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Abstract

Let X be a compact subspace of the real line equipped with the standard topology. We prove that there exists a chaotic homeomorphism of X onto itself if and only if X is homeomorphic to the Cantor set, and show a family of chaotic homeomorphisms of the Cantor set onto itself.

Introduction

Let X be a compact space with metric. A continuous map f of X onto itself is said to be ***chaotic*** if f satisfies the following conditions:

(C-1) Periodic points are dense in X.

(C-2) f is topologically transitive.

(C-3) f has sensitive dependence on initial conditions.

This definition was introduced by Devaney (cf. [3. § 1.8, Definition 8.5]). Needless to say, though the conditions (C-1) and (C-2) are topological properties, the condition (C-3) is not a topological property but a metric one. However, as mentioned in [2], if X is a compact metric space, the condition (C-3) also becomes a topological property. Moreover, in [2], it was proved that the condition (C-3) is derived from two conditions (C-1) and (C-2) under the assumption that X has infinitely many points. This means that chaotic

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property is a property of topology. Our attention is restricted to continuous maps on comapct subspaces of the real line equipped with the standard topology. We know a lot of chaotic maps in those continuous maps. As typical examples, there exist the tent map on the unit interval and the unilateral shift map on the Cantor set. These two examples have deep relationship in the sense of operator theory in L^1-space, namely the operators associated with two chaotic maps are considered as the same one in L^1-space. This was precisely shown in a recent paper by the authors [1, Proposition 3.3]. In the paper, we recognized that the chaotic phenomina appears by the property on non-injectivity of two maps. Now, in addition to the unilateral shift, the bilateral shift on the Cantor set is also a typical chaotic map, which is of course a homeomorphism of the Cantor set. It is quite natural to seek a corresponding chaotic map on the unit interval to the bilateral shift as in the case of unilateral shift. It is our purpose to show that that is impossible. We prove that there exists a chaotic homeomorphism of X onto itself if and only if X is homeomorphic to the Cantor set. Furthermore we show a family of chaotic homeomorphisms of the Cantor set onto itself. Here we note that, in [4], Melo and Strien have had a lot of detailed discussion on the behavior of orbits for one-dimensional dynamics. In contrast to it, in the present paper, we have discussion on it from the point of view of chaotic property in the sense of Devaney's definition.

Now we give some notation. We denote by $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{N}$ the real line with Euclidean metric d, the set of all integers and the set of all positive integers respectively. For $k \in \mathbf{N}$, the sets of periodic points for a map $f : X \to X$ are denoted as follows:

(i) $P_k(f) = \{x \in X : f^k(x) = x\}$.

(ii) $Q_k(f) = \{x \in X : f^k(x) = x \text{ but } f^i(x) \neq x \text{ for } i = 1, 2, \ldots, k-1\}$.

Moreover we put

(iii) $Per(f) = \bigcup_{k=1}^{\infty} P_k(f)$.

Namely, $P_k(f)$ is the set of all k-periodic points, $Q_k(f)$ is the set of those k-periodic points whose prime period is k and $Per(f)$ is the set of all periodic points. Obviously we have that $Per(f) = \cup_{k=1}^{\infty} Q_k(f)$ and $\{Q_k(f)\}_{k=1}^{\infty}$ is a family of mutually disjoint subsets of X. Using these notation, the definition of chaotic map is re-written as follows:

(C-1) $Per(f)$ is dense in X.

(C-2) For any pair of non-empty open sets U and V in X there exists $k \in \mathbf{N}$ such that $f^k(U) \cap V \neq \emptyset$.

(C-3) There exists $\delta > 0$ which satisfies that for any $x \in X$ and any neighborhood U of x there exist $y \in U$ and $k \in \mathbf{N}$ such that $d(f^k(x), f^k(y)) \geq \delta$.

Moreover we remark that all totally disconnected subspaces of **R** which have no isolated points are homeomorphic and called the Cantor set. Furthermore we note that sometimes we use the term *homeomorphism of X* instead of *homeomorphism of X onto itself.*

§ 1. The chaotic homeomorphisms of compact subspace of R.

First we show three lemmas and an example which are devoted to proving our theorem.

Lemma 1.1. *Let* $f : [0, 1] \to [0, 1]$ *be a continuous map. If f satisfies two conditions* (C-1) *and* (C-2), *then it is not a homeomorphism.*

Proof. Suppose that f is a homeomorphism and satisfies the condition (C-1). Then, $Per(f)$ is dense in [0, 1]. Since $Per(f) = P_1(f) \cup P_2(f) = P_2(f)$ and each $P_i(f)$ $(i = 1, 2)$ is closed, we have $Per(f)$ is closed and thus it follows that $Per(f) = [0, 1]$. In the case where $P_1(f) = [0, 1]$, we have $f(x) = x$ $(x \in [0, 1])$, so that f does not satisfy the condition (C-2). Next, suppose that $P_1(f) = Q_1(f)$ is a proper subset of [0, 1]. Since $Per(f) = Q_1(f) \cup Q_2(f)$, the set $Q_2(f)$ is not empty. Thus f is a monotonically decreasing map on [0, 1] with $f(0) = 1$ and $f(1) = 0$. Therefore $P_1(f) = Q_1(f)$ consists of only one point $\{x_0\}$. Thus we have $Q_2(f) = [0, x_0) \cup (x_0, 1]$. We put $U = (0, x_0/2)$ and $V = (x_0/2, f(x_0/2))$. Then we have $f^{2n}(U) = U$ and $f^{2n+1}(U) = (f(x_0/2), 1)$ for all $n \in \mathbf{N} \cup \{0\}$. Consequently we have $f^n(U) \cap V = \emptyset$ for all $n \in \mathbf{N}$, that is, f does not satisfy the condition (C-2). q.e.d.

It goes without saying that Lemma 1.1 means that no homeomorphism of $[a, b]$ satisfy both conditions (C-1) and (C-2) for all a and b with $a < b$.

Lemma 1.2. *Let X be a compact subspace of* **R** *which contains a non-empty open interval* (a, b) *and* $f : X \to X$ *a continuous map. If f satisfies two conditions* (C-1) *and* (C-2), *then it is not a homeomorphism.*

Proof. Suppose that f is a homeomorphism and satisfies the condition (C-1). We show that f does not satisfy the condition (C-2) as in the above proof. Let (c, d) be the largest open interval in X such that (c, d) contains (a, b). Since X is closed in **R**, the closed interval $[c, d]$ is contained in X. In the case where $f^n([c, d]) \cap [c, d] = \emptyset$ for all $n \in \mathbf{N}$, f does not satisfy the condition (C-2).

Next, we consider the case where $f^n([c, d]) \cap [c, d] \neq \emptyset$ for some $n \in \mathbf{N}$. Let k be the smallest positive integer such that $f^k([c, d]) \cap [c, d] \neq \emptyset$. Since $[c, d]$ is a connected component in X and f^k is a homeomorphism of X onto itself, the set $f^k([c, d])$ is also a connected component in X. Thus we have $f^k([c, d]) = [c, d]$. Now we have that the restriction of f^k to $[c, d]$ is a homeomorphism of $[c, d]$ onto itself and $Per(f) \cap [c, d] = Per(f^k) \cap [c, d]$. Since $Per(f)$ is dense in X, we have that $Per(f) \cap [c, d]$ is dense in $[c, d]$ and thus $Per(f^k) \cap [c, d]$ is dense in $[c, d]$. Hence, by Lemma 1.1, f^k is not topologically transitive on $[c, d]$, that is, there exist two non-empty open intervals U and V in $[c, d]$ such that $f^{nk}(U) \cap V = \emptyset$ for all $n \in \mathbf{N}$. Moreover, by the property of k, we have

$$f^{nk+i}(U) \cap V \subset f^i([c, d]) \cap [c, d] = \emptyset$$

for $n \in \mathbf{N} \cup \{0\}$ and $i = 1, 2, \ldots, k-1$. Since U and V are non-empty open sets in X, this means that f does not satisfy the condition (C-2). q.e.d.

Lemma 1.3. *Let X be a compact subspace of $\mathbf{R}$ which has an isolated point and $f : X \to X$ a continuous map. Then it does not satisfy the condition* (C-3).

Proof. Let x be an isolated point. Then there exist $\epsilon > 0$ such that

$$\{y \in X | d(y, x) < \epsilon\} \cap X = \{x\}.$$

Hence, for this ϵ, the condition $d(y, x) < \epsilon$ implies $y = x$. Thus trivially f does not satisfy the condition (C-3). q.e.d.

Now let X be a compact subspace of $\mathbf{R}$ and put $C = \Pi_{n \in \mathbf{Z}} \{0, 1\}$. As mentioned in the introduction, if X is a totally disconnected subspace of $\mathbf{R}$ which has no isolated points then X is homeomorphic to C. As is well-known, there exists a chaotic homeomorphism of C onto itself, say, the bilateral shift map, which we note in the following.

Example 1.4. We denote by $S : C \to C$ the bilateral shift map of C onto itself, that is, S is the map defined by $y = S(x)$ where $x = (x_n)_{n \in \mathbf{Z}}$, $y = (y_n)_{n \in \mathbf{Z}}$ and $y_n = x_{n+1}$ for each $n \in \mathbf{Z}$. Then S is a chaotic homeomorphism.

Combining Lemmas 1.2, 1.3 and Example 1.4, we get our theorem as follows.

Theorem 1.5. *Let X be a compact subspace of* $\mathbf{R}$. *Then there exists a chaotic homeomorphism $f: X \to X$ if and only if X is homeomorphic to the Cantor set.*

In general case, the conditions (C-1) and (C-2) are independent each other. This independence between two conditions valids in the case of compact subspaces of the real line. In fact, as appeared in the proof of Lemma 1.1, the identity map $f(x) = x$ on $[0, 1]$ satisfies the condition (C-1) but does not satisfy the condition (C-2). Moreover the following example satisfies the condition (C-2) but does not satisfy the condition (C-1). Let $X = \{1/n \mid n \in \mathbf{N}\} \cup \{0\}$ and f be the map defined by $f(0) = 0$, $f(1) = 1/2$ and $f(1/(2n+1)) = 1/(2(n-1)+1)$, $f(1/(2n)) = 1/(2(n+1))$ for $n \in \mathbf{N}$.

§ 2. A kind of chaotic homeomorphism of the Cantor set.

In Example 1.4, we saw that there exists at least one chaotic homeomorphism of the Cantor set onto itself, which is the bilateral shift map. In this section, we show that this chaotic map is a map in a family of chaotic homeomorphism. For a subset M of $\mathbf{Z}$, we put

$$C_M = \prod_{(m,n)\in M\times\mathbf{Z}} \{0, 1\} = \{x = (x_{(m,n)})_{(m,n)\in M\times\mathbf{Z}} \mid x_{(m,n)} \in \{0, 1\}\},$$

where C_M has the canonical product topology. Moreover we denote by S_M the homeomorphism of C_M onto itself defined by $y = S_M(x)$, where

$$x = (x_{(m,n)})_{(m,n)\in M\times\mathbf{Z}},\ y = (y_{(m,n)})_{(m,n)\in M\times\mathbf{Z}} \quad \text{and} \quad y_{(m,n)} = x_{(m,n+1)}$$

for each $(m, n) \in M \times \mathbf{Z}$. Then, it is easy to see that C_M is homeomorphic to the Cantor set and we have the following.

Lemma 2.1. *The map S_M is a chaotic homeomorphism.*

Proof. For $x = (x_{(m,n)})_{(m,n)\in M\times\mathbf{Z}} \in C_M$, A $\subset M$ and $N \in \mathbf{N}$, we put $\mathbf{Z}(N) = \{n \in Z \mid -N \leq n \leq N\}$ and

$$U(x, A, \mathbf{Z}(N)) = \{y = (y_{(m,n)})_{(m,n)\in M\times\mathbf{Z}} \mid y_{(m,n)} = x_{(m,n)} \text{ for } (m, n) \in A \times \mathbf{Z}(N)\}.$$

Moreover we put

$$\mathcal{B} = \{ U(x, A, Z(N)) \mid x \in C_M, A \text{ is a finite subset of } M \text{ and } N \in \mathbf{N} \}.$$

Then $\mathcal{B}$ is a base of open sets in C_M. First we show that $Per\,(S_M)$ is dense in C_M. For $x = (x_{(m,n)})_{(m,n)\in M\times\mathbf{Z}}$ and $U(x, A, \mathbf{Z}(N)) \in \mathcal{B}$, let $y = (y_{(m,n)})_{(m,n)\in M\times\mathbf{Z}}$ be an element of C_M such that $y_{(m,(2N+1)k+i)} = x_{(m,i)}$ for $(m, i) \in A \times \mathbf{Z}(N)$ and $k \in \mathbf{Z}$. Then we have $y \in U(x, A, \mathbf{Z}(N)) \cap Per_{(2N+1)}(S_M)$. Next we show that S_M is topologically transitive. For $U(x, A_1, \mathbf{Z}(N_1))$ and $U(y, A_2, \mathbf{Z}(N_2))$ in $\mathcal{B}$, where $x = (x_{(m,n)})_{(m,n)\in M\times\mathbf{Z}}$ and $y = (y_{(m,n)})_{(m,n)\in M\times\mathbf{Z}}$, let $z = (z_{(m,n)})_{(m,n)\in M\times\mathbf{Z}}$ be the element of $U(x, A_1, Z(N_1))$ determined by

$$z_{(m,i)} = \begin{cases} x_{(m,i)} & \text{for } (m, i) \in A_1 \times \mathbf{Z}(N_1), \\ y_{(m,i-N_1-N_2-1)} & \text{for } m \in A_2 \text{ and } i = N_1 + 1, N_1 + 2, \ldots, N_1 + 2N_2 + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Put $z' = S^{N_1+N_2+1}(z)$. Then, for $i \in \mathbf{Z}(N_2)$, we have

$$z'_{(m,i)} = z_{(m,i+N_1+N_2+1)} = y_{(m,i+N_1+N_2+1-N_1-N_2-1)} = y_{(m,i)}.$$

Namely, we have $z' \in S^{N_1+N_2+1}(U(x, A_1, Z(N_1))) \cap U(y, A_2, \mathbf{Z}(N_2))$. q.e.d.

Let $\varphi : \mathbf{Z} \to \mathbf{Z}$ be a bijective map and $S_\varphi : C \to C$ be the homeomorphism defined by $y = S_\varphi(x)$, where $x = (x_n)_{n\in\mathbf{Z}}$, $y = (y_n)_{n\in\mathbf{Z}}$ and $y_n = x_{\varphi(n)}$ for $n \in \mathbf{Z}$. For the map φ, we denote by $O(n)$ the orbit of a point $n \in \mathbf{Z}$, that is, $O(n) = \{\varphi^i(n)\}_{i\in\mathbf{Z}}$. Now we prove our result.

Proposition 2.2. *S_φ is chaotic if and only if $O(n)$ is an infinite set for all $n \in \mathbf{Z}$.*

Proof. First we suppose that there exists $m \in \mathbf{Z}$ such that $O(m)$ is a finite set. Namely $O(m) = \{\varphi^i(m)\}_{i=0}^{k-1}$ for some $k \in \mathbf{N}$. We put

$$U = \{x = (x_n)_{n\in\mathbf{Z}} : x_{\varphi^i(m)} = 0 \text{ for } i = 0, 1, \ldots, k-1\},$$

and

$$V = \{x = (x_n)_{n\in\mathbf{Z}} : x_{\varphi^i(m)} = 1 \text{ for } i = 0, 1, \ldots, k-1\}.$$

Then U and V are open sets in C_M with $S_\varphi^k(U) \cap V = U \cap V = \emptyset$ for all $k \in \mathbf{N}$. Thus S_φ is not topologically transitive.

Next, we suppose that $O(n)$ is an infinite set for all $n \in \mathbf{Z}$. Since the relation $m \sim n$ defined by $O(m) = O(n)$ is an equivalence relation, we can take a set M of all representative elements of $O(n)$'s. Namely we have

$$Z = \bigcup_{m \in M} O(m) \quad \text{and} \quad O(m) \cap O(n) = \emptyset \text{ for } m \neq n \text{ in } M,$$

and two topological spaces $C = \Pi_{n \in \mathbf{Z}} \{0, 1\}$ and $C_M = \Pi_{(m,i) \in M \times \mathbf{Z}} \{0, 1\}$ are identifies by the homeomorphism Φ defined by $y = \Phi(x)$, where $x = (x_n)_{n \in \mathbf{Z}} \in C$, $y = (y_{(m,i)})_{(m,i) \in M \times \mathbf{Z}} \in C_M$ and $y_{(m,i)} = x_{\varphi^i(m)}$ for each (m, i). Moreover we have the following commutative diagram:

$$\begin{array}{ccc} & S_\varphi & \\ C & \rightarrow & C \\ \Phi \downarrow & & \downarrow \Phi \\ C_M & \rightarrow & C_M \\ & T_\varphi & \end{array}$$

where the homeomorphism T_φ is the map defined by $z = T_\varphi(y)$, where $y = (y_{(m,i)})_{(m,i) \in M \times \mathbf{Z}}$, $z = (z_{(m,i)})_{(m,i) \in M \times \mathbf{Z}}$ and $z_{(m,i)} = y_{(m,i+1)}$ for each $(m, i) \in M \times \mathbf{Z}$. By virtue of Lemma 2.1, T_φ is chaotic and so is $S_\varphi = \Phi^{-1} \circ T_\varphi \circ \Phi$. q.e.d.

In the above proposition, if $\varphi(n) = n + 1$, then $O(n) = \mathbf{Z}$ for all $n \in \mathbf{Z}$ and S_φ is just the bilateral shift map.

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