi

further by A, we get

$$\frac{\mathbf{r}}{(\mathbf{r}\mathbf{e}_A)}\frac{1}{A}=\frac{\mathbf{r}}{(\mathbf{rA})}\frac{A}{A}=\frac{\mathbf{r}}{(\mathbf{rA})}. \tag{180}$$

$\mathbf{A}$-*component of this has leugth of* A^{-1}. So this is a representation of general solution of inverse vector of type (169). That this fits (65) is verified at once by substitution. Actually decomcosing $\mathbf{r}$ into $\|$el and $\perp$ component for $\mathbf{A}$, we get

$$\frac{\mathbf{r}}{(\mathbf{rA})}=\frac{\mathbf{r}_\parallel+\mathbf{r}_\perp}{(\mathbf{r}_\parallel+\mathbf{r}_\perp,\mathbf{A})}=\frac{\mathbf{r}_\parallel}{(\mathbf{r}_\parallel\mathbf{A})}+\frac{\mathbf{r}_\perp}{(\mathbf{r}_\parallel\mathbf{A})}=\frac{r_\parallel}{r_\parallel A}\mathbf{e}_A+\frac{\mathbf{r}_\perp}{(\mathbf{r}_\parallel\mathbf{A})}=\mathbf{A}^{-1}+\frac{\mathbf{r}_\perp}{(\mathbf{r}_\parallel\mathbf{A})} \tag{181}$$

This shows that (180) is a general solution of type (169).
As this is directed to $\mathbf{r}$, we would write this $\mathbf{A}_r^{-1}$, namely

$$\mathbf{A}_r^{-1} \stackrel{D}{=} \frac{\mathbf{r}}{(\mathbf{Ar})} = A^{-1} \sec (\mathbf{Ar})\, \mathbf{e}_r, \ (\mathbf{Ar}) \neq O. \tag{182}$$

The generality is checked also algebraically from the number of unknowns and equations.

Art. 39 General inverse vector,

(132) contains null-divisor, but though one add this any vertical vector to $\mathbf{A}$, it satisfies also (65). We would call this type "General inverse vector" and denote it by $\mathbf{A}_g^{-1}$, namely

$$\mathbf{A}_g^{-1} \stackrel{D}{=} \frac{\mathbf{r}+p\,[\mathbf{qA}]}{(\mathbf{Ar})} = \mathbf{A}_r^{-1} + p\ [\mathbf{q}\,\mathbf{r}_A^{-1}], \quad p, \mathbf{q}, : \text{arbitrary}. \tag{183}$$

(183) can be deformed to type $\mathbf{A}_r^{-1}$ by inverse operation of decomposition of (181). Also, if we adopt $\mathbf{A}$ as $\mathbf{r}$, we get $\mathbf{A}_A^{-1} = \mathbf{A}^{-1}$, (184)

$$\mathbf{A}_g^{-1} = \mathbf{A}^{-1} + p\ [\mathbf{qA}^{-1}]. \tag{185}$$

Mr. Hisazue calls $\mathbf{A}^{-1}$ "principal value", null-divisor "subsidiary value". Surely

$$(\mathbf{AA}_r^{-1}) = (\mathbf{A}_r^{-1}\mathbf{A}) = (\mathbf{AA}_g^{-1}) = (\mathbf{A}_g^{-1}\mathbf{A}) = 1. \tag{168}$$

$$\text{But } [\mathbf{AA}_r^{-1}] = -[\mathbf{A}_r^{-1}\mathbf{A}] = \frac{[\mathbf{Ar}]}{(\mathbf{Ar})} = [\mathbf{r}_A^{-1}\mathbf{r}] = -[\mathbf{r}\mathbf{r}_A^{-1}] \neq o, \tag{187}$$

$$[\mathbf{AA}_g^{-1}] = -[\mathbf{A}_g^{-1}\mathbf{A}] = \frac{[\mathbf{Ar}]+p[\mathbf{A}\ [\mathbf{qA}]]}{\mathbf{Ar})} \neq o, \tag{188}$$

From these we knew the characteristic of $\mathbf{A}^{-1}$ in $\mathbf{A}_r^{-1}$ and $\mathbf{A}_g^{-1}$, namely the one of $\mathbf{A}_g^{-1}$ or $\mathbf{A}_r^{-1}$ which satisfies (102) is $\mathbf{A}^{-1}$.

Art. 40 Application of general inverse vector.

The following results owe entirely to Mr. J. A. Schouten to SDG. I. Problem 2.14 and 2.15, they are nothing else than the direct vector representation of them. Respecting the text, we write unknown $\mathbf{v}$ and not x, and we use only -1 as exponential index and denote the coefficient vectors $\mathbf{P}^h$: $\mathbf{P}^1, \mathbf{P}^2, \cdots\cdots, \mathbf{P}^m$.
also inner products value, w^h : $w^1, w^2, \cdots\cdots, w^m$.

In these, upper indices h, are contravariant, so -1 does not appear in these. We decide the order of examples by the rank r of $\mathbf{P}^h$.

$p, q, p_i, p_{ih} = -p_{hi}$: any scalars, $\mathbf{q}, \mathbf{r}$: any vectors — which do not make the denominators zero in the case they exist.

We adopted as the sign of null-divisor + on the contrary to SDG. by Art. 17 (27).

Example. 0, 11. (62) : $(\mathbf{Pv}) = o$, $\mathbf{v} = p\ [\mathbf{q\,P}]$. (0,11)

Example. 0. 12.

$$\left.\begin{aligned} &(\mathbf{P}^1\mathbf{v}) = O,\ (\mathbf{P}^2\mathbf{v}) = O, \\ &\mathbf{P}^1 \parallel \mathbf{P}^2 \text{ namely } [\mathbf{P}^1\mathbf{P}^2] = O \\ &\mathbf{v} = [\mathbf{q},\ p_1\mathbf{P}^1 + p_2\mathbf{P}^2] \end{aligned}\right\} \tag{0,12}$$

Example. 0.1m.

$$\left.\begin{aligned} &(\mathbf{P}^1\mathbf{v}) = O, \cdots\cdots, (\mathbf{P}^m\mathbf{v}) = O, \\ &\mathbf{P}^1 \parallel \mathbf{P}^2 \parallel \cdots\cdots \parallel \mathbf{P}^m : [\mathbf{P}^1\mathbf{P}^2] = \cdots\cdots = [\mathbf{P}^{m-1}\mathbf{P}^m] = O, \\ &\mathbf{v} = [\mathbf{q},\ p_1\mathbf{P}^1 + \cdots\cdots + p_m\mathbf{P}^m]. \end{aligned}\right\} \tag{0,1m}$$

Example. 0.22. $(\mathbf{P}^1\,\mathbf{v})=0, (\mathbf{P}^2\,\mathbf{v})=0,\ \mathbf{P}^1 \nparallel \mathbf{P}^2:\ [\mathbf{P}^1\,\mathbf{P}^2]\neq 0$
$\mathbf{v}=pq\,[\mathbf{P}^1\,\mathbf{P}^2]$. (0.22)

Example. 0.23. $(\mathbf{P}^1\,\mathbf{v})=0,\ (\mathbf{P}^2\,\mathbf{v})=0,\ (\mathbf{P}^3\,\mathbf{v})=0,$
$\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3$ are coplanar but are not $\parallel$ el,
$(\mathbf{P}^1[\mathbf{P}^2\mathbf{P}^3])=0,\quad [\mathbf{P}^h\,\mathbf{P}^i]\neq 0.$
$\mathbf{v}=q\,\{p_{12}[\mathbf{P}^1\,\mathbf{P}^2]+\cdots\cdots+p_{m-1\,m}[\mathbf{P}^{m-1}\,\mathbf{P}^m]\}$ (0.23)

Example. 0.2 m. $(\mathbf{P}^1\,\mathbf{v})=0.\cdots\cdots,\ (\mathbf{P}^m\,\mathbf{v})=0.$
$\mathbf{P}^1\cdots, \mathbf{P}^m$ are coplanar but are not $\parallel$ el,)
$(\mathbf{P}^h\,[\mathbf{P}^i\,\mathbf{P}^j])=0.\quad [\mathbf{P}^h\,\mathbf{P}^i]\neq 0,$
$\mathbf{v}=q\,\{p_{12}[\mathbf{P}^1\,\mathbf{P}^2]+\cdots\cdots+p_{m-1\,m}[\mathbf{P}^{m-1}\,\mathbf{P}^m]\}$ (0.2m)

Example, 0.33, $(\mathbf{P}^1\,\mathbf{v})=0, (\mathbf{P}^2\,\mathbf{v})=0,\ (\mathbf{P}^3\,\mathbf{v})=0, (\mathbf{P}^1[\mathbf{P}^2\mathbf{P}^3])\neq 0, \mathbf{v}=0.$ (0.33)

Following are inner quotient vectors and so contain null-divisors (as said above, sign of them is changed to + from the original SDG.)

Example. 11 $(\mathbf{P}\mathbf{v})=w,\ \mathbf{v}=\dfrac{\mathbf{r}}{(\mathbf{P}\mathbf{r})}w+p\,[\mathbf{q}\dfrac{\mathbf{P}}{(\mathbf{P}\,\mathbf{r})}]=\mathbf{P}_r^{-1}w+p[\mathbf{q}\,\mathbf{r}_p^{-1}]$ (e11)

or compared with (183) $\mathbf{v}=\mathbf{P}_g^{-1}\,w=\dfrac{\mathbf{r}}{(\mathbf{P}\,\mathbf{r})}w+p\,\dfrac{[\mathbf{q}\mathbf{P}]}{(\mathbf{P}\,\mathbf{r})}w.$ (e11′)

pw of the second term in (e11′) coresponds to p in (e11).

Example. 12. $(\mathbf{P}^1\,\mathbf{v})=w^1,\ (\mathbf{P}^2\,\mathbf{v})=w^2,$
$\mathbf{P}^1\parallel\mathbf{P}^2$ namely $[\mathbf{P}^1\,\mathbf{P}^2]=0,\ w^1\,\mathbf{P}^2-w^2\,\mathbf{P}^1=0,$

$$v=\frac{p_1w^1+p_2w^2}{(p_1\mathbf{P}^1+p_1\mathbf{P}^2,\mathbf{r})}\mathbf{r}+p\ [\mathbf{q}\,,\frac{p_1\mathbf{P}^1+p_2\mathbf{P}^2}{(p_1\mathbf{P}^1+p_2\mathbf{P}^2,\mathbf{r})}] \tag{e12}$$

$$=\{p_1\mathbf{P}^1+p_2\mathbf{P}^2\}_r^{-1}\ \{p_1\omega^1+p_2w^2\}\ +p[\mathbf{q}\,,\mathbf{r}^{-1}{}_{\{p_1\mathbf{P}^1+p_2\mathbf{P}^2\}}].$$

If we put the linear combination of $\mathbf{P}^1$ and $\mathbf{P}^2$ anb w^1 and w^2,

$$p_1\mathbf{P}^1+p_2\mathbf{P}^2=\mathbf{P}\ ,\ p_1w^1+p_2w^2=w, \tag{e12a}$$

$$\mathbf{v}=\mathbf{P}_r^{-1}\,w+p\,[\mathbf{q}\,\mathbf{r}_P^{-1}], \tag{e12b}$$

$$\mathbf{v}=\mathbf{P}_g^{-1}\,w. \tag{e12c}$$

Example. lm. $(\mathbf{P}^1\,\mathbf{v})=w^1,\cdots\cdots,(\mathbf{P}^m\mathbf{v})=w^m,$

$\mathbf{P}^1\parallel\mathbf{P}^2\parallel\cdots\cdots\parallel\mathbf{P}^m:\ [\mathbf{P}^1\mathbf{P}^2]=\cdots\cdots=[\mathbf{P}^{m-1}\,\mathbf{P}^m]=0,$

$w^1\,\mathbf{P}^2-w^2\mathbf{P}^1=0,\cdots\cdots,\ w^{m-1}\,\mathbf{P}^m-w^m\mathbf{P}^{m-1}=0,$

$$\mathbf{v}=\{p_1\mathbf{P}^1+\cdots\cdots+p_m\mathbf{P}^m\}_r^{-1}\{p_1w^1+\cdots\cdots+p_mw^m\}$$
$$+p[\mathbf{q},\mathbf{r}^{-1}{}_{\{p_1\mathbf{P}^1+\cdots\cdots p_m\mathbf{P}^m\}}]. \tag{elm}$$

If we put $p_1\mathbf{P}^1+\cdots\cdots+p_m\mathbf{P}^m=\mathbf{P}$, $p_1w^1+\cdots\cdots+p_mw^m=w$, (elma)
then $\mathbf{v}=\mathbf{P}_r^{-1}w+p[\mathbf{q}\mathbf{r}_p^{-1}]=\mathbf{P}_g^{-1}w.$ (elm′)

Example. 22. $(\mathbf{P}^1\,\mathbf{v})=w^1,(\mathbf{P}^2\,\mathbf{v})=w^2,$

$\mathbf{P}^1\nparallel\mathbf{P}^2:\ [\mathbf{P}^1\,\mathbf{P}^2]\neq 0$

$$\mathbf{v}=\frac{[\mathbf{P}^2\mathbf{r}]\,w^1-[\mathbf{P}^1\mathbf{r}]\,w^2+pq\,[\mathbf{P}^1\,\mathbf{P}^2]}{([\mathbf{P}^1\mathbf{P}^2]\,\mathbf{r})}. \tag{e22}$$

This can be got from the following Ex. 33 by putting $\mathbf{P}^3=\mathbf{r}$, $w^3=pq$ in it.

Namely $(\mathbf{r}\,\mathbf{v})=\mathrm{pq}$. (e22a)

From it we can also represent

$$\mathbf{v}=\{\mathbf{P}^1\}^{-1}_{[\mathbf{p}^2\,\mathbf{r}]}\,w^1+\{\mathbf{P}^2\}^{-1}_{[\mathbf{r}\,\mathbf{p}^1]}\,w^2+\{\mathbf{r}\}^{-1}_{[\mathbf{p}^1\mathbf{p}^2]}\,\mathrm{pq}. \qquad (e22')$$

Further, (e22) can be written also

$$\mathbf{v}=[\mathbf{P}^2w^1-\mathbf{P}^1w^2,\,[\mathbf{P}^1\mathbf{P}^2]^{-1}_{r}]+\mathrm{pq}\;\mathbf{r}^{-1}_{[\mathbf{P}^1\mathbf{P}^2]} \qquad (e22'')$$

Putting $[\mathbf{P}^1\mathbf{P}^2]=\mathbf{P}$, we get

$$\mathbf{v}=[\mathbf{P}^2w^1-\mathbf{P}^1w^2,\,\mathbf{P}^{-1}_{r}]+\mathrm{pq}\;\mathbf{r}^{-1}_{\mathrm{p}}=[\mathbf{P}^2w^1-\mathbf{P}^1w^2,\mathbf{P}^{-1}_{g}] \qquad (e22m)$$

It becomes clear that this also satisfies the problem.

Example. 23. $(\mathbf{P}^1\,\mathbf{v})=w^1,\ (\mathbf{P}^2\,\mathbf{v})=w^2,\ (\mathbf{P}^3\,\mathbf{v})=w^3.$

$\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3$, are coplanar but any two of them are not parallel, namely $(\mathbf{P}^1[\mathbf{P}^2\mathbf{P}^3])=0,\ [\mathbf{P}^2\mathbf{P}^3]\neq 0,\ [\mathbf{P}^3\mathbf{P}^1]\neq 0,\ [\mathbf{P}^1\mathbf{P}^3]\neq 0,$

$$w^1[\mathbf{P}^2\mathbf{P}^3]+w^2[\mathbf{P}^3\mathbf{P}^1]+w^3[\mathbf{P}^1\mathbf{P}^2]=0.$$

$$\mathbf{v}=\left[\begin{array}{ll}\mathbf{P}^1(\ \cdot\ \ -\mathrm{p}_{12}w^2-\mathrm{p}_{13}w^3) & \left\{\mathrm{p}_{12}[\mathbf{P}^1\mathbf{P}^2]+\mathrm{p}_{13}[\mathbf{P}^1\mathbf{P}^3]\right.\\ +\mathbf{P}^2(\mathrm{p}_{12}w^1\ \ \cdot\ \ -\mathrm{p}_{23}\,w^3) & \left.\qquad+\mathrm{p}_{23}[\mathbf{P}^2\mathbf{P}^3]\right\}^{-1}_{r}\\ +\mathbf{P}^3(\mathrm{p}_{13}w^1+\mathrm{p}_{23}w^2\ \ \cdot\ ), & \end{array}\right] \qquad (e23)$$

$$+\mathrm{q}\mathbf{r}^{-1}_{\{\mathrm{p}_{12}[\mathbf{P}^1\mathbf{P}^2]+\mathrm{p}_{13}[\mathbf{P}^1\mathbf{P}^3]+\mathrm{p}_{23}[\mathbf{P}^2\mathbf{P}^3]\}},$$

$$\text{Or }\mathbf{v}=\left[\begin{array}{ll}\mathbf{P}^1(\ \cdot\ \ -\mathrm{p}_{12}w^2-\mathrm{p}_{23}w^3) & \left\{\mathrm{p}_{12}[\mathbf{P}^1\mathbf{P}^2]+\mathrm{p}_{13}[\mathbf{P}^1\mathbf{P}^3]\right.\\ +\mathbf{P}^2(p_{12}w^1\ \ \cdot\ \ -\mathrm{p}_{13}w^3) & \left.\qquad+\mathrm{p}_{23}[\mathbf{P}^2\mathbf{P}^3]\right\}^{-1}_{g}\\ +\mathbf{P}^3(p_{13}w^1+\mathrm{p}_{23}w^2\ \ \cdot\ ), & \end{array}\right] \qquad (e23')$$

Exanple. 33. $(\mathbf{P}^1\mathbf{v})=w^1,\ (\mathbf{P}^2\mathbf{v})=w^2,\ (\mathbf{P}^3\mathbf{v})=w^3,$

$\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3$ are not coplanar : $([\mathbf{P}^1\mathbf{P}^2]\mathbf{P}^3)\neq 0$,

$$\mathbf{v}=\frac{[\mathbf{P}^2\mathbf{P}^3]\,w^1+[\mathbf{P}^3\mathbf{P}^1]\,w^2+[\mathbf{P}^1\mathbf{P}^2]\,w^3}{(\mathbf{P}^1[\mathbf{P}^2\mathbf{P}^3])} \qquad (e33)$$

$$=\{\mathbf{P}^1\}^{-1}_{[\mathbf{p}^2\mathbf{p}^3]}\,w^1+[\mathbf{P}^2]^{-1}_{[\mathbf{p}^3\mathbf{p}^1]}\,w^2+\{\mathbf{P}^3\}^{-1}_{[\mathbf{p}^1\mathbf{p}^2]}\,w^3 \qquad (e33')$$

Of course this coincides with Cramer's rule by determinants.
It is interesting that this expression has the form of sum of individual inverse operations of the 3 equations of the problem.

Next we put $\mathbf{P}^1=\mathbf{a},\ \mathbf{P}^2=\mathbf{b},\ \mathbf{P}^3=\mathbf{c}$,

$$\mathbf{a}^{-1}{}_{[\mathbf{bc}]}=\frac{[\mathbf{bc}]}{(\mathbf{a}[\mathbf{bc}])}\overset{D}{=}\mathbf{a}^{-1},\ \mathbf{b}^{-1}{}_{[\mathbf{ca}]}=\frac{[\mathbf{ca}]}{(\mathbf{a}[\mathbf{bc}])}\overset{D}{=}\mathbf{b}^{-1},\ \mathbf{c}^{-1}{}_{[\mathbf{ab}]}=\frac{[\mathbf{ab}]}{(\mathbf{a}[\mathbf{bc}])}\overset{D}{=}\mathbf{c}^{-1}. \qquad (189)$$

These $\mathbf{a}^{-1}, \mathbf{b}^{-1}, \mathbf{c}^{-1}$ are not principal inverse vectors, but general ones, so these do not satisfy (102) ; these have following properties;

$$\left.\begin{array}{l}(\mathbf{aa}^{-1})=(\mathbf{bb}^{-1})=(\mathbf{cc}^{-1})=1.\\ (\mathbf{ab}^{-1})=(\mathbf{ba}^{-1})=(\mathbf{bc}^{-1})=(\mathbf{cb}^{-1})=(\mathbf{ca}^{-1})=(\mathbf{ac}^{-1})=0,\\ (\mathbf{a}[\mathbf{bc}])\,(\mathbf{a}^{-1}[\mathbf{b}^{-1}\mathbf{c}^{-1}])=1.\end{array}\right\} \qquad (190)$$

If we substitute left sides of the problem as w^h in (e33)

$$\mathbf{v}=(\mathbf{va})\mathbf{a}^{-1}+(\mathbf{vb})\mathbf{b}^{-1}+(\mathbf{vc})\mathbf{c}^{-1}. \qquad (191)$$

In the definition formulae (190) of $\mathbf{a}^{-1}, \mathbf{b}^{-1}, \mathbf{c}^{-1}$, $\mathbf{a}$ and $\mathbf{a}^{-1}$ are symmetric, we get

$$\frac{[\mathbf{b}^{-1}\,\mathbf{c}^{-1}]}{(\mathbf{a}^{-1}[\mathbf{b}^{-1}\,\mathbf{c}^{-1}])}=\mathbf{a}\,,\ \frac{[\mathbf{c}^{-1}\,\mathbf{a}^{-1}]}{(\mathbf{a}^{-1}[\mathbf{b}^{-1}\,\mathbf{c}^{-1}])}=\mathbf{b}\,,\ \frac{[\mathbf{a}^{-1}\mathbf{b}^{-1}]}{(\mathbf{a}^{-1}[\mathbf{b}^{-1}\mathbf{c}^{-1}])}=\mathbf{c} \qquad (192)$$

for (191), we get

$$\mathbf{V}=(\mathbf{V}\mathbf{a}^{-1})\,\mathbf{a}+(\mathbf{V}\mathbf{b}^{-1})\mathbf{b}+(\mathbf{V}\,\mathbf{c}^{-1})\,\mathbf{c}\,.$$

By this consideration we knew that so-called inverse vector system is really a kind of inverse vector system in the meaning of (182).

Example. 34. $(\mathbf{P}^1\,\mathbf{v})=w^1,\cdots\cdots,\ (\mathbf{P}^4\,\mathbf{v})=w^4$.

Any three of $\mathbf{P}^h$ are not coplanar: $(\mathbf{P}^1\,[\mathbf{P}^2\,\mathbf{P}^3)\neq 0,\cdots\cdots,(\mathbf{P}^2[\mathbf{P}^3\,\mathbf{P}^4\,])\neq 0$,

$$w^1(\mathbf{P}^2\,[\mathbf{P}^3\,\mathbf{P}^4])+w^2\,(\mathbf{P}^3\,[\mathbf{P}^1\,\mathbf{P}^1])+w^3(\mathbf{P}^4[\mathbf{P}^1\mathbf{P}^2])+w^4(\mathbf{P}^1[\mathbf{P}^2\mathbf{P}^3])=0,$$

$$\mathbf{v}=\frac{\left\{\begin{array}{r}(-p_3\,w^4+p_4w^3)\,[\mathbf{P}^1\,\mathbf{P}^2]+(-p_2w^4+p_4w^2)[\mathbf{P}^1\mathbf{P}^3]+(-p_2w^3+p_3w^2)[\mathbf{P}^1\mathbf{P}^4]\\ +(-p_1w^4+p_4w^1)[\mathbf{P}^2\mathbf{P}^3]+(-p_1w^3+p_3w^1)[\mathbf{P}^2\mathbf{P}^4]\\ +(-p_1w^2+p_2w^1)[\mathbf{P}^3\mathbf{P}^4]\end{array}\right\}}{p_1(\mathbf{P}^2[\mathbf{P}^3\mathbf{P}^4])+p_2(\mathbf{P}^3[\mathbf{P}^4\mathbf{P}^1])+p_3(\mathbf{P}^4[\mathbf{P}^1\mathbf{P}^2])+p_4(\mathbf{P}^1[\mathbf{P}^2\mathbf{P}^3])} \qquad (e34)$$

In algebraic topology, incidence matrices (boundary matrices) of neighbouring dimension are null-divisors each other (see. e. g. VAS) and also in the algebraic electromagnetism which uses fully this algebraic topology, null-divisor and inverse operation appear in various rank and order, therefore there are used the above results and their extension to any dimension, so we described them rather lengthy. See OAI.

§5. Outer quotient vector.

Outer quotient vector, namely $\mathbf{X}$ satisfying (87) or (19) is

$$\mathbf{X}=X\mathbf{e}_2\,,\ X=\frac{B}{A}\,,$$

if we assume $\mathbf{A}=A\,\mathbf{e}_1\,,\ \mathbf{B}=B\mathbf{e}_3\,,$

As $\mathbf{B}\perp\mathbf{A}$, $\mathbf{e}_2=[\mathbf{e}_3\mathbf{e}_1]=\dfrac{[\mathbf{BA}]}{BA}$,

$$\mathbf{X}=X\,\mathbf{e}_2=\frac{B}{A}\,\frac{[\mathbf{BA}]}{BA}=[\mathbf{B}\,,\frac{\mathbf{A}}{A^2}]=[\mathbf{B}\,\mathbf{A}^{-1}]=-[\mathbf{A}^{-1}\,\mathbf{B}], \qquad (194)$$

or $\mathbf{A}\backslash\mathbf{B}=-\mathbf{B}/\mathbf{A}=-[\mathbf{A}^{-1}\,\mathbf{B}]=[\mathbf{B}\,\mathbf{A}^{-1}]$. (195)

putting (194) into (80),

$$[\mathbf{AX}]=[\mathbf{A}[\mathbf{BA}^{-1}]]=(\mathbf{AA}^{-1})\mathbf{B}-(\mathbf{AB})\mathbf{A}^{-1} \qquad (196)$$

The first term equals $\mathbf{B}$ by the definiton (101), and the second term is zero by $\mathbf{A}\perp\mathbf{B}$ independent of $\mathbf{A}^{-1}$.

So we can use $\mathbf{A}_g^{-1}$ instead ef $\mathbf{A}^{-1}$ in (196). Using (183) as $\mathbf{A}_g^{-1}$

$$\mathbf{X}_g=\mathbf{A}\backslash_g\,\mathbf{B}=[\mathbf{BA}_g^{-1}]=[\mathbf{B}\,,\frac{\mathbf{r}+[\mathbf{qA}]}{(\mathbf{Ar})}]=[\mathbf{BA}_r^{-1}]-(\mathbf{Bq})\,\mathbf{r}_A^{-1}\,. \qquad (197)$$

By experience of Art. 39, $\mathbf{X}_g$ can be also

$$[\mathbf{BA}_r^{-1}]=\frac{[\mathbf{B\,r}]}{(\mathbf{Ar})},\ [\mathbf{BA}_g^{-1}]+(\mathbf{Bq})\ \mathbf{A}^{-1}=[\mathbf{BA}^{-1}]+k\mathbf{A}^{-1},\ k:\text{arbitrary} \qquad (198)$$

These are general solutions containing outer null-divisor. The general solution of

(91) is simply

$$\mathbf{A}_g\backslash[\mathbf{AC}]=[[\mathbf{AC}]\mathbf{A}_g^{-1}]=\mathbf{C}-(\mathbf{CA}_g^{-1})\,\mathbf{A}=\mathbf{C}+\mathrm{k}\mathbf{A} \text{ or } \mathbf{C}+\mathrm{k}\mathbf{A}^{-1}. \tag{199}$$

$\mathbf{C}$ is the one solution without null-divisor.

After all generality of outer product caused by null-divisor is covered by the generality of inner inverse vector $\mathbf{A}_g^{-1}$.

Example 1. If $\mathbf{P}_1 \perp \mathbf{Q}_1$ or $(\mathbf{P}_1\mathbf{Q}_1)=0$, and $\mathbf{P}_2 \perp \mathbf{Q}_2$ or $(\mathbf{P}_2\mathbf{Q}_2)=0$, and also $\mathbf{Q}_1 \parallel \mathbf{Q}_2$ or $[\mathbf{Q}_1\mathbf{Q}_2]=0$ and $\mathbf{P}_1 \perp \mathbf{P}_2$ or $(\mathbf{P}_1\mathbf{P}_2)=0$, (200)

the solution of $[\mathbf{P}_1\mathbf{v}]=\mathbf{Q}_1$, $[\mathbf{P}_2\mathbf{v}]=\mathbf{Q}_2$ is

$$\mathbf{v}=[\mathbf{Q}_1\mathbf{P}_1^{-1}]+[\mathbf{Q}_2\mathbf{P}_2^{-1}]. \tag{201}$$

The general condition which enables to have solution of the problem (200) can be got by decomposing this into indivisual scalar equations and applying the extension of (e34).

Example 2. $(\mathbf{AX})=w$, $[\mathbf{BX}]=\mathbf{C}$, $\mathbf{B}\perp\mathbf{C}$ or $(\mathbf{BC})=0$, $\mathbf{A}\not\perp\mathbf{B}$ or $(\mathbf{AB})\neq 0$, (202)

The terminal points of the first equation lie on a plane, points of the second equation on a straight line. So if these are not parallel or $[\mathbf{AB}]\neq 0$, the solution is determined uniquely as the intersection. At first a solution of the former is

$$\mathbf{X} = [\mathbf{CB}^{-1}] + a\mathbf{B}.$$

Determining a by putting this into the latter,

$$\mathbf{X}=\mathbf{A}_B^{-1}w+[\mathbf{CB}^{-1}]-(\mathbf{A}[\mathbf{CB}^{-1}])\,\mathbf{A}_B^{-1}. \tag{203}$$

The invariance of this solution under transformation $\mathbf{B}^{-1}\rightarrow\mathbf{B}_g^{-1}$ is shown by using the form (185).

Example 3. $(\mathbf{AX})=w$, $[\mathbf{AX}]=\mathbf{B}$, $\mathbf{A}\perp\mathbf{B}$: $(\mathbf{AB})=0$. (204)

putting $\mathbf{B}\rightarrow\mathbf{A}$, $\mathbf{C}\rightarrow\mathbf{B}$ in Example 2, $\mathbf{X}=\mathbf{A}^{-1}w+[\mathbf{BA}^{-1}]$. (205)

The identity putting w and $\mathbf{B}$ of (204) into (205) is nothing else than the decomposition formula of the above (132).

Example 4. From computation of $[\mathbf{D}_g^{-1}[\mathbf{DA}]]$, $[\mathbf{D}[\mathbf{D}_g^{-1}\mathbf{A}]]$, we get

$$\mathbf{A}=\mathbf{D}\,(\mathbf{D}_r^{-1}\mathbf{A})-[\mathbf{D}_r^{-1}[\mathbf{DA}]] \text{ or } \mathbf{A}=\mathbf{D}\,(\mathbf{D}_g^{-1}\mathbf{A})-[\mathbf{D}_g^{-1}[\mathbf{DA}]]; \qquad (205a), (205b)$$

$$\mathbf{A}=\mathbf{D}_r^{-1}(\mathbf{DA})-[\mathbf{D}[\mathbf{D}_r^{-1}\mathbf{A}]] \text{ or } \mathbf{A}=\mathbf{D}_g^{-1}(\mathbf{DA})-[\mathbf{D}[\mathbf{D}_g^{-1}\mathbf{A}]]. \qquad (205c), (205d)$$

Art. 41. Extension of the solution.

The problem (51) is

$$\mathbf{D}a=\mathbf{A},\ a=(\mathbf{D}_g^{-1}\mathbf{A}=\frac{(\mathbf{rA})}{(\mathbf{Dr})}=(\mathbf{D}^{-1}+[\mathbf{qD}^{-1}],\ \mathbf{A}), \tag{206}$$

if $\mathbf{D}\parallel\mathbf{A}$.

Also (166); $(\mathbf{DX})=a$, $\mathbf{X}=\mathbf{D}_g^{-1}a=\dfrac{\mathbf{r}}{(\mathbf{Dr})}a=\mathbf{D}^{-1}a+[\mathbf{qD}^{-1}]\,a$. (207)

(198): $[\mathbf{DX}]=\mathbf{A}$, $\mathbf{X}=-[\mathbf{D}_g^{-1}\mathbf{A}]=-\dfrac{[\mathbf{rA}]}{(\mathbf{Dr})}=-[\mathbf{D}^{-1}\mathbf{A}]-(\mathbf{qA})\,\mathbf{D}^{-1}$. (208)

In these problems, there existed the constraints:

for the first problem: $\mathbf{D}\parallel\mathbf{A}$ or $[\mathbf{DA}]=0$, (209)

for the third problem: $\mathbf{D}\perp\mathbf{A}$ or $(\mathbf{DA})=0$, (210)

However the solutions do not loss their mathematical meaning, even if we omit the above constraints. Then what meaning have the solutions ? Decomposing $\mathbf{A}$ into parallel and vertical components $\mathbf{A}_{\parallel}$ and $\mathbf{A}_{\perp}$ to $\mathbf{D}$, for principal inverse value,

$$\mathbf{D}a=\mathbf{D}(\mathbf{D}^{-1}\mathbf{A})=\mathbf{D}(\mathbf{D}^{-1}\mathbf{A}_{\parallel})+\mathbf{D}(\mathbf{D}^{-1}\mathbf{A}_{\perp})=\mathbf{A}_{\parallel}.$$

namely the above answer fulfils $\mathbf{D}a=\mathbf{A}_{\parallel}$ and not $\mathbf{D}a=\mathbf{A}$, (211)

Answer using (206) does not fulfil this relation. Also for (280), it fulfils

$$[\mathbf{D},\ -[\mathbf{D}^{-1}\mathbf{A}]-(\mathbf{q}\mathbf{A})\mathbf{D}^{-1}]=\mathbf{A}_{\perp}$$

Namely this fulfis $[\mathbf{DX}]=\mathbf{A}_{\perp}$ and not $[\mathbf{DX}]=\mathbf{A}$. (212)

We owe these (211) and (212) to Mr. Hisazue.

For the principal outer quotients

$$(\mathbf{A}\backslash\mathbf{A},\ \mathbf{B}\backslash\mathbf{A})=(-[\mathbf{A}^{-1}\mathbf{B}],-[\mathbf{B}^{-1}\mathbf{A}])=-1, \quad (213)$$

$$(\mathbf{A}\backslash\mathbf{B},\ \mathbf{A}/\mathbf{B})=+1, \quad (214)$$

namely $\mathbf{A}\backslash\mathbf{B}=-\{\mathbf{B}\backslash\mathbf{A}\}^{-1}=\{\mathbf{A}/\mathbf{B}\}^{-1}$; (215)

Also $[\mathbf{A}\backslash\mathbf{B},\ \mathbf{B}/\mathbf{A}]=-[\mathbf{A}\backslash\mathbf{B}]=[\mathbf{A}\cdot\mathbf{B}]=0$, (216)

this distinguishes the principal outer quotient from the other general ones.

§ 6 Algebra of operators.

Let M^p be operators to set of one variable function

$$M^f \ni M^{re}, f_0=x, f_1 f_2, \cdots\cdots, M^\infty, \cdots\cdots.$$

M^f is a commutative field on M^{re}, matrices on this M^f are non-commutative ring, field etc.. M^p and M^{re} are commutative, but M^p and M^f are not,

left distributive: $p^i(f_1+f_2)=p^i f_1+p^i f_2$, (217)

right distributive: $(p^i+p^k)f=p^i f+p^k f$, (218)

non-associative: $p\{f_1 f_2\}=\{pf_1\}f_2+f_1\{pf_2\}$, $p^n\{f_1 f_2\}$: Leibniz (219)

non-associative: $p^{-1}\{f_1 f_2\}=f_1\{p^{-1}f_2\}-p^{-1}\{(pf_1)p^{-1}f_2\}$ (partial integral), (220)

exponential laws: $p^i p^k=p^k p^i=p^{i+k}$, (221)

$(p^i)^k=(p^k)=p^{ik}$, (222)

$p^0=1$. (223)

We treat the Hamilton's operator

$$\nabla=\nabla_1+\nabla_2+\nabla_3=\mathbf{e}_1\partial_1+\mathbf{e}_2\partial_2+\mathbf{e}_3\partial_3,\ \partial_i=\frac{\partial}{\partial x_i}.$$

as a vectorial operator in M^3.

Let A, B be representatives of both scalar M^1 and vector M^3,

$\divideontimes_i$ be either of (38), (55), (75), ($i=2,3,4$),

$\divideontimes_k$ be either of (37, or $\divideontimes_i$ ($k=1,2,3,4$),

noncommutative: $\nabla\divideontimes_i \mathrm{A} \neq \mathrm{A}\divideontimes_i\nabla$, (224)

left distributive: $\nabla\divideontimes_i(\mathrm{A}+\mathrm{B})=\nabla\divideontimes_i\mathrm{A}+\nabla\divideontimes_i\mathrm{B}$, (225)

non-associatiative: $\nabla\divideontimes_i(\mathrm{A}\divideontimes_k\mathrm{B})=\nabla_A\divideontimes_i(\mathrm{A}\divideontimes_k\mathrm{B})+\nabla_B\divideontimes_i(\mathrm{A}\divideontimes_k\mathrm{B})$, (226)

where ∇_A indicates operation only to A, imitating ∂_1. With these and val^{+1} we get van^{+1}, e. g. from

$$[\mathbf{A}[\mathbf{BC}]]=(\mathbf{CA})\mathbf{B}-(\mathbf{AB})\mathbf{C}.$$

$\mathbf{A}=\mathbf{B}=\nabla$: rot rot $\mathbf{A}=\nabla$ div $\mathbf{A}-\nabla^2\mathbf{A}$. (227)

§ 7 Algebraification of integral vector laws.

Art. 42 Defininition. From the experience for electrostatic field by Coulomb, if betweeen scalar potential field V and scalar field

$a = -\frac{\rho}{\varepsilon}$, ρ: charge density, ε : permittivity.

there is a relation: $\text{div } \nabla \text{V} = \nabla^2 \text{V} = a,$ (228)

then we have $\text{V} = -\int \frac{dv}{4\pi r} a,$ (229)

The field point searching V must avoid all the points where a distributes otherwise this integral (229) diverges, in other case there can hold only integral relation (229) by Lebesgue, Stieltjes etc., where differential law (228) no more holds by discontinuities of V etc., and in case where connectivity of the domain under consideration is not topologically simple, only differnetial law holds, and integral law does not hold as described in MEM. I. p. 17-8 by J. C. Maxwell. However, let us treat only the case whre (228) and (229) are quite equivalent, then

putting (229) into (228) : $\nabla^2 \left(-\int \frac{dv}{4\pi r} a \right) = a,$ (230)

putting (228) into (229) : $\text{V} = -\int \frac{dv}{4\pi r} \nabla^2 \text{V},$ (231)

thus ∇^2 and $-\int \frac{dv}{4\pi r}$ construct inverse operation each other, so for any scalar field a, let us define

$$\nabla^{-2} a \overset{D}{=} -\int \frac{dv}{4\pi r} a, \tag{232}$$

$$Pot\ a = \int \frac{dv_2}{r} a, \quad \nabla^{-2} = -\frac{1}{4\pi} Pot, \tag{GV22}$$

where (GV22) shows the formula (22) of the Chapter 4 in Gibbs' book GV, then from (230), (231)

$$\nabla^2 \nabla^{-2} a = \nabla^{-2} \nabla^2 a = a, \tag{233}$$

$$\nabla^2 Pot\ a = Pot\ \nabla^2 a = -4\pi a. \tag{GV30, 52. 66}$$

Applying the above operation to each component A_i of any vector field $\mathbf{A}$,

$$\nabla^{-2} \mathbf{A} \overset{D}{=} -\int \frac{dv}{4\pi r} \mathbf{A} = \mathbf{e}_1 \nabla^{-2} \text{A}_1 + \mathbf{e}_2 \nabla^{-2} \text{A}_2 + \mathbf{e}_3 \nabla^{-2} \text{A}_3, \tag{234}$$

$$Pot\ \mathbf{A} = \int \frac{dv_2}{r} \mathbf{A} = \mathbf{e}_1 Pot\ \text{A}_1 + \mathbf{e}_2 Pot\ \text{A}_2 + \mathbf{e}_3 Pot\ \text{A}_3. \tag{GV23, 24}$$

$$\nabla^2 \nabla^{-2} \mathbf{A} = \nabla^{-2} \nabla^2 \mathbf{A} = \mathbf{A}, \tag{235}$$

$$\nabla^2 Pot\ \mathbf{A} = Pot\ \nabla^2 \mathbf{A} = -4\pi \mathbf{A}, \tag{GV31, 52, 66}$$

$$\nabla \nabla^{-2} a = \nabla^{-2} \nabla a = \int \frac{dv_2}{4\pi r^2} a\, \mathbf{e}_{21}, \tag{236}$$

$$\nabla\ Pot\ a = Pot\ \nabla\ a = \int \frac{dv_2}{r^3} \mathbf{r}_{12}\, a \tag{GV27, 39}$$

where $\mathbf{e}_{21} = \frac{\mathbf{r}_{21}}{r}$ is the unit vector directing from the point 2 at a to the field point 1. Here we define

$$\nabla^{-1} a \overset{D}{=\!=} \int \frac{dv_2}{4\pi r^2}\, a\, \mathbf{e}_{21}\,, \tag{237}$$

$$New\ a = \int \frac{dv_2}{r^3}\, r_{12}\, a\,. \tag{GV42}$$

In electrostatic field,

$$div\ \mathbf{D} = (\nabla \mathbf{D}) = \rho\,.\ \mathbf{D} = \nabla^{-1}\rho\,,$$

which coincides with (166-7) , therefore (237) can be written also $\mathrm{div}^{-1} a$

Summarizing,

$$\nabla\, \nabla^{-2} a = \nabla^{-2}\nabla\, a = \nabla^{-1} a \overset{D}{=\!=} div^{-1} a = -\frac{1}{4\pi} New\ a = \int \frac{dv_2}{4\pi r^2} a\, \mathbf{e}_{21}\,, \tag{238}$$

$$\nabla\, Pot\, a = Pot\, \nabla\, a = New\ a = \int \frac{dv_2}{r^3}\, \mathbf{r}_{12}\, a\,, \tag{GV27, 42, 45, 61}$$

∇^2, ∇^{-2} are scalar operator $M^1 \rightarrow M^1$, but ∇^{-1} is an operator vector, so $\nabla^{-1} a$ is a vector and there are two kinds of operation to any vector field.

$$(\nabla, \nabla^{-2}\mathbf{A}) = \nabla^{-2}(\nabla\mathbf{A}) = \int \frac{dv_2}{4\pi r^2}\, (\mathbf{A}\, \mathbf{e}_{21}\,). \tag{239}$$

So we define

$$(\nabla^{-1}\mathbf{A}) \overset{D}{=\!=} \int \frac{dv_2}{4\pi r^2} (\mathbf{A}\, \mathbf{e}_{21})\,, \tag{240}$$

again in electrostatic field,

$$V = -\nabla^{-2}\frac{\rho}{\varepsilon} = -\nabla^{-2} div\ \frac{\mathbf{D}}{\varepsilon} = -\nabla^{-2} div\ \mathbf{E} = -\nabla^{-2}(\nabla\mathbf{E}) = -(\nabla^{-1}\mathbf{E})\,,$$

and $\mathbf{E} = -\ grad\ V$

are reciprocal operation (GV63), so we can write $grad^{-1}\mathbf{A}$ for $(\nabla^{-1}\mathbf{A})$.

$$(\nabla, \nabla^{-2}\mathbf{A}) = \nabla^{-2}(\nabla\mathbf{A}) = (\nabla^{-1}\mathbf{A}) \overset{D}{=} grad^{-1}\mathbf{A} = -\frac{1}{4\pi} Max\mathbf{A} = \int \frac{dv_2}{4\pi r^2} (\mathbf{A}\mathbf{e}_{21})\,. \tag{241}$$

$$div\ Pot\ \mathbf{A} = Pot\ div\ \mathbf{A} = Max\ \mathbf{A} = \int \frac{dv_2}{r^3} (\mathbf{r}_{12}\mathbf{A})\,. \tag{GV29, 41, 44, 45, 63}$$

Next, $$[\nabla, \nabla^{-2}\mathbf{A}] = \nabla^{-2}[\nabla\mathbf{A}] = -\int \frac{dv_2}{4\pi r^2} [\mathbf{A}\, \mathbf{e}_{21}] \tag{242}$$

So we define, $$[\nabla^{-1}\mathbf{A}] \overset{D}{=\!=} -\int \frac{dv_2}{4\pi r^2} [\mathbf{A}\, \mathbf{e}_{21}]\,, \tag{243}$$

then from the experience of magnetic field by conduction current,

when $rot\,\mathbf{H} = [\nabla\mathbf{H}] = \mathbf{i}$, *by* (194) , $\mathbf{H} = -[\nabla^{-1}\mathbf{i}\,]$.

Therefore we can write $-[\nabla^{-1}\mathbf{A}]$ also $rot^{-1}\mathbf{A}$.

$$-[\nabla\nabla^{-2}\mathbf{A}] = -\nabla^{-2}[\nabla\mathbf{A}] = -[\nabla^{-1}\mathbf{A}] = rot^{-1}\mathbf{A} = \frac{1}{4\pi} Lap\mathbf{A} = \int \frac{dv_2}{4\pi r^2} [\mathbf{A}\, \mathbf{e}_{21}]\,, \tag{244}$$

$$rot\ Pot\ \mathbf{A} = Pot\ rot\ \mathbf{A} = Lap\ \mathbf{A} = \int \frac{dv_2}{r^3} [\mathbf{r}_{12}\mathbf{A}]\,. \tag{GV28, 40, 43, 45, 60}$$

Art. 43 Algebraification of van^{-1}.

By the definitions (232) , (234) , (237) , (240) , (243) integral operator is reduced to notation of val^{-1}. Rules of principal inverse vector in §3 can be applied to these all, especially if we interpret $\mathbf{D}$ in Art. 35 (118) and other as ∇, these equations represent directly well known integral vectorial laws. Namely

(118) : $a = \nabla^2\nabla^{-2} a = \nabla^{-2}\nabla^2 a = \nabla^{-2} div\, \nabla\, a = div\, \nabla^{-2}\nabla a$ (245)

(119) : $= div\ div^{-1} a = grad^{-1}\ grad\, a\,,$ (GV46, 53, 63) (246)

(120): $(\nabla^2)^{-1} a = \nabla^{-2} a,$ (127)

(121): $\mathbf{A} = \nabla^2\nabla^{-2}\mathbf{A} = \nabla^{-2}\nabla^2\mathbf{A} = \nabla^{-2}(\nabla\nabla)\mathbf{A} = (\nabla\nabla^{-1})\mathbf{A} = (\nabla^{-1}\nabla)\mathbf{A},$ (GV66) (248)

(122): $(\nabla^2)^{-1}\mathbf{A} = \nabla^{-2}\mathbf{A},$ (249)

(123)→(238): $div^{-1} a = grad\nabla^{-2} a = \nabla^{-2} grad\ a,$ (250)

(124)→(239): $grad^{-1}\mathbf{A} = div\,\nabla^{-2}\mathbf{A} = \nabla^{-2} div\,\mathbf{A},$ (251)

(125)→(240): $rot^{-1}\mathbf{A} = -rot\,\nabla^{-2}\mathbf{A} = -\nabla^{-2} rot\,\mathbf{A},$ (252)

(126): $rot\ div^{-1} a = -rot^{-1} grad\ a = rot\,\nabla^{-2} grad\ a = \nabla^{-2} rot\ grad\ a = O$ (GV50)(253)

(127): $\nabla\, div\nabla^{-2}\mathbf{A} = grad\ grad^{-1}\mathbf{A} = \nabla\nabla^{-2} div\,\mathbf{A} = div^{-1} div\,\mathbf{A} = \nabla^{-2}\nabla\, div\,\mathbf{A},$
(GV32,47) (254)

(128): $-div\ rot^{-1}\mathbf{A} = ([\nabla\nabla^{-1}]\mathbf{A}) = div\,\nabla^{-2}\ rot\,\mathbf{A} = grad^{-1} rot\mathbf{A} = \nabla^{-2} div\ rot\mathbf{A} = o,$
(GV49) (255)

(129): $-rot\,rot^{-1}\mathbf{A} = rot\,\nabla^{-2} rot\,\mathbf{A} = -rot^{-1} rot\,\mathbf{A} = div^{-1}\ div\,\mathbf{A} - (\nabla\nabla^{-1})\mathbf{A},$
(GV48) (256)

(130): $= div^{-1} div\,\mathbf{A} - \mathbf{A} = grad\ grad^{-1}\mathbf{A} - \mathbf{A}$ (257)

(131): $rot\ rot\,\nabla^{-2}\mathbf{A} = \nabla^{-2} rot\,rot\,\mathbf{A} = \nabla^{-2}\{\nabla div\,\mathbf{A} - \nabla^2\mathbf{A}\}.$ (GV33) (258)

Art. 40. Decomposition of vector field (Helmholtz)

From this (132): $\mathbf{A} = div^{-1}\ div\ \mathbf{A} + rot^{-1} rot\,\mathbf{A}$ (GV55) (259)

(133): $= grad\ grad^{-1}\mathbf{A} + rot\ rot^{-1}\ \mathbf{A}$ (GV54) (260)

(134): $= \nabla^{-2}\{\nabla div\ \mathbf{A} - rot\,rot\ \mathbf{A}\} = grad\nabla^{-2} div\mathbf{A} - rot\nabla^{-2} rot\mathbf{A},$ (261)

$div^{-1} div \overset{D}{=} div^0 = grad\ grad^{-1} \overset{D}{=} grad^0$ (262)

and $rot^{-1} rot = rot\ rot^{-1} = rot^0$ (263)

are mutually null-divisor and each is idempotent (Art. 10(16)), namely

$div^0\, div^0 = div^0$ (264), $grad^0\ grad^0 = grad^0$ (265), $rot^0\ rot^0 = rot^0$ (266)

$div^0\, rot^0 = rot^0\ div^0 = grad^0\ rot^0 = rot^0\, grad^0 = O,$ (267)

At (135) of Art. 36 we got the formula to decompose $\mathbf{A}$ to parallel and vertical component $\mathbf{A}_\parallel$ and $\mathbf{A}_\perp$ to another vector $\mathbf{D}$, corresponding it we got here the formulae to decompose any vector field $\mathbf{A}$ into divergent and rotational component $\mathbf{A}_d$ and $\mathbf{A}_r$, namely

(136): $\mathbf{A} = \mathbf{A}_d + \mathbf{A}_r,$ (268)

(137): $[\nabla\mathbf{A}_d] = rot\mathbf{A}_d = O,$ (GV61) (269)

(138): $(\nabla\mathbf{A}_r) = div\ \mathbf{A}_r = O,$ (270)

Where (141): $\mathbf{A}_d = div^0\,\mathbf{A} = grad^0\,\mathbf{A} = \mathbf{A} - \mathbf{A}_r$ (GV57, 62. 65) (271)

(142): $\mathbf{A}_r = rot^0\,\mathbf{A} = rot^0\,\mathbf{A}_r = \mathbf{A} - \mathbf{A}_d$ (GV56, 60, 64) (272)

Also $\mathbf{A}_d$: $div\,\mathbf{A}_d = (\nabla\nabla^{-1})\ div\,\mathbf{A} = div\,\mathbf{A},\ rot\ \mathbf{A}_d = [\nabla\nabla^{-1}]\ div\,\mathbf{A} = O,$ (GV59, 61)

$\mathbf{A}_r$: $rot\,\mathbf{A}_r = rot^0\, rot\,\mathbf{A} = rot\,\mathbf{A},\ div\,\mathbf{A}_r = div\ rot^0\,\mathbf{A} = O,$ (GV58)

$rot\ rot\ \mathbf{A}_r = -\nabla^2\mathbf{A}_r$ (GV67), $\nabla\ div\,\mathbf{A}_d = \nabla^2\mathbf{A}_d$ (GV68)

or (143): $div\,\mathbf{A}_d = div\ \mathbf{A},\ rot\,\mathbf{A}_d = O$ (273); (144): $rot\ \mathbf{A}_r = rot\ \mathbf{A},\ div\,\mathbf{A}_r = O.$ (274)

Art. 45 Discussion of null-divisors.

For a vector field $\mathbf{A}$, if we give only its divergence $div\ \mathbf{A}$, rotational component is indefinite, so we can adopt as null-divisor not only $rot_0\,\mathbf{A}$ but also one multiplied by any scalar field p : $rot^{-1} p\; rot\; \mathbf{A}$, also by any vector field $\mathbf{q}$, we can use also $rot^{-1}\mathbf{q}$ in stead of nondivergent $rot\,\mathbf{A}$, Namely

$$div\;\mathbf{A}=a,\; Answer:\; \mathbf{A}=div^{-1}\;a+rot^{-1}\,rot\,\mathbf{A}\;\text{or}\;=div^{-1}\,a+rot^{-1}\,\mathbf{q},$$

Similar to Art. 38 we would call the second term of (259) "normal type". On the contrary when we are given only rotation of a vector field $\mathbf{A}$, we would call the first term of (259) or (260) "normal type" of outer null-divisor.

Also (145): $div^{-1}\nabla^2\, a=\nabla^2\, div^{-1}=grad\; a\,,$ (275)

(146): $grad^{-1}\nabla^2\,\mathbf{A}=\nabla^2\; grad^{-1}\,\mathbf{A}=div\,\mathbf{A},$ (276)

(147): $-rot^{-1}\nabla^2\,\mathbf{A}=-\nabla^2\, rot^{-1}\,\mathbf{A}=rot\,\mathbf{A},$ (277)

(148): $\nabla^{-2} a=(\nabla^{-1}\nabla^{-1})\,a=\;grad^{-1}\,div^{-1} a,$ (GV69) (278)

(149): $\nabla^{-2}\,\mathbf{A}=(\nabla^{-1}\nabla^{-1})\,\mathbf{A}$

$=\nabla^{-2}\{\nabla grad^{-1}\,\mathbf{A}+rot\;rot^{-1}\,\mathbf{A}\}=div^{-1}\,grad^{-1}\,\mathbf{A}+rot^{-1}\,rot^{-1}\,\mathbf{A},$ (GV70) (279)

(150): $\nabla^{-2}\,\mathbf{A}_d=div^{-1}\,grad^{-1}\;\;\mathbf{A}=div^{-1}\;grad^{-1}\;\mathbf{A}_d\,,$ (280)

(151): $\nabla^{-2}\,\mathbf{A}_r=-rot^{-1}\,rot^{-1}\,\mathbf{A}=-rot^{-1}\,rot^{-1}\,\mathbf{A}_r\,,$ (281)

(152): $rot^{-1}\;div^{-1}\,a=0,$ (282)

(153): $([\nabla^{-1}\nabla^{-1}]\mathbf{A})=-grad^{-1}\;rot^{-1}\,\mathbf{A}=0\,,$ (283)

(154): $[[\nabla^{-1}\nabla^{-1}]\mathbf{A}]=0_b^*,$ (284)

(155): $(\nabla^{-1})^{-1}\;a=grad\;a,\;((\nabla^{-1})^{-1}\,\mathbf{A})=div\mathbf{A},\;[(\nabla^{-1})^{-1}\,\mathbf{A}]\;=rot\,\mathbf{A}\,,$ (285)

$(\nabla^{-2})^{-1} a=\nabla^2\, a,\;(\nabla^{-2})^{-1}\,\mathbf{A}=\nabla^2\,\mathbf{A}\,.$

Art. 46 Helmholtzian operator.

(156): $\nabla^{-2}\nabla^{-2}\;a\overset{D}{=}\nabla^{-4}\,a\overset{D}{=}-\int\frac{dv_2}{8\pi}r_{12}\,a\,,$ (GV89, 96) (286)

(157): $\nabla^{-2}\nabla^{-2}\,\mathbf{A}\overset{D}{=}\nabla^{-4}\,\mathbf{A}\overset{D}{=}-\int\frac{dv_2}{8\pi}r_{12}\mathbf{A}\,,$ (GV90, 97) (287)

(158): $grad\,\nabla^{-4} a=\nabla^{-4}\,grad\;\;a=div^{-1}\,\frac{r^2}{2}a\overset{D}{=}\nabla^{-3}a\,,$ (GV91) (288)

(159): $div\,\nabla^{-4}\mathbf{A}=\nabla^{-4}\,div\,\mathbf{A}=grad^{-1}\,\frac{r^2}{2}\mathbf{A}\overset{D}{=}(\nabla^{-3}\,\mathbf{A})\,,$ (GV92) (289)

(160): $rot\,\nabla^{-4}\,\mathbf{A}=\nabla^{-4}\,rot\,\mathbf{A}=-rot^{-1}\,\frac{r^2}{2}\mathbf{A}\overset{D}{=}[\nabla^{-3}\,\mathbf{A}]\,,$ (GV93) (290)

(161): $\nabla^2\nabla^{-4}\,a=\nabla^{-4}\,\nabla^2\, a=grad^{-1}\,\frac{r^2}{2}grad\;\;a=\nabla^{-2}\;a\,,$ (GV94) (291)

(162): $\nabla^2\nabla^{-4}\,\mathbf{A}=\nabla^{-4}\,\nabla^2\,\mathbf{A}=\nabla^{-2}\,\mathbf{A}\,,$ (GV95) (292)

(163): $\nabla^{-2}\,\mathbf{A}=rot\;rot\,\nabla^{-4}\,\mathbf{A}-\nabla\,div\,\nabla^{-4}\,\mathbf{A}\,,$ (GV98) (293)

(286)-(293) are algebraification of Helmholtzian operator. $H=Hel=-8\pi\nabla^{-4}$ described in GV. p. 258-9.

Example 1. $div\;\mathbf{D}=(\nabla\mathbf{D})=\rho,\;\mathbf{D}=div\;\rho+rot^{-1}\,rot\,\mathbf{D}\,;$

$$if\;rot\;\mathbf{D}=\frac{1}{\varepsilon}\,rot\,\mathbf{E}=0,\;then\;\mathbf{D}=div^{-1}\,\rho=\nabla^{-1}\,\rho=\int\frac{dv}{4\pi r^2}\,\rho\,\mathbf{e}_{21}.\;(Coulomb)\quad(294)$$

See Art. 45.

Example 2. $rot\;\mathbf{H}=[\nabla\mathbf{H}]=\mathbf{i}\,,\;\mathbf{H}=div^{-1}\,div\,\mathbf{H}+rot^{-1}\,\mathbf{i}\,;$

$$\text{if } div\ \mathbf{H} = \frac{1}{\mu} div\ \mathbf{B} = 0, \text{ then } \mathbf{H} = rot^{-1}\mathbf{i} = -[\nabla^{-1}\mathbf{i}] = \int \frac{dv}{4\pi r^2}[\mathbf{i}\mathbf{e}_{21}]\ , \quad (\textit{Biot-Savart}). \tag{295}$$

See Art. 45.

Example 3. $\nabla^2 V = -\frac{\rho}{\varepsilon}$, $V = \nabla^2\left(-\frac{\rho}{\varepsilon}\right) = -\int \frac{dv}{4\pi r}\left(-\frac{\rho}{\varepsilon}\right) = \int \frac{dv}{4\pi r}\frac{\rho}{\varepsilon}$,

(*Potential law*) or $\mathbf{E} = -grad\ V$, $V = -grad^{-1}\mathbf{E} = -(\nabla^{-1}\mathbf{E}) = -\nabla^{-2}(\nabla\mathbf{E})$

$$= -\nabla^{-1} div\mathbf{E} = -\nabla^{-1}\frac{div\,\mathbf{D}}{\varepsilon} = -\nabla\frac{\rho}{\varepsilon} = \int \frac{dv}{4\pi r}\frac{\rho}{\varepsilon}. \tag{296}$$

Example 4. $\nabla^2\mathbf{A} = -\mu\,\mathbf{i}$, $\mathbf{A} = \nabla^{-2}(-\mu\,\mathbf{i}) = -\int \frac{dv}{4\pi r}(-\mu\mathbf{i}) = \int \frac{dv\mu\,\mathbf{i}}{4\pi r}$

(SE241-2) (297)

Example 5. $\mathbf{b} = \mu_0(\mathbf{h} + \mathbf{M})$, $\mathbf{M}$: *magnetization*.

$div\,\mathbf{b} = \mu_0\ div\ (\mathbf{h} + \mathbf{M}) = 0$, $rot\ \mathbf{b} = 0$, $\mathbf{h} = -\nabla v$,

$div\ \mathbf{h} = -div\ \nabla v = -\nabla^2 v = -div\,\mathbf{M}$,

$$v = \nabla^{-2} div\,\mathbf{M} = grad^{-1}\ \mathbf{M} = \int \frac{dv}{4\pi r^2}(\mathbf{M}\,\mathbf{e}_{21}). \tag{298}$$

Example 6. $rot\ \mathbf{E} = \mathbf{f}$, $div\ \mathbf{D} = \rho$, $\mathbf{D} = \varepsilon\mathbf{E}$,

$\mathbf{D} = div^{-1} div\,\mathbf{D} + rot^{-1} rot\,\mathbf{D} = div^{-1}\rho + \varepsilon\, rot^{-1}\mathbf{f}$

$$= \int \frac{dv}{4\pi r^2}\rho\,\mathbf{e}_{21} + \varepsilon\int \frac{dv}{4\pi r^2}\ [\mathbf{f}\,\mathbf{e}_{21}]. \tag{299}$$

§ 8. References.

1. inverse vector:

Abb.

HV Keiitiro HISAZUE : Elementary Vector Analysis (1932) p. 33-41 (in Japanese)

2. grad^{-1}, div^{-1}, rot^{-1}:

GV J. W. Gibbs : Vector Analysis (1901) p. 237-8.

3. abstract algebra:

WCG H. Weyl : Classical Groups (1939)

4. lattice and vector lattice:

BLT G. Birkhoff : Lattice Theory (1940 Rev. 1949)

5. axiomatic construction of vector, algebraic properties:

SGA O. Schreier, E. Sperner : Einfuhrung in die analytische Geometrie und Algebra. Bd. I (1931), Bd. II (1935)

SDG J. A. Schouten, D. J. Struik : Einfuhrung in die neueren Methoden der Differentialgeometrie. Bd. I (1935), Bd. II (1938)

6. electromagnetic theory:

SE J. A. Stratton : Electromagnetic Thery (1941)

MEM J. C. Maxwell : A Treatise on Electricity and Magnetism, I (1904) II (1904)

7. lattice (grid or net) theory of physical field:

OE S. Okada, M. Iwasita, T. Huzi : Electromagnetic Theory (No. 1-4) Journal of Elec. Com. Eng. 28. 8&9 (1944) p. 11-8.

OP　S. Okada : Lattice theory of physical Field,
Denki Hyooron 35. 5/6(1947) p. 65-9. (in Japanese)

8. subspace : besides SDG.,

OS　S. Okada, R. Onodera : Theory of flat subspaces and their application,
Bulletin of the Yamagata University, Nat. Sci. No. 1 (March 1950) p. 5-13.

9. transverse vector (in 2-dimension) (in Appendix) :

NEM　J. Nielsen : Elementare Mechanik (1935) p. 10, 142, 288, 293.

10. Algebraic theory of electromagnetism :

OA.I. S. Okada, R. Onodera : Algebraic Electromagnetism (No. 1),
Bulletin of the Yamagata Univ., Nat. Sci. No.2 (1950) in press.

11. algebraic topology :

VAS　O. Veblen : Analysis Situs (1946)

LAT　S. Lefschetz : Algebraic Topology (1948)

APPENDIX. Inner Null-divisor and Inverse Vector in 2-dimensional space.

In 2-dimensional case

$$(\mathbf{AB})=A_1B_1+A_2B_2=(\mathbf{BA}). \tag{300}$$

Let us call a counter-clockwise $\frac{\pi}{2}$ radian rotated vector of any vector:

$$\mathbf{A}=A_1\mathbf{e}_1+A_2\mathbf{e}_2$$

transverse vector and denote it with $\hat{\mathbf{A}}$. Algebraically (NEM p. 10):

$$\hat{\mathbf{A}}=-A_2\mathbf{e}_1+A_1\mathbf{e}_2 \tag{301}$$

Following product corresponds to the vector product of $\mathbf{A}$ and $\mathbf{B}$ in 2-dimensional case:

$$(\hat{\mathbf{A}}\mathbf{B})=-A_2B_1+A_1B_2=\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}=-(\mathbf{A}\hat{\mathbf{B}}), \tag{302}$$

$$(\mathbf{A}\hat{\mathbf{A}})=0, \tag{303}$$

$$(\hat{\mathbf{A}}\hat{\mathbf{B}})=(\mathbf{AB}), \tag{304}$$

$$\mathbf{A}^{-1}=\frac{\mathbf{A}}{(\mathbf{A}\ \mathbf{A})}=\frac{A_1\mathbf{e}_1+A_2\mathbf{e}_2}{A_1A_1+A_2A_2}, \tag{305}$$

$$\mathbf{A}_r^{-1}=\frac{\mathbf{r}}{(\mathbf{A}\ \mathbf{r})}. \tag{306}$$

$$\mathbf{A}_g^{-1}=\frac{\mathbf{r}+q\hat{\mathbf{A}}}{(\mathbf{Ar})}=\mathbf{A}_r^{-1}+q\,\hat{\mathbf{r}}_{\hat{\mathbf{A}}}^{-1} \tag{307}$$

$$or\ \mathbf{A}_g^{-1}=\mathbf{A}^{-1}+q\,\hat{\mathbf{A}}^{-1} \tag{308}$$

If we put the third axis of right hand system $\mathbf{e}_3$ for $\mathbf{e}_1\,\mathbf{e}_2$ surface in 3-dimensional space, the transverse vector becomes:

$$\hat{\mathbf{A}}=[\mathbf{e}_3\mathbf{A}]. \tag{309}$$

Null-divivisors.

Example. 0. 1　$(\mathbf{Pv})=0,\ \mathbf{v}=pq\ \hat{\mathbf{P}}.$　(0. 1)

Example. 0. 2　$(\mathbf{P}^1\mathbf{v})=0,\ (\mathbf{P}^2\mathbf{v})=0,$

$\mathbf{P}^1 \parallel \mathbf{P}^2$ namely zero area : $(\mathbf{P}^1\hat{\mathbf{P}}^2)=-(\hat{\mathbf{P}}^1\mathbf{P}^2)=0,$

$$\mathbf{v}=\mathrm{q}\,(\mathrm{p}_1\hat{\mathbf{P}}^1+\mathrm{p}_2\hat{\mathbf{P}}^2)\,. \tag{0. 2}$$

Example. 0. m $(\mathbf{P}^1\mathbf{v})=\mathrm{O},\cdots\cdots(\mathbf{P}^m\mathbf{v})=\mathrm{O}.$

$$\mathbf{P}^h \parallel \mathbf{P}^i:\ (\mathbf{P}^h\hat{\mathbf{P}}^i)=\mathrm{O},$$

$$\mathbf{v}=\mathrm{q}\ (\mathrm{p}_1\hat{\mathbf{P}}^1+\cdots\cdots+\mathrm{p}_m\hat{\mathbf{P}}^m)\,. \tag{0. m}$$

Quotient vector

Example 11 $(\mathbf{P}\,\mathbf{v})=w,\ \mathbf{v}=\dfrac{\mathbf{r}w+q\hat{\mathbf{P}}}{(\mathbf{P}\,\mathbf{r})}=\mathbf{P}_r^{-1}\,w+q\,\hat{\mathbf{r}}_{\hat{P}}^{-1}\,,$ (ee11)

or $\mathbf{v}=\mathbf{P}_g^{-1}\,w\,.$

Example 12 $(\mathbf{P}^1\mathbf{v})=w^1,(\mathbf{P}^2\mathbf{v})=w^2\,.$ (ee11′)

$$\mathbf{P}^1 \parallel \mathbf{P}^2:\ (\mathbf{P}^1\hat{\mathbf{P}}^2)=-(\hat{\mathbf{P}}^1\mathbf{P}^2)=\mathrm{O},\ w^1\mathbf{P}^2-w^2\mathbf{P}^1=\mathrm{O},$$

$$\mathbf{v}=\frac{p_1w^1+p_2w^2}{(p_1\mathbf{P}^1+p_2\mathbf{P}^2,\mathbf{r})}\,\mathbf{r}+q\,\frac{p_1\hat{\mathbf{P}}^1+p_2\hat{\mathbf{P}}^2}{(p_1\mathbf{P}^1+p_2\mathbf{P}^2,\mathbf{r})}$$

$$=\{p_1\mathbf{P}^1+p_2\mathbf{P}^2\}_r^{-1}\,\{p_1w^1+p_2w^2\}+q\,\hat{\mathbf{r}}^{-1}_{\{p_1\hat{\mathbf{P}}^1+p_2\hat{\mathbf{P}}^2\}} \tag{ee12}$$

$$\text{or}\quad \mathbf{v}=\{p_1\mathbf{P}^1+p_2\mathbf{P}^2\}_g^{-1}\,\{p_1w^1+p_2w^2\}\,. \tag{ee12′}$$

Fxample 1m $(\mathbf{P}^1\mathbf{v})=w^1,\cdots\cdots,\ (\mathbf{P}^m\mathbf{v})=w^m.$

$$\mathbf{P}^1 \parallel \mathbf{P}^2 \parallel\ \cdots\cdots \parallel \mathbf{P}^m:\ (\mathbf{P}^1\hat{\mathbf{P}}^2)=\cdots\cdots(\mathbf{P}^{m-1}\hat{\mathbf{P}}^m)=\mathrm{O}\,,$$

$$w^h\,\mathbf{P}^i-w^i\,\mathbf{P}^h=\mathrm{O}\,,$$

$$\mathbf{v}=\{p_1\mathbf{P}^1+\cdots+p_m\mathbf{P}^m\}_r^{-1}\,\{p_1w^1\cdots+p_mw^m\}+q\,\mathbf{r}^{-1}\ \{p_1\hat{\mathbf{P}}^1+\cdots p_m\hat{\mathbf{P}}^m\}\,, \tag{ee1m}$$

$$\text{or}\quad \mathbf{v}=\{p_1\mathbf{P}^1+\cdots+p_m\mathbf{P}^m\}_g^{-1}\,w\,,\quad w=p_1w^1+\cdots\cdots+p_mw^m \tag{ee1m}$$

Example 22 $(\mathbf{P}^1\mathbf{v})=w^1,\ (\mathbf{P}^2\mathbf{v})=w^2,$

$$\mathbf{P}^1 \nparallel \mathbf{P}^2:\ (\mathbf{P}^1\hat{\mathbf{P}}^2)\neq\mathrm{O}\,.$$

$$\mathbf{v}=\frac{\hat{\mathbf{P}}^2w^1-\hat{\mathbf{P}}^1w^2}{(\mathbf{P}^1\hat{\mathbf{P}}^2)}=\frac{\hat{\mathbf{P}}^2}{(\mathbf{P}^1\hat{\mathbf{P}}^2)}\,w^1+\frac{\hat{\mathbf{P}}^1}{(\mathbf{P}^1\hat{\mathbf{P}}^2)}\,w^2$$

$$=\{\mathbf{P}^1\}^{-1}_{\hat{\mathbf{p}}^2}\,w^1+\{\mathbf{P}^2\}^{-1}_{\hat{\mathbf{p}}^1}\,w^2\,. \tag{ee22}$$

Example 2m $(\mathbf{P}^1\mathbf{v})=w^1,\cdots\cdots,(\mathbf{P}^m\mathbf{v})=w^m.$

$$\mathbf{P}^h \nparallel \mathbf{P}^i:\ (\mathbf{P}^h\hat{\mathbf{P}}^i)\neq\mathrm{O}\,,$$

$$w^1(\mathbf{P}^2\hat{\mathbf{P}}^3)+w^2(\mathbf{P}^3\hat{\mathbf{P}}^1)+w^3(\mathbf{P}^1\hat{\mathbf{P}}^2)=\mathrm{O},\cdots\cdots,$$

$$w^{m-2}\,(\mathbf{P}^{m-1}\hat{\mathbf{P}}^m)+w^{m-1}\,(\mathbf{P}^m\hat{\mathbf{P}}^{m-2})+w^m(\mathbf{P}^{m-2}\,\hat{\mathbf{P}}^{m-1})=\mathrm{O}$$

$$\mathbf{v}=\frac{\left\{\begin{array}{l}\hat{\mathbf{P}}^1(\ \ \bullet\ \ -p_{12}w^2-\cdots\cdots\ \ -p_{1m}\,w^m)\\ +\hat{\mathbf{P}}^1(p_{12}w_1\ \ \bullet\ \ \ -p_{2m}\,w^m)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ +\hat{\mathbf{P}}^m(p_{1m\,w_1}+\cdots\cdots+p_{m-1\,m}w^{m-1}\,,\qquad)\end{array}\right\}}{\left\{\begin{array}{l}p_{12}\,(\mathbf{P}^1\,\hat{\mathbf{P}}^2)+\cdots\cdots+p_{1m}\,(\mathbf{P}^1\hat{\mathbf{P}}^m)\\ \qquad\cdots\cdots\\ \qquad\qquad+p_{m-1m}\,(\mathbf{P}^{m-1}\,\hat{\mathbf{P}}^m)\end{array}\right\}} \tag{ee2m}$$

Other example: Regularity of complex function $\mathbf{w}=f\,(\mathbf{z})$, (Cauchy-Riemann).

put: $\nabla=\mathbf{e}_1\,\partial_1+\mathbf{e}_2\,\partial_2$

Conjugate number: $\bar{\mathbf{w}}=\mathbf{e}_1w_1-\mathbf{e}_2w_2(\mathbf{e}_1=1\,,\ \mathbf{e}_2=i)$

Regularity (Cauchy-Riemann): $(\nabla\bar{\mathbf{w}})=\mathrm{O}$ *and* $(\nabla\hat{\bar{\mathbf{w}}})=\mathrm{O},$ (310)

$$\hat{\bar{\mathbf{w}}}=\bar{\hat{\mathbf{w}}}=-\mathbf{e}_1w_2-\mathbf{e}_2w_1$$

商 vector と夫による vector 積分法則の代数化

岡田幸雄　　保科正吉

梗　概　3次元 vector 算に於ける数値積, 内積, 外積等種々の積の逆算を考察し之に依り内零因子, 外零因子, 内逆元, 内商, 外商等を求め内逆元から内積に関し vector の一般整数巾（ベキ）を定義し vector 代數に除法を導入した, 又 Hamilton 作用素（演算子）∇ (nabla) の逆元 ∇^{-1} によつて從來 Pot, New, Max, Lap として知られていた積分演算子の法則を vector 除法の代數に歸する事が出來た, 就中

$$[\mathbf{A}[\mathbf{BC}]] = (\mathbf{CA})\,\mathbf{B} - (\mathbf{AB})\,\mathbf{C}$$

に於て $\mathbf{A} = \mathbf{B} = \nabla$ として $\nabla^2\mathbf{A} = \nabla\,\mathrm{div}\,\mathbf{A} - \mathrm{rot\ rot}\mathbf{A}$

を得ると同様に, $\mathbf{A} = \mathbf{D}^{-1}$, $\mathbf{B}=\mathbf{D}$, $\mathbf{C}=\mathbf{X}$ として

$$[\mathbf{D}^{-1}[\mathbf{DX}]] = \mathbf{D}\,(\mathbf{D}^{-1}\mathbf{X}) - (\mathbf{D}^{-1}\mathbf{D})\,\mathbf{X} = \mathbf{D}\,(\mathbf{D}^{-1}\mathbf{X}) - \mathbf{X}$$

から vector $\mathbf{X}$ の $\mathbf{D}$ に平行, 垂直兩成分への分解式

$$\mathbf{X} = \mathbf{X}_{\parallel} + \mathbf{X}_{\perp} = \mathbf{D}\,(\mathbf{D}^{-1}\mathbf{X}) - [\mathbf{D}^{-1}[\mathbf{DX}]]$$

を得, 更に此 $\mathbf{D}$ を ∇ (nabla) と解釋して任意の vector 場を泉成分 $\mathbf{X}_a$ と渦成分 $\mathbf{X}_r$ に分解する公式

$$\mathbf{X} = \mathbf{X}_a + \mathbf{X}_r = \nabla\,(\nabla^{-1}\mathbf{X}) - [\nabla^{-1}[\nabla\mathbf{X}]] = \mathrm{grad\ grad}^{-1}\,\mathbf{X} - \mathrm{rot}^{-1}\,\mathrm{rot}\,\mathbf{X}$$

$$= \mathrm{div}^{-1}\mathrm{div}\,\mathbf{X} - \mathrm{rot\ rot}^{-1}\,\mathbf{X} = \nabla^{-2}\,\{\nabla\mathrm{div}\,\mathbf{X} - \mathrm{rot\ rot}\mathbf{X}\}$$

を得, 之等の代数的同一性を示し, 周知の逆 vector 系も本逆 vecter の1種である事を示し, vector は和と外積に關し Lie 環である事も指摘した。

	19	竪て	扨て
149	15	電力 $V^{a0}\hat{I}_{a0}$	$V_{a0}\hat{I}_{a0}$
	28	及び$\frac{2\pi}{3}$(Racian)	及び$2\frac{2\pi}{3}$(Radian)
150	10	運算から	運算が
	30	Department-	Department
152		第三図説明	変圧器一次電圧/二次電圧$=\frac{200}{50000}$
155		第六図(一)説明	同上
156		第六図(二)説明	同上
156		第七図説明	變圧器一次電圧/二次電圧$=\frac{200}{30000}$
157		オツシログラム No.1. No.2, No.3,	同上
159		第八図説明	同上
		第九図説明	同上
160		オツシログラムI	Vは變圧器一次電圧の読み
161		オツシログラムII, III	故に実際の電圧は此の
162		オツシログラムVI	$\frac{30000}{200}=150$倍である
181	10	represnted	represented
182	−3	$x\ni X$	$x\in X$
183	13	matrlx	matrix
184	9	(A/B)*A	(A\B)*A
185	15	*namely	*
	−3	Theminimun number which satisfy	—
186	18	associtve	associative
	−6	M	M^1
	−3	Additvely	Additively
	−2	(34)	(35)
187	23	vector	vectors
188	−15	vectet	vector
189	16	quotieut	quotient
	−5	e_1 e_2 e_3	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$
	−1	(24)	(22)
190	(83)	$-\mathbf{A}/\mathbf{O}$	$-\mathbf{A}\backslash\mathbf{O}$
	(84)	X	$\mathbf{X}$
	(85)	$=O$ is equivaleut	$=\mathbf{O}$ is equivalent.
191	13	clossed	closed
192	(112)	$\stackrel{D}{=}\mathbf{A}^2$	$=\mathbf{A}^{-2}$
	−12	s=1	s=−1
	−9	used	use

193	3	from	from right side
	−10	$(\mathbf{D}\,\mathbf{D})=\mathbf{D}^{-2}$	$(\mathbf{D}\,\mathbf{D})^{-1}=\mathbf{D}^{-2}$
	(135)	$\mathbf{e}_D[\mathbf{e}_D\mathbf{A}]$	$\mathbf{e}_D(\mathbf{e}_D\mathbf{A})$
194	(143)	$\mathbf{A}])$	$\mathbf{A}]])$
	(151)	$\mathbf{A}]$	$\mathbf{A}]]$
195	9		(164)
	10		(165)
	−15	(160)	(138)
196	4	$\mathbf{X}_1$	$\mathbf{X}^1$
	18	a inner	an inner
	20	of	or
197	7	(132)	(182)
	15	(168)	(186)
198	6	$\mathbf{v}=q\{p_{12}[\mathbf{P}^1\mathbf{P}^2]+\cdots+p_{m-1m}[\mathbf{P}^{m-1}\mathbf{P}^m]$	$\mathbf{v}=q\{p_{13}[\mathbf{P}^1\mathbf{P}^2]+p_{13}[\mathbf{P}^1\mathbf{P}^2]+p_{23}[\mathbf{P}^2\mathbf{P}^3]\}$
	11	$\mathbf{v}=0$ (0.33)	$\mathbf{v}=\mathbf{0}$ (0.33)
	16	coresponds	corresponds
198	(e12)	v	$\mathbf{v}$
	−15	anb	and
200	3		(193)
	20	or (19)	or (91)
	−6	(80)	(87)
201	16	former	latter
	18	latter	former
	(206)	$(\mathbf{D}^{-1}_{g}\mathbf{A}$	$(\mathbf{D}^{-1}_{g}\mathbf{A})$
	−5	(198)	(170)
202	(213)	$\mathbf{A}\backslash\mathbf{A}$	$\mathbf{A}\backslash\mathbf{B}$
	(216)	$=-[\mathbf{A}\backslash\mathbf{B}]=[\mathbf{A}\backslash\mathbf{B}]=0$	$=-[\mathbf{A}\backslash\mathbf{B},\ \mathbf{A}\backslash\mathbf{B}]=\mathbf{0}$
	(222)	(p^k)	$(p^k)^i$
	−9	(75)	(74)
203	11	differnetial	differential
205	1	(127)	(247)
	16	Art, 40	Art 44
	(263)	$=\mathrm{rot}^0$	$\overset{\mathbf{D}}{=}\mathrm{rot}^0$
	−5	yiv	div
	−4	yiv	div
206	1	rot_0	rot^0
	(277)	rvt	rot
	(279)	$+\mathrm{rot}^{-1}\,\mathrm{rot}^{-1}\,\mathbf{A}$	$-\mathrm{rot}^{-1}\,\mathrm{rot}^{-1}\,\mathbf{A}$
207	3	$\nabla^2\left(-\frac{\rho}{\varepsilon}\right)$	$\nabla^{-2}\left(-\frac{\rho}{\varepsilon}\right)$

	5	$-\nabla \frac{\rho}{\varepsilon}$	$-\nabla^{-1} \frac{\rho}{\varepsilon}$
	—10	fuhr	führ
	—8	fuhr	führ
208	10	Nat. Sci.	Eng.
	—4	divivisors	divisors
209	(ee22)	$\frac{\hat{\mathbf{P}}^1}{(\mathbf{P}^1 \hat{\mathbf{P}}^2)} w^2$	$\frac{\hat{\mathbf{P}}^1}{(\hat{\mathbf{P}}^1 \mathbf{P}^2)} w$
	—11	$p_{12}\ w_1$	$p_{12}\ w^1$
	—9	$p_{1m} w_1$	$p_{1m} w^1$
	—5	Caucby	Cauchy