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A Classification of Double Circulant Hermitian Self-Dual Codes over $\mathbb{F}_4$ of Lengths up to 26

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Abstract

In this paper, we give a classification of optimal double circulant Hermitian self-dual codes over $\mathbb{F}_4$ of lengths up to 26.

1 Introduction

For a Hermitian self-dual $[n, n/2, d]$ code over $\mathbb{F}_4$, the following upper bound on the minimum weight is known [7]:

$$d \le 2\left\lfloor \frac{n}{6} \right\rfloor + 2.$$

A Hermitian self-dual code of minimum weight $d_e(n) = 2\lfloor n/6 \rfloor + 2$ is called *extremal*. For instance, extremal codes are known to exist for admissible lengths $n \le 10$, $14 \le n \le 22$, and $n = 28, 30$, while there is no extremal code for lengths $n = 12, 24, 26$ [6], [7]. Let $d_h(n)$ be the highest minimum weight among all Hermitian self-dual codes of length n. A Hermitian self-dual code of minimum weight $d_h(n)$ is called *optimal*. All Hermitian self-dual codes were classified for lengths $n \le 16$ [1], [5] and all extremal codes were classified for lengths 18 and 20 [3].

Let $Cir(\boldsymbol{x})$ be the $n \times n$ circulant matrix with the first row $\boldsymbol{x} \in \mathbb{F}_4^n$ and I_n be the identity matrix of order n (≥ 1). A *pure double circulant* $[2k, k]$ code has a generator matrix of the form

$$P(\boldsymbol{r}) = (I_k \ Cir(\boldsymbol{r})) \tag{1}$$

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where $\boldsymbol{r}$ is a vector of $\mathbb{F}_4^k$ $(k \geq 1)$. A $[2k, k]$ code with a generator matrix of the form

$$B(\boldsymbol{r}, \alpha, \beta, \gamma) = \left(\begin{array}{c|cccc} & \alpha & \beta & \cdots & \beta \\ & \gamma & & & \\ I_k & \vdots & & Cir(\boldsymbol{r}) & \\ & \gamma & & & \end{array} \right), \qquad (2)$$

where $\boldsymbol{r}$ is a vector of $\mathbb{F}_4^{k-1}$ $(k \geq 2)$ and borders α, β, γ are elements of $\mathbb{F}_4$, is called a *bordered double circulant* code. These two families of codes are collectively called *double circulant* codes. Both pure and bordered double circulant Hermitian self-dual codes exist for all even lengths. We denote the highest minimum weight among all double circulant Hermitian self-dual $[n, n/2]$ codes by $\rho_h(n)$. Clearly, $d_e(n) \geq d_h(n) \geq \rho_h(n)$.

Gulliver [2] determined $\rho_h(n)$ for admissible lengths $n \leq 40$. For lengths $n \leq 30$, $n \neq 18$, $\rho_h(n)$ is equal to $d_h(n)$. But he did not give a classification of double circulant Hermitian self-dual $[n, n/2, \rho_h(n)]$ codes, and for lengths 24 and 26 he only obtained all weight distributions for which such a code exists. This is the motivation of this paper, and we have the following classification of optimal double circulant Hermitian self-dual codes of lengths up to 26.

Theorem 1. *There are* 3 *inequivalent double circulant Hermitian self-dual* $[22, 11, 8]$ *codes and* 4 *inequivalent double circulant Hermitian self-dual* $[24, 12, 8]$ *codes. Also there are* 19 *inequivalent double circulant Hermitian self-dual* $[26, 13, 8]$ *codes.*

In Table 1, we give the currently known values of $d_e(n)$, $d_h(n)$ and $\rho_h(n)$ for lengths $n \leq 40$, and we also provide the references for these results.

2 Fundamental Concepts

In this section, we give some basic definitions and properties. Let $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$ be the Galois field with four elements, where $\overline{\omega} = \omega^2 = \omega + 1$. A (linear) code over $\mathbb{F}_4$ of length n and dimension k is a k-dimensional subspace of $\mathbb{F}_4^n$ and denoted an $[n, k]$ code. An element of a code is called a codeword. A $k \times n$ matrix whose rows consist of a base of an $[n, k]$ code C is called a generator matrix for C. A generator matrix G is in standard form if $G = (I_k\ A)$ where A is some $k \times (n-k)$ matrix. Two codes C and C' are *equivalent* if there is some monomial matrix M over $\mathbb{F}_4$ such that $C' = CM = \{\boldsymbol{c}M \mid \boldsymbol{c} \in C\}$. A monomial matrix which maps C to itself is called an automorphism of C and the set of all automorphisms of C forms the automorphism group $\mathrm{Aut}(C)$ of C.

Table 1: The highest minimum weights

n	$d_e(n)$	$d_h(n)$		$\rho_h(n)$	
2	2	2	[7]	2	[2]
4	2	2	[7]	2	[2]
6	4	4	[7]	4	[2]
8	4	4	[7]	4	[2]
10	4	4	[7]	4	[2]
12	6	4	[7]	4	[2]
14	6	6	[7]	6	[2]
16	6	6	[7]	6	[2]
18	8	8	[7]	6	[2]
20	8	8	[7]	8	[2]
22	8	8	[7]	8	[2]
24	10	8	[4], [7]	8	[2]
26	10	8	[6]	8	[2]
28	10	10	[7]	10	[2]
30	12	12	[7]	12	[2]
32	12	10 or 12	[2]	10	[2]
34	12	10 or 12	[2]	10	[2]
36	14	12 or 14	[2]	12	[2]
38	14	12 or 14	[2]	12	[2]
40	14	12 or 14	[2]	12	[2]

The weight $wt(\boldsymbol{x})$ of a vector $\boldsymbol{x} \in \mathbb{F}_4^n$ is the number of non-zero components of $\boldsymbol{x}$. The minimum non-zero weight of all codewords in C is called the minimum weight of C and an $[n, k]$ code of minimum weight d is denoted an $[n, k, d]$ code. Let A_i $(i = 0, \dots, n)$ be the number of codewords of weight i, then $A_0, A_1, \dots, A_n$ are called the weight distribution of C. Usually, only the non-zero A_i are listed. A code with only even weight codewords is called even.

For two vectors $\boldsymbol{x} = (x_1, \dots, x_n), \boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{F}_4^n$, the following inner product

$$\boldsymbol{x} \star \boldsymbol{y} = \sum_{i=1}^{n} x_i \overline{y_i},$$

where $\overline{}$ is given by $\overline{0} = 0, \overline{1} = 1$, and $\overline{\overline{\omega}} = \omega$, is known as the Hermitian inner product. The *dual code* $C^\perp$ of C is defined as

$$C^\perp = \{\boldsymbol{x} \in \mathbb{F}_4^n \mid \boldsymbol{x} \star \boldsymbol{c} = 0 \text{ for all } \boldsymbol{c} \in C\}.$$

A generator matrix for $C^\perp$ is called a parity check matrix for C. If a $k \times n$ matrix $(I_k \ A)$, where $A = (a_{ij})$, is a generator matrix for an $[n, k]$ code C, then the $(n - k) \times n$ matrix $(\overline{A}^T \ I_{n-k})$ is a parity check matrix for C where $\overline{A} = (\overline{a_{ij}})$. A code C is called *Hermitian self-dual* if $C = C^\perp$. It is known that an $[n, k]$ code C is Hermitian self-dual if and only if C is even and $n = 2k$ [5].

3 Preliminaries

We have to check all odd weight vectors in $\mathbb{F}_4^k$ as $\boldsymbol{r}$ in $P(\boldsymbol{r})$, all addmissible weight vectors in $\mathbb{F}_4^{k-1}$ as $\boldsymbol{r}$ and all elements in $\mathbb{F}_4$ as α, β, γ in $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ to complete a classification of double circulant Hermitian self-dual codes of length $2k$. So we use some ideas to reduce the number of possible vectors and borders.

For length 2, the code generated by $P((1)) = (1\ 1)$ is the unique double circulant Hermitian self-dual code up to equivalence. The code generated by

$$P((10)) = B((1), 1, 0, 0) = \left(\begin{array}{ccc} I_2 & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array} \right)$$

is the unique double circulant Hermitian self-dual code of length 4 up to equivalence. Therefore, for the remainder of this section, we consider the cases of lengths $2k$ not less than 6. Let $A[i]$ denote the i-th row of a matrix A. For a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_4^n$, $\boldsymbol{x}^\sigma$, $\boldsymbol{x}^\tau$ and $\overline{\boldsymbol{x}}$ provide the vectors $(x_n, x_1, \ldots, x_{n-1})$, $(x_n, x_{n-1}, \ldots, x_1)$ and $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$, respectively. Also let $t(\boldsymbol{x})$ be the summation of all components of $\boldsymbol{x}$, i.e.

$$t(\boldsymbol{x}) = \sum_{i=1}^{n} x_i.$$

Following lemmas are useful for the achievement of our aim. Lemmas 2, 3, 5 and 10 are obvious, so we omit their proofs. The other lemmas are a bit complicated, therefore we give their proofs.

Lemma 2. *Let $P(\boldsymbol{r})$ and $B(\boldsymbol{r'}, \alpha, \beta, \gamma)$ be generator matrices for pure and bordered double circulant Hermitian self-dual codes, respectively. Then both $Cir(\boldsymbol{r})$ and $Cir(\boldsymbol{r'})$ have no repeated rows.*

Lemma 3. *(i) Let C and C' be pure double circulant codes with generator matrices $P(\boldsymbol{r})$ and $P(\boldsymbol{r'})$, respectively. If $\boldsymbol{r'} = \boldsymbol{r}^\sigma$, then C and C' are equivalent.*

(ii) Let C and C' be bordered double circulant codes with generator matrices $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ and $B(\boldsymbol{r'}, \alpha, \beta, \gamma)$, respectively. If $\boldsymbol{r'} = \boldsymbol{r}^\sigma$, then C and C' are equivalent.

Lemma 4. *(i) Let C and C' be pure double circulant codes with generator matrices $P(\boldsymbol{r})$ and $P(\boldsymbol{r'})$, respectively. If $\boldsymbol{r'} = \boldsymbol{r}^\tau$, then C and C' are equivalent.*

(ii) Let C and C' be bordered double circulant codes with generator matrices $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ and $B(\boldsymbol{r'}, \alpha, \beta, \gamma)$, respectively. If $\boldsymbol{r'} = \boldsymbol{r}^\tau$, then C and C' are equivalent.

Proof. Note that

$$\begin{pmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{pmatrix} Cir(\boldsymbol{r}) \begin{pmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{pmatrix} = Cir(\boldsymbol{r})^T = Cir(\boldsymbol{r}^{\tau\sigma}).$$

From Lemma 3, the result follows. $\square$

Lemma 5. *Let C and C' be pure double circulant codes with generator matrices $P(\boldsymbol{r})$ and $P(\boldsymbol{r}')$, respectively. If $\boldsymbol{r}' = s\boldsymbol{r}$ for some non-zero element s of $\mathbb{F}_4$, then C and C' are equivalent.*

Lemma 6. *Let C and C' be pure double circulant Hermitian self-dual codes with generator matrices $P(\boldsymbol{r})$ and $P(\boldsymbol{r}')$, respectively. If $\boldsymbol{r}' = \overline{\boldsymbol{r}}$, then C and C' are equivalent.*

Proof. Since C is Hermitian self-dual, a parity check matrix $(\overline{Cir(\boldsymbol{r})}^T \ I_k) = (Cir(\overline{\boldsymbol{r}})^T \ I_k)$ generates C itself. As seen in the proof of Lemma 4, $Cir(\overline{\boldsymbol{r}}^{\tau\sigma}) = Cir(\overline{\boldsymbol{r}})^T$. Therefore, $P(\boldsymbol{r})$ and $P(\overline{\boldsymbol{r}}^{\sigma\tau})$ generate equivalent codes. Moreover, by Lemmas 3 and 4, $P(\overline{\boldsymbol{r}}^{\sigma\tau})$ and $P(\overline{\boldsymbol{r}})$ generate equivalent codes. The proof is complete. $\square$

Lemma 7. *Let $P(\boldsymbol{r})$ be a generator matrix for a pure double circulant Hermitian self-dual code. Then $t(\boldsymbol{r}) \neq 0$.*

Proof. Suppose that $t(\boldsymbol{r}) = 0$. Remark that the summation of any column of $Cir(\boldsymbol{r})$ is $t(\boldsymbol{r})$ since $Cir(\boldsymbol{r})$ is circulant. Hence, C has the following codeword $\boldsymbol{c}$ of weight k, which is obtained from the summation of all rows of $P(\boldsymbol{r})$:

$$\boldsymbol{c} = (1, \dots, 1, 0, \dots, 0).$$

But this implies $\boldsymbol{c} \star P(\boldsymbol{r})[1] = 1$. This gives a contradiction. $\square$

By Lemmas 2–7, the number of pure double circulant codes which must be checked further for equivalences is reduced.

Lemma 8. *Any bordered double circulant Hermitian self-dual code is equivalent to some code which has a generator matrix of one of the following forms:*

Type A: $B(\boldsymbol{r}, 1, 0, 0)$ *where the first element of* $\boldsymbol{r}$ *is* 1,

Type B: $B(\boldsymbol{r}, 0, 1, 1)$ *where the first element of* $\boldsymbol{r}$ *is* 1, *Type C:* $B(\boldsymbol{r}, 1, 1, 1)$ *where the first element of* $\boldsymbol{r}$ *is* 1,

Type D: $B(\boldsymbol{r}, \omega, 1, 1)$ *where the first element of* $\boldsymbol{r}$ *is* 1.

Type B occurs only for even k *and Types C and D occur only for odd* k. *Type A appears in both cases.*

Proof. Let $G = B(\boldsymbol{r}, \alpha, \beta, \gamma)$ be a generator matrix for a bordered double circulant Hermitian self-dual code C. First, assume that $\beta = 0$. Since C is Hermitian self-dual, $G[1] \star G[2] = \alpha\overline{\gamma} = 0$. If $\alpha = 0$, then $wt(G[1]) = 1$, which contradicts that C is Hermitian self-dual. Thus $\gamma = 0$. Next, suppose $\gamma = 0$ and $\beta \neq 0$. Since $G[2] \star G[1] = t(\boldsymbol{r})\overline{\beta} = 0$, $\beta \neq 0$ implies $t(\boldsymbol{r}) = 0$. Thus, C has the following codeword $\boldsymbol{c}$ which is obtained from the summation of all rows of G:

$$\boldsymbol{c} = (1, \ldots, 1, \alpha, \beta, \ldots, \beta),$$

where the number of 1's is k. Since $t(\boldsymbol{r})\overline{\beta} = 0$, $G[2] \star \boldsymbol{c} = 1 + t(\boldsymbol{r})\overline{\beta} = 1$ and which againsts the assumption of C. Hence, $\gamma = 0$ leads to $\beta = 0$. So $\beta = 0$ if and only if $\gamma = 0$. Therefore α, β, γ are restricted within the following three cases:

(i) $\alpha \neq 0$, $\beta = \gamma = 0$,

(ii) $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$,

(iii) $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$.

Let us think cases (i), (ii) and (iii).

(i) Multiply the $(k+1)$-th column of G by $\overline{\alpha}$. Moreover, from Lemma 2, $\boldsymbol{r}$ is not the zero vector. If necessary, by multiplying all the $(k+2)$-th to the $2k$-th columns of G by some appropriate non-zero element of $\mathbb{F}_4$ and performing σ appropriate times, one can get a generator matrix of Type A.

(ii) If k is odd, then $wt(G[1])$ is odd. Thus k is even. First, if $\boldsymbol{r}$ does not have component of 1, multiply all the $(k+2)$-th to the $2k$-th columns of G by some appropriate non-zero element of $\mathbb{F}_4$, and furthermore, perform σ appropriate times, and one can make the first element of $\boldsymbol{r}$ be 1. Next, multiply the $(k+1)$-th column of G by $\overline{\gamma}$, the first row of G by $\overline{\beta}$ and the first column of G by β. Then one can get a generator matrix of Type B.

(iii) If k is even, then $wt(G[1])$ is odd. Thus k is odd. Since $G[2] \star G[1] = \overline{\alpha}\gamma + t(\boldsymbol{r})\overline{\beta} = 0$, $t(\boldsymbol{r}) = \overline{\alpha}\beta\gamma \neq 0$. Hence, there exists some $\chi \in \mathbb{F}_4$ such that the number of χ's in $\boldsymbol{r}$ is odd. Furthermore, $wt(\boldsymbol{r})$ is even, then $\boldsymbol{r}$ has at least two non-zero elements of $\mathbb{F}_4$ in its components. This implies that one can obtain two matrices such as $B(\boldsymbol{r}_1, a, 1, 1)$ and $B(\boldsymbol{r}_2, b, 1, 1)$ where a and b are distinct non-zero elements of $\mathbb{F}_4$ and the first elements of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are both 1. Then one can get a generator matrix of Type C or Type D. □

Remark 9. *If we delete the condition about the first element of $\boldsymbol{r}$ in Lemma 8, then any bordered double circulant Hermitian self-dual code is equivalent to a code with generator matrix of Type A, Type B or Type C.*

Lemma 10. *Let C and C' be bordered double circulant codes with generator matrices $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ and $B(\boldsymbol{r}', \alpha, \beta, \gamma)$, respectively. Suppose these two matrices are of the same type which is either Type A or Type B in Lemma 8. If $\boldsymbol{r}' = s\boldsymbol{r}$ for some non-zero element s of $\mathbb{F}_4$, then C and C' are equivalent.*

Lemma 11. *Let C and C' be bordered double circulant Hermitian self-dual codes with generator matrices $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ and $B(\boldsymbol{r}', \alpha, \beta, \gamma)$, respectively. Suppose these two matrices are of the same type which is one of Type A, Type B and Type C in Lemma 8. If $\boldsymbol{r}' = \overline{\boldsymbol{r}}$, then C and C' are equivalent.*

Proof. From the same discussion seen in the proof of Lemma 6, $B(\overline{\boldsymbol{r}}, \overline{\alpha}, \overline{\beta}, \overline{\gamma})$ generates a code which is equivalent to C. In addition, $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = (\alpha, \beta, \gamma)$ since α, β, γ are 1 or 0, respectively. This completes the proof. □

Lemma 12. *Let C and C' be bordered double circulant Hermitian self-dual codes with generator matrices $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ and $B(\boldsymbol{r}', \alpha, \beta, \gamma)$, respectively. Suppose these two matrices are of Type D in Lemma 8. If $\boldsymbol{r}' = \omega\overline{\boldsymbol{r}}$, then C and C' are equivalent.*

Proof. From the same discussion seen in the proof of Lemma 6, $B(\overline{\boldsymbol{r}}, \overline{\omega}, 1, 1)$ generates a code which is equivalent to C. Furthermore, it is easily verified that $B(\overline{\boldsymbol{r}}, \overline{\omega}, 1, 1)$ and $B(\omega\overline{\boldsymbol{r}}, \omega, 1, 1)$ generate equivalent codes. The result follows. □

Lemma 13. *Let $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ be a generator matrix for a bordered double circulant Hermitian self-dual code. If $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ is a matrix of Type A in Lemma 8, then $t(\boldsymbol{r}) \neq 0$. If $B(\boldsymbol{r}, \alpha, \beta, \gamma)$ is a matrix of Type B, Type C or Type D in Lemma 8, then $t(\boldsymbol{r}) = \overline{\alpha}$.*

Proof. Let $G = B(\boldsymbol{r}, \alpha, \beta, \gamma)$ be a matrix of Type A and consider the codeword $\boldsymbol{c}$ which obtained from the summation of all rows of G. Since G is a generator matrix for a Hermitian self-dual code, $G[2] \star \boldsymbol{c} = 0$, which implies the conclusion. Let G be a matrix of Type B, Type C or Type D. Since $G[2] \star G[1] = \overline{\alpha}\gamma + t(\boldsymbol{r})\overline{\beta} = 0$, hence $t(\boldsymbol{r}) = \overline{\alpha}\beta\gamma$, and $\beta = \gamma = 1$ implies the conclusion. □

From Lemmas 2, 3, 4 and 8–13, the number of bordered double circulant codes which must be checked further for equivalences is reduced.

4 Results

We explain how to complete our classification. By exhaustive search with MAGMA, we have found all double circulant optimal Hermitian self-dual codes of lengths up to 26 which must be checked in order to complete the classification under reductions based on the lemmas in Section 3. After that, we completed the classification by using `IsEquivalent(A,B)` which is one of the build-in-functions of MAGMA in order to determine the equivalence of given two codes `A` and `B`.

We give the results of the classifications of pure (resp. bordered) double circulant Hermitian self-dual codes with parameters $[22, 11, 8]$, $[24, 12, 8]$ and $[26, 13, 8]$ in Table 2 (resp. Table 3). In Table 2 (resp. Table 3), n gives the lengths of the codes and $\boldsymbol{r}$ gives the first rows of $Cir(\boldsymbol{r})$ in $P(\boldsymbol{r})$ (resp. $B(\boldsymbol{r}, \alpha, \beta, \gamma)$), respectively. WD and $|\mathrm{Aut}|$ give the weight distributions and the orders of the automorphism groups, respectively. For the bordered double circulant codes, the borders (α, β, γ) are also listed in Table 3.

For length 22, there are three inequivalent extremal double circulant Hermitian self-dual codes. These codes are all pure double circulant, in other words, there is no extremal bordered double circulant Hermitian self-dual code of length 22.

For length 24, there are four inequivalent optimal pure double circulant Hermitian self-dual codes and two inequivalent optimal bordered double circulant Hermitian self-dual codes. It was shown that there exist exactly two inequivalent Hermitian self-dual $[24, 12, 8]$ codes with an automorphism of order 11 [8]. Thus, any Hermitian self-dual $[24, 12, 8]$ code with an automorphism of order 11 is equivalent to either $C_{24,B1}$ or $C_{24,B2}$. From computing by MAGMA, $C_{24,P1}$ and $C_{24,B1}$ are equivalent. Furthermore, $C_{24,P1}, C_{24,P2}, C_{24,P3}$ and $C_{24,B2}$ are inequivalent to each other since they have distinct automorphism groups. Hence there are exactly four inequivalent optimal double circulant Hermitian self-dual codes of length 24.

Table 2: Pure optimal double circulant Hermitian self-dual codes

n	Code	$\boldsymbol{r}$	WD	$\lvert\mathrm{Aut}\rvert$
22	$C_{22,P1}$	$(1,1,1,\overline{\omega},1,1,\overline{\omega},1,\overline{\omega},\overline{\omega},\overline{\omega})$	η_{22} in [5]	1330560
	$C_{22,P2}$	$(1,\overline{\omega},1,\overline{\omega},1,\omega,1,\overline{\omega},\overline{\omega},1,\omega)$	η_{22} in [5]	66
	$C_{22,P3}$	$(1,1,\overline{\omega},0,0,1,1,0,0,\overline{\omega},1)$	η_{22} in [5]	66
24	$C_{24,P1}$	$(1,1,0,1,1,1,1,0,1,0,0,0)$	$W_{24,1}$ in [2]	734469120
	$C_{24,P2}$	$(1,\omega,\omega,\overline{\omega},1,\omega,1,\overline{\omega},\omega,\omega,1,0)$	$W_{24,2}$ in [2]	144
	$C_{24,P3}$	$(1,\omega,\omega,\omega,\omega,\omega,\overline{\omega},\overline{\omega},\omega,1,1,0)$	$W_{24,3}$ in [2]	72
26	$C_{26,P1}$	$(1,\overline{\omega},0,\overline{\omega},\overline{\omega},0,\overline{\omega},1,1,0,\omega,0,1)$	$W_{26,1}$ in [2]	156
	$C_{26,P2}$	$(1,\overline{\omega},\overline{\omega},0,\overline{\omega},\omega,0,0,\omega,\overline{\omega},0,\overline{\omega},\overline{\omega})$	$W_{26,1}$ in [2]	78
	$C_{26,P3}$	$(1,0,0,\omega,\overline{\omega},\omega,0,0,\omega,\overline{\omega},\omega,0,0)$	$W_{26,1}$ in [2]	78
	$C_{26,P4}$	$(1,\overline{\omega},\omega,\omega,0,1,0,\overline{\omega},1,1,\overline{\omega},\omega,\omega)$	$W_{26,1}$ in [2]	39
	$C_{26,P5}$	$(1,0,\omega,\omega,0,0,\omega,\overline{\omega},1,\overline{\omega},\overline{\omega},0,\omega)$	$W_{26,1}$ in [2]	39
	$C_{26,P6}$	$(1,\omega,\overline{\omega},0,0,1,1,1,\omega,0,\omega,0,\overline{\omega})$	$W_{26,2}$ in [2]	39
	$C_{26,P7}$	$(1,1,0,1,0,1,1,\overline{\omega},0,\omega,\omega,0,\overline{\omega})$	$W_{26,3}$ in [2]	156
	$C_{26,P8}$	$(1,\omega,1,\omega,\omega,0,1,\overline{\omega},\overline{\omega},1,0,\omega,\omega)$	$W_{26,4}$ in [2]	78
	$C_{26,P9}$	$(1,\omega,\overline{\omega},0,\omega,1,1,\overline{\omega},\overline{\omega},0,1,\overline{\omega},1)$	$W_{26,4}$ in [2]	39
	$C_{26,P10}$	$(1,1,\omega,0,0,\omega,\omega,1,\omega,1,\overline{\omega},1,\overline{\omega})$	$W_{26,4}$ in [2]	39
	$C_{26,P11}$	$(1,1,\omega,\omega,1,\omega,1,1,1,1,1,\omega,1)$	$W_{26,5}$ in [2]	16848

Table 3: Bordered optimal double circulant Hermitian self-dual codes

n	Code	$\boldsymbol{r}$	(α,β,γ)	WD	$\lvert\mathrm{Aut}\rvert$
24	$C_{24,B1}$	$(1,0,1,0,0,0,1,1,1,0,1)$	$(0,1,1)$	$W_{24,1}$ in [2]	734469120
	$C_{24,B2}$	$(1,0,1,0,\overline{\omega},0,1,1,0,\overline{\omega},0)$	$(0,1,1)$	$W_{24,4}$ in [2]	66
26	$C_{26,B1}$	$(1,0,\overline{\omega},0,1,\overline{\omega},\overline{\omega},0,\omega,0,\overline{\omega},\overline{\omega})$	$(1,1,1)$	$W_{26,1}$ in [2]	144
	$C_{26,B2}$	$(1,\overline{\omega},0,\overline{\omega},0,0,0,1,\omega,0,\overline{\omega},0)$	$(1,1,1)$	$W_{26,1}$ in [2]	36
	$C_{26,B3}$	$(1,\omega,0,0,1,0,0,\omega,1,0,\omega,0)$	$(\omega,1,1)$	$W_{26,6}$ in [2]	432
	$C_{26,B4}$	$(1,1,\overline{\omega},1,1,0,0,\omega,\omega,\omega,0,0)$	$(1,1,1)$	$W_{26,7}$ in [2]	72
	$C_{26,B5}$	$(1,\omega,\omega,0,1,\omega,0,0,\overline{\omega},\omega,0,\omega)$	$(1,1,1)$	$W_{26,8}$ in [2]	72
	$C_{26,B6}$	$(1,0,1,\omega,0,0,0,0,0,\omega,\omega,\overline{\omega},0)$	$(1,1,1)$	$W_{26,9}$ in Table 4	72
	$C_{26,B7}$	$(1,\omega,\omega,0,\omega,\overline{\omega},\omega,\omega,0,\overline{\omega},\overline{\omega},1)$	$(1,1,1)$	$W_{26,10}$ in Table 4	72
	$C_{26,B8}$	$(1,\overline{\omega},\overline{\omega},\overline{\omega},\omega,0,\overline{\omega},\omega,0,1,\omega,\overline{\omega})$	$(1,1,1)$	$W_{26,11}$ in Table 4	864

Table 4: New weight distributions of $[26, 13, 8]$ codes

	$W_{26,9}$	$W_{26,10}$	$W_{26,11}$
Weight	Number	Number	Number
0	1	1	1
8	210	417	705
10	9675	8640	7200
12	152940	153768	154920
14	1374780	1380576	1388640
16	6819825	6799539	6771315
18	18028890	18060768	18105120
20	23934540	23905560	23865240
22	13974060	13989792	14011680
24	2738820	2734059	2727435
26	75123	75744	76608

Table 5: Numbers of the classified $[n, n/2, \rho_h(n)]$ codes

n	$\rho_h(n)$	N	N_P	N_B	n	$\rho_h(n)$	N	N_p	N_b
2	2	1	1	–	16	6	2	2	1
4	2	1	1	1	18	6	5	3	2
6	4	1	1	1	20	8	1	1	0
8	4	1	1	1	22	8	3	3	0
10	4	1	1	1	24	8	4	3	2
12	4	3	3	1	26	8	19	11	8
14	6	1	1	1					

For length 26, there are 11 inequivalent optimal pure double circulant Hermitian self-dual codes and 8 inequivalent optimal bordered double circulant Hermitian self-dual codes. Furthermore these 19 codes are inequivalent to each other. Therefore we have Theorem 1. Note that Gulliver [2] gave eight weight distributions of double circulant Hermitian self-dual $[26, 13, 8]$ codes and mentioned that any such code has one of them as its weight distribution. But there exist three more weight distributions $W_{26,9}, W_{26,10}$ and $W_{26,11}$ given in Table 4.

Finally, we give the numbers N of all inequivalent double circulant Hermitian self-dual $[n, n/2, \rho_h(n)]$ codes of lengths $n \leq 26$ in Table 5, where N_P and N_B give the numbers of all inequivalent pure and bordered double circulant Hermitian self-dual $[n, n/2, \rho_h(n)]$ codes, respectively.

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