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GAP program for uniform constructions of some finite simple groups

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Abstract

Let H be a finite group with an involution in $Z(H)$. By the Brauer-Fowler theorem, there are only finitely many non-isomorphic simple groups which have H as a centralizer of the involution. We will explain our automatic GAP [7] program for Michler's algorithm [6] which constructs finite simple groups from this H.

0 Introduction

For the classification of finite simple groups, the Brauer-Fowler theorem is essential.

Theorem 1 (Brauer-Fowler [2]) *Let H be a finite group having center $Z(H)$ of even order. Then there are only finitely many non-isomorphic simple groups G which contain a 2-central involution t for which $C_G(t) \cong H$.*

From this theorem, we can see that there are only finitely many non-isomorphic simple groups which have a given centralizer H of an involution. But there is no concrete way to construct these simple groups from H in general. Our GAP program which uses Michler's algorithm [6] constructs some finite groups which satisfies some conditions on the group H.

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0.1 Michler's Algorithm

Before we start to explain our program, we will review Michler's algorithm in [6]. There are 3 main parts in the algorithm. Let H be a finite permutation group with an involution t in $Z(H)$.

1. Find an elementary abelian normal subgroup A of a Sylow 2-subgroup S of H. Construct $D := N_H(A)$ and $C := C_H(A)$.

2. Find a group extension E of D which satisfies the following 3 conditions.

   i A is normal in E.

   ii $D = C_E(t)$.

   iii $|E : D|$ is odd.

3. Construct G as an epimorphism image of a free product $H *_D E$ with the common subgroup D. (Amalgamation)

The first part is done by only GAP commands. The second part is a main calculation of our program. Since $C_E(A) = C$, E/C is embedded in $\mathrm{Aut}(A)$. The point of the second part is getting a group extension E/C of D/C in $\mathrm{Aut}(A) \cong GL(d, 2)$ where $|A| = 2^d$. We can construct some group extension E as a permutation group from the elementary abelian 2-group A and the subgroup E/C of $\mathrm{Aut}(A)$ by using computational group theory [4]. The next diagram shows relations of subgroups in G.

$$
\begin{tikzcd}
 & H & & E \arrow[dr] & & GL(d,2) \cong \mathrm{Aut}(A) \\
S \arrow[ur] & & D \arrow[ul] \arrow[ur] \arrow[dr] & & E/C \arrow[ur] & \\
 & C \arrow[ur] \arrow[dr] & & D/C \arrow[ur] & & \\
A \arrow[ur] & & 1 \arrow[ur] & & &
\end{tikzcd}
$$

The last part is also not so easy, but we can construct G by using representation theory. We call this last part amalgamation. We need to find a pair of compatible characters. (see Section 2 and [5])

In the first section, we will explain our automatic program. To do everything automatically, we put additional conditions for G. In the second section, the detail of amalgamation is described. The second part of Michler's algorithm (the main part of our program) is explained in the third section. We will show an example of a calculation in the last section.

1 The program MNGs

Our GAP program is made up of some independent functions of GAP. Thus it is also possible to use these functions for getting a partial result. A resulting group G is obtained by combination of these functions.

1.1 The output of the MNGs

We named our program **MNGs** from a phrase "**M**ake **N**ew **G**roup**s**". The purpose of this program is to construct a new group G from the centralizer of an involution t of G. Let p be a prime which does not divide the order of H and E. Let k be a finite field of characteristic p. The group G is constructed as a matrix group over k. But we also need many local results for D, C and E in process. We put each information in the output as a record. Since different pairs (A, E) can produce different G, all data for i th A and j th E are obtained by "`OutputName[i][j].RecordName`" in GAP. For example, if `gs` is the name of an output of **MNGs**, we can get a group G by `gs[1][1].G` for the first A and the first E. This is a table of names of records which our program stored.

`H`	an input group	`CharTableOfH`	the character table of H
`t`	an involution	`A`	an elementary abelian normal subgroup of Sylow 2-subgroup of H
`D`	the normalizer of A in H	`CharTableOfD`	the character table of D
`Ep`	a group extension E of D	`CharTableOfE`	the character table of E
`C`	the centralizer of A in H	`delta`	$\Delta = D/C$ as a subgroup of $\mathrm{Aut}(A)$
`eta`	a homomorphism from D to Δ	`GroupHomFromD`	an isomorphism from D to a subgroup E
`Phi`	$\Phi = E/C$	`CP`	a pair of compatible characters
`G`	a newly constructed group	`Ambs`	an ambiguity of G

The next example shows that the MNGs constructs the McLaughlin group McL as the first new group from $H \cong 2.A_8$. The output `gs[1][1].G` is a simple matrix subgroup of $GL(22, 11)$ whose order is 898,128,000.

```
gap> tom:=TableOfMarks("2.A8");;
gap> H:=UnderlyingGroup(tom);
<permutation group of size 40320 with 2 generators>
```

```
gap> gs:=MNGs(H,22,300,0);;
       < many messages ... >
gap> Size(gs[1][1].G);
898128000
gap> IsSimple(gs[1][1].G);
true
gap> IsomorphismTypeFiniteSimpleGroup(gs[1][1].G);
rec( series := "Spor", name := "Mc" )
```

MNGs needs a permutation group H and 3 numbers (like 22, 300 and 0 in the above example) as an input data. We will explain these numbers in Section 2.

1.2 Additional Conditions for MNGs

For an automatic calculation, we put additional 5 conditions.

C.1 The order of A is at most 2^6.

C.2 Both D and E are split with A.

C.3 C is equal to A.

C.4 The restriction of a pair of compatible characters to D is multiplicity free.

C.5 The characteristic p of the field k in 1.1 is small enough. ($p < 23$)

The order of $\mathrm{Aut}(A)$ is limited by the condition C.1. The conditions C.2 and C.3 make it possible to construct an extended group E automatically by the semidirect product of A by E/C. The condition C.4 means each constituent of the restriction of characters has multiplicity one. The condition C.4 restricts the possibility of embedding D in E. We put the last condition C.5 for the GAP function `BrauerCharacterValue`. If p is too big, this function does not work.

In the next table, we show groups which are constructed automatically by MNGs. We use the notations of the Atlas [3] of finite groups in this table. The group G is a target simple group. H is constructed as a centralizer of an involution in G. The program MNGs does not only reconstruct G but also constructs some satellite groups (F_1, F_2, F_3).

H	G	F_1	F_2	F_3
$2^4{:}S_4$	M_{22}	$2^4{:}A_6$	$2^4{:}A_7$	$2^3.L_3(2){:}2$
$2^{4+3}{:}L_3(2)$	M_{24}	He	$2^4{:}A_8$	
$2^3{:}S_4$	A_8	$2^3{:}L_3(2)$	$(3^7 \times 2^3){:}L_3(2)$	
$2^5{:}S_6$	A_{12}	$Sp_6(2)$		
$2.A_8$	McL			
$2 \times A_5$	J_1			

There are some simple groups which are not constructed automatically by MNGs because some conditions are not satisfied. But MNGs constructs the following groups with some satellite groups by adding some manually operated steps.

H	G	F_1	Conditions not satisfied
$2^4{:}L_3(2)$	M_{23}	$2^4{:}A_7$	C.4 or C.5
$2^{1+4}{:}A_5$	J_2	$3.J_3$	C.2
$2.A_{11}$	Ly		C.2

2 Amalgamation

In this section, we will explain the third part of Michler's algorithm. Using a pair of compatible characters (see [5]) of H and E, our program tries to construct irreducible p-modular representations of G. This means the output G of our program is presented as a matrix group over a finite field of characteristic p.

2.1 Outline of amalgamation

We can construct G as an image of the free product $H *_D E$ of H and E with a common subgroup D by the following steps.

1. Construct the character tables of H, E and D.

2. Find pairs of compatible characters (χ_H, τ_E) such that $\chi_D = \tau_D$. (the restriction of characters to D)

3. Construct corresponding kH and kE-modules V_H and W_E such that the actions of D on these modules coincide.

The first step is done by some commands in GAP. The second step is a combinatorial problem. The detail of this step is explained in [5]. In the last step, we construct p-modular representation V of G whose restrictions to H and E correspond to the character χ_H and τ_E, respectively. A set of representation matrices for both V_H and W_E can be generators of G. In general, we cannot pick up a unique module from a character. But since the order of H and E are not divisible by p, we can construct V_H and W_E by the direct sum of simple modules which correspond to irreducible constituents in χ_H and τ_E.

As we said in 1.1, MNGs needs 3 numbers as an input data. The first number is the upper bound of the degree of pairs of compatible characters. If this number is 0, MNGs calculates all pairs. We will explain the second number in 2.2. The third number is the characteristic p of the finite field k. If this number is 0, MNGs sets the smallest prime which does not divide the order of H and E.

2.2 Construction of simple kH or kE -modules

In the last step in 2.1, MNGs needs to construct simple modules which correspond to irreducible characters in χ_H and τ_E. Since H and E are permutation groups, we can get permutation modules for any finite filed k. By a tensor product of composition factors in the permutation modules, it is possible to construct all simple modules which we need. We set an upper limit for a dimension of tensor products. This is the second number in the input of MNGs. If this number is 0, no-limit is set for the dimension of tensor products.

3 Group Extension

Now we start to explain the method to find E from D automatically. This is the second part of Michler's algorithm. Since we set the conditions C.2 and C.3 in 2.1, we can construct E by the semi-direct product of A and E/C. Thus it is enough to find E/C in $GL(d, 2)$ for construction of E automatically.

3.1 In case d is small

Since A is elementary abelian, we can see that E/C as a subgroup of $GL(d, 2)$ where d is the rank of A.

$$1 \to C \to E \to E/C \hookrightarrow GL(d, 2)$$

If d is **less or equal to 4** then the order of $GL(d,2)$ is **at most 20,160**. Then we can search E/C which satisfies our conditions in 0.1 from **ALL** subgroups of $GL(d,2)$. But the order of $GL(d,2)$ increases dramatically.

$$\begin{array}{rcr} |GL(4,2)| & = & 20,160 \\ |GL(5,2)| & = & 9,999,360 \\ |GL(6,2)| & = & 20,158,709,760 \end{array}$$

Thus if d is more than 4, it is impossible to search E/C from **all** subgroups of $GL(d,2)$.

3.2 In case $d = 5$ or 6

As we have seen in 3.1, we need a smaller group which contains E/C. Let ω be an orbit of t under E/C. Since D should be $C_E(t)$, the index $|E:D|$ is equal to the length of the orbit ω. Moreover ω is the union of some orbits o_i of D on A. So we can make a list of candidates for the orbit ω by combinations of o_i. Let S_ω be the stabilizer of ω in $GL(d,2)$. Since E/C stabilizes ω, E/C must be a subgroup of S_ω. The next diagram shows subgroups and its action domains.

$$\begin{array}{rcccc} & D/C \subset & E/C \subset & S_\omega \subset & GL(d,2) \\ \text{Action}: & \swarrow \cdots \downarrow & \downarrow & & \downarrow \\ t \in o_1 \cup o_2 \cup \ldots & = & \omega & \subset & A \end{array}$$

We can hope S_ω is much smaller than whole $GL(d,2)$. Indeed we can see S_ω is small enough in case $d = 6$ from the next table.

G	$\lvert\omega\rvert$	$\lvert E/C\rvert$	$\lvert S_\omega\rvert$	d	$\lvert GL(d,2)\rvert$
A_{12}	3^3	$2^3.3^4$	$2^4.3^4$	6	$2^{15}.3^4.5.7^2.31$
$Sp_6(2)$	7	$2^3.3.7$	$2^4.3^2.5.7$	6	$2^{15}.3^4.5.7^2.31$
M_{24}	3.7	$\mathbf{2^4.3^2.7}$	$\mathbf{2^4.3^2.7}$	6	$2^{15}.3^4.5.7^2.31$
McL	3.5	$2^3.3^2.5.7$	$\mathbf{2^6.3^2.5.7}$	4	$\mathbf{2^6.3^2.5.7}$

On the other side, if ω consists of all elements of A except identity, S_ω becomes $GL(d,2)$ as one can see in the last line of the table.

3.3 Transitive Extension

As d increases, the number of the candidates of ω also increases. For example,

- In case $D = 2^6:(S_4 \times 2)$ for M_{24}, there are **63** candidates of ω.

- In case $D = 2^5 : S_6$ for A_{12}, there are **255** candidates of ω.
- In case $D = 2^7 : S_8$ for A_{16}, there are **16,383** candidates of ω.

Thus we also need to take off some hopeless candidates. Let us observe E/C from another point of view. Since E/C acts on ω transitively and the one point stabilizer of t in E/C must be D/C, E/C must be **a transitive extension** of D/C on ω.

$$\left(\begin{array}{c} \text{Exists } E \text{ with} \\ \text{the orbit } \omega \text{ of } t \end{array} \right) \Leftrightarrow \left(\begin{array}{c} D/C \text{ has a transitive} \\ \text{extension on } \omega \end{array} \right)$$

Thus if we know a good criterion where ω induces a transitive extension of D/C, we can delete hopeless candidates of ω from the list.

Let $\bar{D}$ be a permutation group acting on Ω. Let $\bar{E}$ be a transitive extension of $\bar{D}$. A subset X of $\bar{D}$ is called an S-subset with respect to the pair $(\bar{D}, \bar{E})$ if and only if all $\bar{E}$-conjugates Y of X are $\bar{D}$-conjugate with X. Let $I(X) := \{i \in \Omega | \forall x \in X, i^x = i\}$ and c_X denotes the number of $\bar{D}$-conjugates of X.

Criterion 1 (Bannai [1]) *Let $\bar{D}$ be a permutation group on Ω. If $\bar{D}$ has a transitive extension $\bar{E}$ then for any S-subset X with respect to the pair $(\bar{D}, \bar{E})$, $\frac{|\Omega|+1}{|I(X)|+1} c_X$ is an integer.*

Let x in $\bar{D}$. We can say $\{x\}$ is an S-subset if all y in $\bar{D}$ such that $|I(x)| = |I(y)|$ and the order of y equal with the order of x are $\bar{D}$-conjugate with x. This means that we can find some S-subsets without $\bar{E}$. So let $\Omega = \omega \backslash t$. If there is some x such that $\{x\}$ is an S-subset and $\frac{|\Omega|+1}{|I(\{x\})|+1} c_x$ is **NOT** an integer, then this ω is **OUT**! (You can forget it.) By using Bannai's criterion, we can decrease the number of candidates of ω as follows.

- In case $D = 2^6 : (S_4 \times 2)$ for M_{24}, **63** candidates of ω. $\Rightarrow$ **48** candidates
- In case $D = 2^5 : S_6$ for A_{12}, **255** candidates of ω. $\Rightarrow$ **76** candidates
- In case $D = 2^7 : S_8$ for A_{16}, **16,383** candidates of ω. $\Rightarrow$ **206** candidates

4 Demonstration

Let $H := \langle n_1, n_2, n_3, n_4, n_5, v_1, v_2, v_3, v_4, v_5 \rangle$ be $2^5 : S_6$ by the following relations.

- $(n_i)^2 = 1$ for $(i = 1 \ldots 5)$, $(n_i n_{i+1})^3 = 1$ for $(i = 1 \ldots 4)$,
  $[n_i, n_j] = 1$ for $(1 \leq i+1 < j \leq 5)$
- $(v_i)^2 = 1$ for $(i = 1 \ldots 5)$, $[v_i, v_j] = 1$ for $(1 \leq i < j \leq 5)$
- $v_i{}^{n_j} = \begin{cases} v_{i+1} & (i = j) \\ v_{i-1} & (i = j+1) \quad \text{for } (j < 5), \\ v_i & (\text{else}) \end{cases}$
  $v_i{}^{n_5} = v_i v_5$ for $(i = 1 \ldots 4)$, $v_5{}^{n_5} = v_5$

In this section, we will explain the detail of a concrete calculation. Let $t = v_1v_2v_3v_4v_5$. The group H is isomorphic to the centralizer of the involution t in the alternating group A_{12}. MNGs finds two subgroups A of order 64 and 32. But the group A of order 32 is not split. Thus we skip this group. Generators of the group A of order 64 are $\{n_1, n_4, v_3, n_1{}^{v_2}, n_4{}^{v_5}, n_5{}^{n_4n_3}\}$. Let H_0 be the subgroup of H generated by a set $\{n_2, n_3, n_4, n_5, v_2, v_3, v_4, v_5\}$. Then H_0 is isomorphic to $2^4 : S_5$. Since the index of H_0 in H is 12, we can get a permutation representation of degree 12 for H. The generators [n_1, n_2, n_3, n_4, n_5, v_1, v_2, v_3, v_4, v_5] of H correspond to the permutations

```
[ (1,2)( 3, 5), (2,4)( 5, 7), (4,6)( 7, 9), (6,8)( 9,11), ( 8,10)(11,12),
  (1,3)(10,12), (2,5)(10,12), (4,7)(10,12), (6,9)(10,12), ( 8,11)(10,12) ].
```

Let d_1=`(1,2,3,5)(4,9,10,8,7,6,12,11)`, d_2=`(1,9,4,2,11,12)(3,6,7,5,8,10)`. Then $\{d_1, d_2\}$ generates D. Moreover let h=`(1,10,8,6,4,2)(3,12,11,9,7,5)`. Then $\{h, d_1, d_2\}$ are generators of H. MNGs constructs 2 group extensions E_1 and E_2 which are isomorphic to $2^6 : L_3(2)$ and $2^6 : (3^3 : S_4)$, respectively. Let η_1 and η_2 be group homomorphisms from D to E_1 and E_2, respectively. Then the images $\eta_1(d_1) = \eta_2(d_1)$ and $\eta_1(d_2) = \eta_2(d_2)$ are

```
[ ( 1,18,42,59, 7,24,48,61)( 2,26,43,55, 8,32,45,49)( 3,22,44,63, 5,20,46,57)
  ( 4,30,41,51, 6,28,47,53)( 9,19,38,60,15,21,36,62)(10,27,39,56,16,29,33,50)
  (11,23,40,64,13,17,34,58)(12,31,37,52,14,25,35,54),

  ( 1,64,37,30,35,58)( 2,52,53,29,47,42)( 3,60, 5,32,39,26)( 4,56,21,31,43,10)
  ( 6,20,55,25,15,44)( 7,28)( 8,24,23,27,11,12)( 9,16,40,22,19,59)
  (13,48,38,18,51,57)(14,36,54,17,63,41)(33,62)(34,50,49,61,45,46)].
```

Let e_1 =

```
( 1,41,30, 8,48,27)( 2,42,29, 7,47,28)( 3,38,21, 6,35,20)( 4,37,22, 5,36,19)
( 9,44,18,16,45,23)(10,43,17,15,46,24)(11,39,25,14,34,32)(12,40,26,13,33,31)
(49,57,60,56,64,61)(50,58,59,55,63,62)(51,54)(52,53)
```

and $e_2 =$

```
( 3,55,43)( 4,56,44)( 5,38,33)( 6,37,34)( 7,20,11)( 8,19,12)( 9,49,58)
(10,50,57)(13,22,26)(14,21,25)(15,36,52)(16,35,51)(17,62,46)(18,61,45)
(23,47,40)(24,48,39)(27,60,63)(28,59,64)(29,41,53)(30,42,54).
```

Then $\{\eta_1(d_1), \eta_1(d_2), e_1\}$ and $\{\eta_2(d_1), \eta_2(d_2), e_2\}$ are sets of generators of E_1 and E_2, respectively. The degree of a pair of compatible characters (χ_{1H}, τ_{E_1}) is 7. χ_{1H} is a sum of 2 irreducible characters of degree 1 and 6. τ_{E_1} is an irreducible character. The restriction of these characters to D is decomposed into 2 irreducible characters of degree 1 and 6. The degree of a pair of compatible characters (χ_{2H}, τ_{E_2}) is 11. χ_{2H} is a sum of 2 irreducible characters of degree 5 and 6. τ_{E_2} is a sum of 2 irreducible characters of degree 2 and 9. The restriction of these characters to D is decomposed into 3 irreducible characters of degree 2, 3 and 6. The irreducible characters of H with degree 6 in χ_{1H} and χ_{2H} are the same. By using these compatible characters, the program MNGs succeeded to construct $Sp_4(2)$ and A_{12} automatically. The pair H and E_1 induced $G_1 \cong Sp_4(2)$ as a subgroup of $GL(7, 11)$. The pair H and E_2 induced $G_2 \cong A_{12}$ as a subgroup of $GL(11, 7)$. The generators of G_1 which correspond to $\{h, d_1, d_2, e_1\}$ are

$$\begin{pmatrix} X & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & 2 & \cdot & \cdot & \cdot \\ \cdot & X & 4 & 9 & 3 & \cdot & \cdot \\ \cdot & \cdot & X & 5 & 8 & 5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & X & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & X \\ \cdot & \cdot & \cdot & X & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 9 & 2 & 3 & 5 \\ \cdot & X & \cdot & 1 & 6 & 3 & 5 \\ \cdot & \cdot & \cdot & 7 & X & 8 & 7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & X & 3 & 4 & 2 \\ \cdot & \cdot & \cdot & \cdot & X & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} X & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 4 & 1 & 8 & X & 6 \\ \cdot & 1 & 4 & 1 & 8 & \cdot & 5 \\ \cdot & 3 & X & X & 8 & 3 & 7 \\ \cdot & 1 & \cdot & 4 & 9 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & X & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 8 & 7 & 9 \end{pmatrix}, \begin{pmatrix} \cdot & 8 & 8 & 9 & 1 & \cdot & \cdot \\ 2 & 6 & 5 & 3 & 8 & 6 & 4 \\ 9 & 6 & 5 & 3 & 8 & 6 & 4 \\ \cdot & \cdot & \cdot & 5 & 8 & \cdot & 1 \\ \cdot & \cdot & \cdot & 6 & 9 & 3 & 4 \\ \cdot & \cdot & \cdot & 6 & X & 3 & 5 \\ \cdot & \cdot & \cdot & 6 & X & 4 & 4 \end{pmatrix}$$

(X is 10 in $GF(11)$)

The generators of G_2 which correspond to $\{h, d_1, d_2, e_2\}$ are

$$\begin{pmatrix} 1&2&1&3&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 4&6&\cdot&4&6&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 5&6&5&2&4&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&5&1&3&5&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 4&5&\cdot&4&5&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6 \\ \cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot \end{pmatrix}, \begin{pmatrix} 3&6&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&4&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&6&3&6&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&4&3&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&6&3&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix}, \begin{pmatrix} 2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&4&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&4&5&6&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&2&6&5&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&4&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6 \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \begin{pmatrix} 4&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 6&2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&3&5&5&5&\cdot&4&3&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&6&6&\cdot&2&5&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&2&2&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&5&\cdot&\cdot&\cdot&3&3&\cdot \\ \cdot&\cdot&3&1&\cdot&4&3&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&4&6&\cdot&4&3&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&3&3&\cdot \end{pmatrix}.$$

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