A note on the restriction of Fourier multipliers from weighted L^p spaces to Lorentz spaces

勘　甚　裕　一
（金沢大学　理工研究域　機械工学系）
菅　野　絢　子
（東京都立調布北高等学校）
佐　藤　圓　治
（山形大学　理学部数理科学科）

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A note on the restriction of Fourier multipliers from weighted L^p spaces to Lorentz spaces

Yuichi KANJIN, Ayako KANNO and Enji SATO

Abstract

Let p be in $1 \le p < \infty$, $\phi(x)$ a bounded continuous function on $\mathbb{R}$, $T_\phi f(x) = \int_{\mathbb{R}} \phi(\xi)\hat{f}(\xi)e^{ix\xi}d\xi$, and $T_{\phi|\mathbb{Z}} F(x) = \sum_m \phi(m)\hat{F}(m)e^{imx}$. Anderson-Mohanty [1] showed that if T_ϕ is bounded on a weighted L^p space on $\mathbb{R}$ then $T_{\phi|\mathbb{Z}}$ is bounded on the corresponding weighted L^p space on $\mathbb{T}$, whose result is a generalization of Berkson-Gillespie [2]. In this paper, we generalize the result from weighted L^p spaces to Lorentz spaces with an alternative proof.

1. Introduction

Let X be the real line $\mathbb{R}$ or the one dimensional torus $\mathbb{T} = [-\pi, \pi)$ and $w(x)$ a nonnegative function on X. Also let $L^p_w(X)$ be the set of all measurable function f on X with $||f||_{L^p_w(X)} < \infty$, where

$$||f||_{L^p_w(X)} = \left(\frac{1}{2\pi}\int_X |f(x)|^p w(x)dx\right)^{1/p} \quad (1 \le p < \infty).$$

In particular, we denote $L^p(X) = L^p_w(X)$ and $||f||_{L^p(X)} = ||f||_{L^p_w(X)}$ when $w(x) = 1$ on X.

Definition 1. *Let $\phi(x)$ be a bounded continuous function on $\mathbb{R}$ and $\{\Psi(n)\}_{n\in\mathbb{Z}}$ a bounded sequence on the integer group $\mathbb{Z}$. Then, we define*

$$T_\phi f(x) = \int_{\mathbb{R}} \phi(\xi)\hat{f}(\xi)e^{ix\xi}d\xi \;\; (f \in C_c^\infty(\mathbb{R})),$$

and

$$T_\Psi F(x) = \sum_m \Psi(m)\hat{F}(m)e^{imx} \;\; (F \in P(\mathbb{T})),$$

where we denote the Fourier transform of f by $\hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{-i\xi x}dx$, by $C_c^\infty(\mathbb{R})$ the set of all infinitely differentiable functions on $\mathbb{R}$ with compact support, by $\hat{F}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{inx}dx$ the Fourier coefficient of F, and by $P(\mathbb{T})$ the set of all trigonometric polynomials on $\mathbb{T}$. A bounded continuous function ϕ is called an $L^p_w(\mathbb{R})$-multiplier, if there exists a constant C such that $||T_\phi f||_{L^p_w(\mathbb{R})} \le C||f||_{L^p_w(\mathbb{R})}$ $(f \in C_c^\infty(\mathbb{R}))$, and a bounded sequence Ψ is called an $L^p_w(\mathbb{T})$-multiplier, if there exists a constant C such that $||T_\Psi F||_{L^p_w(\mathbb{T})} \le C||F||_{L^p_w(\mathbb{T})}$ $(F \in P(\mathbb{T}))$.

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Here, we denote by $M_{p,w}(X)$ *the set of all* $L^p_w(X)$*-multipliers, by* $||T_\phi||_{M_{p,w}(X)}$ *the operator norm on* $L^p_w(X)$*, and* $||T_\phi||_{M_p(X)} = ||T_\phi||_{M_{p,w}(X)}$*, when* $w(x) = 1$ *on* X*.*

In 1965, de Leeuw [4] proved the follwing:

Theorem A. *If* ϕ *is an* $L^p(\mathbb{R})$*-multiplier for* $1 \leq p < \infty$*, then* $\phi|_{\mathbb{Z}}$ *is an* $L^p(\mathbb{T})$*-multiplier.*

In 2003, Berkson-Gillespie [2] obtained a generalization of de Leeuw's result under the A_p condition (cf. [8]).

We say that for $1 < p < \infty$ a nonnegative function $w(x)$ on $\mathbb{R}$ satisfies A_p condition, if there exists a constant C such that

$$\left(\frac{1}{|Q|}\int_Q w(t)dt\right)\left(\frac{1}{|Q|}\int_Q (w(t))^{\frac{1}{p-1}}dt\right)^{p-1} \leq C$$

for all bounded interval Q, when $|Q|$ is the length of Q. Then we denote $w \in A_p(\mathbb{R})$. Also we denote

$$A_p(\mathbb{T}) = \{w \in A_p(\mathbb{R}) \mid w \text{ is a } 2\pi \text{ periodic function on } \mathbb{R}\}.$$

Berkson-Gillespie's result is the following:

Theorem B. *Let* $1 < p < \infty$ *and* $U \in A_p(\mathbb{T})$*. Put* $u = U|_{\mathbb{T}}$*, the restriction of* U *on* $\mathbb{T}$*. If* ϕ *is an* $L^p_U(\mathbb{R})$*-multiplier, then* $\phi|_{\mathbb{Z}}$ *is an* $L^p_u(\mathbb{T})$*-multiplier with* $||T_{\phi|_{\mathbb{Z}}}||_{M_{p,u}(\mathbb{T})} \leq ||T_\phi||_{M_{p,U}(\mathbb{R})}$*.*

In 2009, Anderson-Mohanty [1] generalized Theorem B by the simple calculation. Their result is the following:

Theorem C. *Let* U *be a nonnegative* 2π *periodic measurable function on* $\mathbb{R}$*, and* $1 < p < \infty$*. Also we assume that* $u = U|_{\mathbb{T}} \in L^1(\mathbb{T})$*. Then we obtain that* $\phi|_{\mathbb{Z}}$ *is in* $M_{p,u}(\mathbb{T})$ *with* $||T_{\phi|_{\mathbb{Z}}}||_{M_{p,u}(\mathbb{T})} \leq ||T_\phi||_{M_{p,U}(\mathbb{R})}$*, if* ϕ *is in* $M_{p,U}(\mathbb{R})$*.*

In this paper, we shall generalize Theorem C to Lorentz spaces with an alternative proof which is different from Anderson-Mohanty [1]. First we introduce Lorentz spaces.

Definition 2. *Let* U *be a nonnegative* 2π *periodic function on* $\mathbb{R}$*,* $u = U|_{\mathbb{T}}$*,* $\mu(E) = \int_E U(x)dx$ *and* $\nu(E) = \int_E u(x)dx$ *for a measurable set* E*. We assume* $u \in L^1(\mathbb{T})$*. For* $1 \leq p < \infty$ *and* $1 \leq q \leq \infty$*, we define the Lorentz space* $L^{p,q}(\mu) = \{f \mid ||f||_{L^{p,q}(\mu)} < \infty\}$*, where*

$$||f||_{L^{p,q}(\mu)} = \begin{cases} \left(q\int_0^\infty \left(t\mu(\{x \in \mathbb{R} \mid |f(x)| > t\})^{\frac{1}{p}}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & (q < \infty) \\ \sup\limits_{t>0} t\mu(\{x \in \mathbb{R} \mid |f(x)| > t\})^{\frac{1}{p}} & (q = \infty) \end{cases}$$

and we define $L^{p,q}(\nu) = \{F \mid ||F||_{L^{p,q}(\nu)} < \infty\}$ *in the same way as* $L^{p,q}(\nu)$*, too. It is known that* $L^{p,q}(\mu) = L^p(\mu)$*, where* $L^p(\mu)$ *is the usual* L^p *space with respect to the measure* μ*, and* $L^{p,q}(\nu) = L^p(\nu)$ *for* $p = q$*. Also we define* $||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)} = \sup_{||f||_{L^p(\mu)}\leq 1} \| T_\phi f \|_{L^{p,q}(\mu)}$ *and* $\| T_{\phi|_{\mathbb{Z}}} \|_{L^p(\nu)\to L^{p,q}(\nu)}$*, too.*

Throughout this paper, the letters C, C_1, C_2 and C_3 will be used to denote positive constants not necessarily the same at each occurrence.

Our main theorem is the following:

Theorem 1. *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. We assume that*

$$||T_\phi f||_{L^{p,q}(\mu)} \leq ||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||f||_{L^p(\mu)} \quad (f \in C_c^\infty(\mathbb{R})).$$

Then, there exists a constant C such that

$$||T_{\phi|_\mathbb{Z}} F||_{L^{p,q}(\nu)} \leq C||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||F||_{L^p(\nu)} \ (F \in P(\mathbb{T})).$$

The following result is a special case of Theorem 1:

Collorary 1. *Let $1 \leq p < \infty$ and ϕ be a bounded continuous function $\mathbb{R}$. Then, if we assume that*

$$||T_\phi f||_{L^{p,\infty}(\mu)} \leq ||T_\phi||_{L^p(\mu)\to L^{p,\infty}(\mu)}||f||_{L^p(\mu)} \quad (f \in C_c^\infty(\mathbb{R})),$$

we obtain that

$$||T_{\phi|_\mathbb{Z}} F||_{L^{p,\infty}(\nu)} \leq C||T_\phi||_{L^p(\mu)\to L^{p,\infty}(\mu)}||F||_{L^p(\nu)} \quad (F \in P(\mathbb{T})).$$

Zafran [10](cf. [3, Remark 3]) showed that if $1 < p < 2$ then there exists a Fourier multiplier operator T from $L^p(\mathbb{R})$ to $L^{p,\infty}(\mathbb{R})$ such that T is not a Fourier multiplier operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$. By this fact, we remark that Corollary 1 is not contained in Anderson-Mohanty [1].

2. The proof of Theorem 1

First we will prove a lemma.

Lemma 1. *When we define $\omega_\delta(x) = e^{-\frac{\delta}{4\pi}x^2}$ $(\delta > 0)$, we have that*

(i) $\lim_{\delta\to 0} \delta^{\frac{1}{2}} \int_\mathbb{R} f(x)\omega_\delta(x)dx = \int_0^{2\pi} f(x)dx \qquad (f \in L^1(\mathbb{T}))$,

(ii) $\lim_{\delta\to 0}(\sqrt{\delta})^{\frac{1}{p}}||\omega_\delta F||_{L^p(\mu)} = \left(\frac{1}{\sqrt{p}}\right)^{\frac{1}{p}}||F||_{L^p(\nu)}$.

Proof. (i) It is easy to prove, but let us give the proof for readers convenience (cf. [1], [9](p.261)).
Since $\delta^{\frac{1}{2}} \int_\mathbb{R} \omega_\delta(x)e^{imx}dx = 2\pi e^{-\frac{\pi}{\delta}m^2}$ $(m \in \mathbb{Z})$, we have

$$\lim_{\delta\to 0} \delta^{\frac{1}{2}} \int_\mathbb{R} P(x)\omega_\delta(x)dx = \int_0^{2\pi} P(x)dx \qquad (P \in P(\mathbb{T})).$$

Also since for $f \in L^1(\mathbb{T})$ and $P \in P(\mathbb{T})$ we have that

$$\begin{aligned}
&\left|\delta^{\frac{1}{2}} \int_{\mathbb{R}} (f(x) - P(x))\omega_\delta(x)dx\right| \\
&\leq \sum_{j\in\mathbb{Z}} \delta^{\frac{1}{2}} \int_{2\pi j}^{2\pi(j+1)} |f(x) - P(x)|\omega_\delta(x)dx \\
&= \sum_{j\in\mathbb{Z}} \delta^{\frac{1}{2}} \int_0^{2\pi} |f(x) - P(x)|\omega_\delta(x + 2\pi j)dx \\
&\leq \delta^{\frac{1}{2}} \int_0^{2\pi} |f(x) - P(x)| \sum_{j\in\mathbb{Z}} \omega_\delta(x + 2\pi j)dx \\
&\leq \delta^{\frac{1}{2}} \int_0^{2\pi} |f(x) - P(x)| \left(2\int_0^\infty \omega_\delta(t)dt\right) dx \\
&\leq 2\pi \int_0^{2\pi} |f(x) - P(x)|dx \\
&= 4\pi^2 ||f - P||_{L^1(\mathbb{T})}.
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
&\left|\delta^{\frac{1}{2}} \int_{\mathbb{R}} f(x)\omega_\delta(x)dx - \int_0^{2\pi} f(x)dx\right| \\
&\leq \left|\delta^{\frac{1}{2}} \int_{\mathbb{R}} (f(x) - P(x))\omega_\delta(x)dx\right| \\
&\quad + \left|\delta^{\frac{1}{2}} \int_{\mathbb{R}} P(x)\omega_\delta(x)dx - \int_0^{2\pi} P(x)dx\right| + \int_0^{2\pi} |f(x) - P(x)|dx.
\end{aligned}$$

By the above facts we get the desired result, since we have that $P(\mathbb{T})$ is dense in $L^1(\mathbb{T})$.
(ii) By $|F(x)|^p U(x) \in L^1(\mathbb{T})$ and (i), we have

$$\begin{aligned}
\delta^{\frac{1}{2}} ||\omega_\delta F||^p_{L^p(\mu)} &= \delta^{\frac{1}{2}} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{p\delta}{4\pi}x^2} |F(x)|^p U(x)dx \\
&\to \frac{1}{2\pi\sqrt{p}} \int_0^{2\pi} |F(x)|^p u(x)dx \quad (\delta \to 0) \\
&= \frac{1}{\sqrt{p}} \parallel F \parallel^p_{L^p(\nu)},
\end{aligned}$$

and $\lim_{\delta\to 0}(\sqrt{\delta})^{\frac{1}{p}} ||\omega_\delta F||_{L^p(\mu)} = \left(\frac{1}{\sqrt{p}}\right)^{\frac{1}{p}} ||F||_{L^p(\nu)}$. □

Proof of Theorem 1. According to Kaneko-Sato [6], we proceed the proof. First we define that

$$F(x) = \sum_{m=-\infty}^{\infty} \hat{F}(m)e^{imx} \quad (F \in P(\mathbb{T})),$$

$$\gamma_\delta(x) = \omega_\delta(x) T_{\phi|_{\mathbb{Z}}} F(x) - T_\phi(\omega_\delta F)(x),$$

and

$$||\gamma_\delta||_\infty = \sup\{t| \ |\{x \in \mathbb{R} \mid |\gamma_\delta(x)| > t\}| > 0\},$$

where $|E|$ is the Lebesgue measure of a measurable set E.

CASE 1. We show the proof in the case $1 \le q < \infty$.
By $\gamma_\delta(x) = \int_{\mathbb{R}} \hat{\gamma_\delta}(\xi) e^{ix\xi} d\xi$, we have that $|\gamma_\delta(x)| \le 2\pi \parallel \hat{\gamma_\delta} \parallel_{L^1(\mathbb{R})}$, and

$$\begin{aligned}\gamma_\delta(x) &= \omega_\delta(x) T_{\phi|_{\mathbb{Z}}} F(x) - T_\phi(\omega_\delta F)(x) \\ &= \omega_\delta(x) \sum_m \phi(m) \hat{F}(m) e^{imx} - T_\phi(\omega_\delta F)(x).\end{aligned}$$

Then, we obtain that

$$\begin{aligned}\hat{\gamma_\delta}(\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} \gamma_\delta(x) e^{-i\xi x} dx \\ &= \sum_m \phi(m)\hat{F}(m)\hat{\omega_\delta}(\xi - m) - \phi(\xi)\widehat{\omega_\delta F}(\xi),\end{aligned}$$

and

$$\widehat{\omega_\delta F}(\xi) = \sum_m \hat{F}(m)\hat{\omega_\delta}(\xi - m). \tag{1}$$

Hence, by (1) we get that

$$\hat{\gamma_\delta}(\xi) = \sum_m \hat{F}(m)\hat{\omega_\delta}(\xi - m)\big(\phi(m) - \phi(\xi)\big),$$

and

$$\begin{aligned}||\hat{\gamma_\delta}||_{L^1(\mathbb{R})} &= \frac{1}{2\pi}\int_{\mathbb{R}} \Big|\sum_m \hat{F}(m)\hat{\omega_\delta}(\xi - m)\big(\phi(m) - \phi(\xi)\big)\Big| d\xi \\ &\le \sum_m |\hat{F}(m)| \cdot \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\omega_\delta}(\xi - m)|\phi(m) - \phi(\xi)| d\xi.\end{aligned} \tag{2}$$

On the other hand, by $w_\delta(x) T_{\phi|_{\mathbb{Z}}} F(x) = \gamma_\delta(x) + T_\phi(w_\delta F)(x)$, we have that

$$\begin{aligned}|\omega_\delta(x) T_{\phi|_{\mathbb{Z}}} F(x)| &\le \parallel \gamma_\delta \parallel_\infty + |T_\phi(\omega_\delta F)(x)| \\ &\le 2\pi \parallel \hat{\gamma_\delta} \parallel_{L^1(\mathbb{R})} + |T_\phi(\omega_\delta F)(x)|,\end{aligned}$$

and

$$\{x \in \mathbb{R} \mid |\omega_\delta(x) T_{\phi|_{\mathbb{Z}}} F(x)| > t\} \subset \{x \in \mathbb{R} \mid |T_\phi(\omega_\delta F)(x)| > t - 2\pi \parallel \hat{\gamma_\delta} \parallel_{L^1(\mathbb{R})}\}. \tag{3}$$

For $a > 2 \cdot 2\pi \parallel \hat{\gamma_\delta} \parallel_{L^1(\mathbb{R})}$, we have

$$\begin{aligned}&\Big(\int_a^\infty \big(t\mu(\{x \in \mathbb{R} \mid |\omega_\delta T_{\phi|_{\mathbb{Z}}} F(x)| > t\})^{\frac{1}{p}}\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\ \le\ &\Big(\int_a^\infty \big(t\mu(\{x \in \mathbb{R} \mid |T_\phi(\omega_\delta F)(x)| > t - 2\pi \parallel \hat{\gamma_\delta} \parallel_{L^1(\mathbb{R})}\})^{\frac{1}{p}}\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\ \le\ &\Big(\int_{a-2\pi||\hat{\gamma_\delta}||_{L^1(\mathbb{R})}}^\infty \big((t + 2\pi||\hat{\gamma_\delta}||_{L^1(\mathbb{R})})\mu(\{x \in \mathbb{R} \mid |T_\phi(\omega_\delta F)(x)| > t\})^{\frac{1}{p}}\big)^q \frac{dt}{t + 2\pi||\hat{\gamma_\delta}||_{L^1(\mathbb{R})}}\Big)^{\frac{1}{q}} \\ \le\ &\Big(\int_{a-2\pi||\hat{\gamma_\delta}||_{L^1(\mathbb{R})}}^\infty \Big(\Big(\frac{t + 2\pi||\hat{\gamma_\delta}||_{L^1(\mathbb{R})}}{t}\Big) t\big(\mu(\{x \in \mathbb{R} \mid |T_\phi(\omega_\delta F)(x)| > t\}\big)^{\frac{1}{p}}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}.\end{aligned}$$

Therefore, by $\frac{t+2\pi||\widehat{\gamma_\delta}||_{L^1(\mathbb{R})}}{t} \leq 1+\frac{2\pi||\widehat{\gamma_\delta}||_{L^1(\mathbb{R})}}{a-2\pi||\widehat{\gamma_\delta}||_{L^1(\mathbb{R})}} \leq 2$ for $t \geq a-2\pi||\widehat{\gamma_\delta}||_{L^1(\mathbb{R})}$, we get that

$$\begin{aligned}
&\Big(\int_a^\infty \big(t\mu(\{x\in\mathbb{R} \mid |\omega_\delta T_{\phi|_\mathbb{Z}}F(x)|>t\})^{\frac{1}{p}}\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
\leq\ & 2\Big(\int_0^\infty \big(t\mu(\{x\in\mathbb{R} \mid |T_\phi(\omega_\delta F)(x)|>t\})^{\frac{1}{p}}\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
=\ & 2||T_\phi(\omega_\delta F)||_{L^{p,q}(\mu)}.
\end{aligned}$$

Also by the assumption of T_ϕ, we obtain that

$$||T_\phi(\omega_\delta F)||_{L^{p,q}(\mu)} \leq ||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||\omega_\delta F||_{L^p(\mu)}, \tag{4}$$

and for $a > 2\cdot 2\pi||\widehat{\gamma_\delta}||_{L^1(\mathbb{R})}$, we get that

$$\begin{aligned}
&\Big(\int_a^\infty \big(t\mu(\{x\in\mathbb{R} \mid |\omega_\delta T_{\phi|_\mathbb{Z}}F(x)|>t\})^{\frac{1}{p}}\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\leq ||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||\omega_\delta F||_{L^p(\mu)}.
\end{aligned}\tag{5}$$

Here, we show $\lim_{\delta\to 0}||\widehat{\gamma_\delta}||_{L^1(\mathbb{R})}=0$. In fact, by $\omega_\delta(x)=e^{-\frac{\delta}{4\pi}x^2}$, we have that $\widehat{\omega_\delta}(\xi)=\delta^{-\frac{1}{2}}\omega_{4\pi^2/\delta}(\xi)$, and by (2)

$$\begin{aligned}
||\widehat{\gamma_\delta}||_{L^1(\mathbb{R})} &\leq \sum_m |\hat{F}(m)|\cdot\frac{1}{2\pi}\int_\mathbb{R}\widehat{\omega_\delta}(\xi-m)|\phi(m)-\phi(\xi)|d\xi \\
&= \sum_m |\hat{F}(m)|\cdot\frac{1}{2\pi}\int_\mathbb{R}\delta^{-\frac{1}{2}}e^{-\frac{\pi}{\delta}(\xi-m)^2}|\phi(m)-\phi(\xi)|d\xi.
\end{aligned}$$

Let $m\in\mathbb{Z}$ be fixed. For $\varepsilon>0$, there exists $\eta_0>0$ such that $|\phi(m)-\phi(\xi)|<\varepsilon$ for $|\xi-m|<\eta_0$. Then, we estimate that

$$\begin{aligned}
&\frac{1}{2\pi}\int_\mathbb{R}\delta^{-\frac{1}{2}}e^{-\frac{\pi}{\delta}(\xi-m)^2}|\phi(m)-\phi(\xi)|d\xi \\
=\ &\frac{1}{2\pi}\int_{|m-\xi|<\eta_0}\delta^{-\frac{1}{2}}e^{-\frac{\pi}{\delta}(\xi-m)^2}|\phi(m)-\phi(\xi)|d\xi \\
&+\frac{1}{2\pi}\int_{|m-\xi|>\eta_0}\delta^{-\frac{1}{2}}e^{-\frac{\pi}{\delta}(\xi-m)^2}|\phi(m)-\phi(\xi)|d\xi \\
=\ &(\alpha)+(\beta),\ \text{say.}
\end{aligned}$$

In (α), we have

$$(\alpha)<\varepsilon\cdot\frac{1}{2\pi}\int_\mathbb{R}\delta^{-\frac{1}{2}}e^{-\frac{\pi}{\delta}(\xi-m)^2}d\xi \leq \frac{\varepsilon}{2\pi}\cdot\frac{1}{\sqrt{\pi}}\int_\mathbb{R}e^{-t^2}dt=\frac{\varepsilon}{2\pi}. \tag{6}$$

In (β), since

$$|\phi(m)-\phi(\xi)|\leq|\phi(m)|+|\phi(\xi)|\leq 2\parallel\phi\parallel_\infty,$$

we obtain that

$$\begin{aligned}
(\beta) &\leq \frac{2\parallel\phi\parallel_\infty}{2\pi}\int_{|m-\xi|\geq\eta_0}\delta^{-\frac{1}{2}}e^{-\frac{\pi}{\delta}(\xi-m)^2}d\xi \\
&= \frac{||\phi||_\infty}{\pi\sqrt{\pi}}\int_{\sqrt{\frac{\pi}{\delta}}\eta_0}^\infty e^{-t^2}dt \to 0 \quad (\delta\to 0).
\end{aligned}\tag{7}$$

Therefore, for $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $(\alpha) + (\beta) < 2\varepsilon$ for $0 < \delta < \delta_0$, and we obtain $\lim_{\delta\to 0} ||\hat{\gamma}_\delta||_{L^1(\mathbb{R})} = 0$.

Also we show

$$\liminf_{\delta\to 0} \sqrt{\delta}\mu(\{x \in \mathbb{R} \mid |\omega_\delta(x) T_{\phi|_{\mathbb{Z}}} F(x)| > t\}) \geq C_1 \nu(\{x \in \mathbb{T} \mid |T_{\phi|_{\mathbb{Z}}} F(x)| > t\}).$$

Putting $G(x) = T_{\phi|_{\mathbb{Z}}} F(x)$, we have $\omega_\delta T_{\phi|_{\mathbb{Z}}} F(x) = \omega_\delta G(x)$. Since

$$\begin{aligned} & \mu(\{x \in \mathbb{R} \mid |\omega_\delta(x) G(x)| > t\}) \\ = & \sum_{j=-\infty}^{\infty} \mu(\{x \in [2\pi j,\ 2\pi(j+1)) \mid e^{-\frac{\delta}{4\pi}x^2}|G(x)| > t\}) \\ = & \sum_{j=-\infty}^{\infty} \mu(\{u \in [0,\ 2\pi) \mid e^{-\frac{\delta}{4\pi}(u+2\pi j)^2}|G(u)| > t\}) \\ \geq & \sum_{j=0}^{\infty} \mu(\{u \in [0,\ 2\pi) \mid e^{-\frac{\delta}{4\pi}(u+2\pi j)^2}|G(u)| > t\}) \end{aligned}$$

and for $s \in [0, 2\pi)$ and $u \in [0, 2\pi)$ we have that $s + 2\pi(j+1) \geq u + 2\pi j$ and $e^{-\frac{\delta}{4\pi}(s+2\pi(j+1))^2} \leq e^{-\frac{\delta}{4\pi}(u+2\pi j)^2}$, we obtain that

$$\begin{aligned} & \sum_{j=0}^{\infty} \mu(\{u \in [0, 2\pi) \mid e^{-\frac{\delta}{4\pi}(u+2\pi j)^2}|G(u)| > t\}) \\ \geq & \sum_{j=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \mu(\{u \in [0, 2\pi) \mid e^{-\frac{\delta}{4\pi}(s+2\pi(j+1))^2}|G(u)| > t\}) ds \\ = & \frac{1}{2\pi} \int_{2\pi}^{\infty} \mu(\{u \in [0, 2\pi) \mid e^{-\frac{\delta}{4\pi}s^2}|G(u)| > t\}) ds \\ = & \frac{1}{\sqrt{\pi\delta}} \int_{\sqrt{\pi\delta}}^{\infty} \mu(\{u \in [0, 2\pi) \mid e^{-x^2}|G(u)| > t\}) dx. \end{aligned}$$

Then, we have that for $0 < \delta < \frac{1}{2\pi}$

$$\begin{aligned} & \sum_{j=0}^{\infty} \mu(\{u \in [0, 2\pi) \mid e^{-\frac{\delta}{4\pi}(u+2\pi j)^2}|G(u)| > t\}) \\ \geq & \frac{1}{\sqrt{\pi\delta}} \int_{\sqrt{\pi\delta}}^{1} \mu(\{u \in [0, 2\pi) \mid |G(u)| > te\}) dx \\ \geq & \frac{C_2}{\sqrt{\pi\delta}} \nu(\{u \in [0, 2\pi) \mid |G(u)| > te\}), \end{aligned}$$

and we get that

$$\begin{aligned} & \sqrt{\pi\delta} \sum_{j=0}^{\infty} \mu(\{u \in [0, 2\pi) \mid e^{-\frac{\delta}{4\pi}(u+2\pi j)^2}|G(u)| > t\}) \\ \geq & C_2 \nu(\{u \in [0, 2\pi) \mid |G(u)| > te\}) \end{aligned}$$

for $0 < \delta < \frac{1}{2\pi}$. Hence, we get that for $0 < \delta < \frac{1}{2\pi}$,

$$\begin{aligned} & \delta^{\frac{1}{2}} \mu(\{x \in \mathbb{R} \mid |\omega_\delta T_{\phi|_{\mathbb{Z}}} F(x)| > t\}) \\ & \geq C_2 \nu(\{x \in [0, 2\pi) \mid |T_{\phi|_{\mathbb{Z}}} F(x)| > te\}) \end{aligned} \tag{8}$$

and we obtain that

$$\begin{aligned}(9)\qquad &\liminf_{\delta\to 0}\sqrt{\delta}\mu(\{x\in\mathbb{R}\mid |\omega_\delta(x)T_{\phi|_{\mathbb{Z}}}F(x)|>t\})\\ &\qquad \geq C_2\nu(\{x\in\mathbb{T}\mid |T_{\phi|_{\mathbb{Z}}}F(x)|>te\}).\end{aligned}$$

Now by (5), (9) and Fatou's Lemma, we have that

$$\begin{aligned}(10)\qquad &\Big(\int_a^\infty\big(t\mu(\{x\in\mathbb{R}\mid |\omega_\delta(x)T_{\phi|_{\mathbb{Z}}}F(x)|>t\})^{\frac{1}{p}}\big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}(\delta^{\frac{1}{2}})^{\frac{1}{p}}\\ &\leq C(\delta^{\frac{1}{2}})^{\frac{1}{p}}||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||\omega_\delta F||_{L^p(\mu)}\end{aligned}$$

and

$$\begin{aligned}&\liminf_{\delta\to 0}\Big(\int_a^\infty\big(t\mu(\{x\in\mathbb{R}\mid |\omega_\delta(x)T_{\phi|_{\mathbb{Z}}}F(x)|>t\})^{\frac{1}{p}}\big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}(\delta^{\frac{1}{2}})^{\frac{1}{p}}\\ \geq\ &\Big(\int_a^\infty\liminf_{\delta\to 0}\Big(t\big(\delta^{\frac{1}{2}}\mu(\{x\in\mathbb{R}\mid |\omega_\delta(x)T_{\phi|_{\mathbb{Z}}}F(x)|>t\})\big)^{\frac{1}{p}}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ \geq\ &\Big(\int_{ae}^\infty\Big(C_3t\nu(\{x\in[0,2\pi)\mid |T_{\phi|_{\mathbb{Z}}}F(x)|>t\})^{\frac{1}{p}}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}},\end{aligned}$$

and by Lemma 1(ii) we have that

$$(11)\qquad (\delta^{\frac{1}{2}})^{\frac{1}{p}}||\omega_\delta F||_{L^p(\mu)}\to\Big(\frac{1}{\sqrt{p}}\Big)^{\frac{1}{p}}||F||_{L^p(\nu)}\quad(\delta\to 0).$$

After all, when $\delta\to 0$, by (10) and (11) we obtain that

$$\begin{aligned}(12)\qquad &\Big(\int_{ae}^\infty\Big(t\nu(\{x\in[0,2\pi)\mid |T_{\phi|_{\mathbb{Z}}}F(x)|>t\})^{\frac{1}{p}}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq C||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||F||_{L^p(\nu)}\end{aligned}$$

and for $a\downarrow 0$ in (12),

$$\begin{aligned}||T_{\phi|_{\mathbb{Z}}}F||_{L^{p,q}(\nu)}\ &=\ \Big(\int_0^\infty\Big(t\nu(\{x\in[0,2\pi)\mid |T_{\phi|_{\mathbb{Z}}}F(x)|>t\})^{\frac{1}{p}}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq\ C||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||F||_{L^p(\nu)}.\end{aligned}$$

CASE 2. We prove the case $q=\infty$.
We can show it in the same way as the case 1. In fact, for $a>2\cdot 2\pi\parallel\hat{\gamma_\delta}\parallel_{L^1(\mathbb{T})}$, we have that

$$\begin{aligned}&\sup_{t>a}t\mu(\{x\in\mathbb{R}\mid |\omega_\delta(x)T_{\phi|_{\mathbb{Z}}}F(x)|>t\})^{\frac{1}{p}}\\ \leq\ &\sup_{u>a-2\pi\|\hat{\gamma_\delta}\|_{L^1(\mathbb{R})}}\Big(\frac{u+2\pi\parallel\hat{\gamma_\delta}\parallel_{L^1(\mathbb{R})}}{u}\Big)u\mu(\{x\in\mathbb{R}\mid |T_\phi(\omega_\delta F)(x)|>u\})^{\frac{1}{p}}\\ \leq\ &2\sup_{t>0}t\mu(\{x\in\mathbb{R}\mid |T_\phi(\omega_\delta F)(x)|>t\})^{\frac{1}{p}}=2\parallel T_\phi(\omega_\delta F)\parallel_{L^{p,\infty}(\mu)}\\ \leq\ &C\parallel T_\phi\parallel_{L^p(\mu)\to L^{p,q}(\mu)}\parallel\omega_\delta F\parallel_{L^p(\mu)},\end{aligned}$$

and we obtain that

$$\begin{aligned}&\sup_{t>a}t\mu(\{x\in\mathbb{R}\mid |\omega_\delta(x)T_{\phi|_{\mathbb{Z}}}F(x)|>t\})^{\frac{1}{p}}\\ &\leq C\parallel T_\phi\parallel_{L^p(\mu)\to L^{p,q}(\mu)}\parallel\omega_\delta F\parallel_{L^p(\mu)}.\end{aligned}$$

Also since by (9)

$$\delta^{\frac{1}{2}}\mu(\{x \in \mathbb{R} \mid |\omega_\delta(x)T_{\phi|_\mathbb{Z}}F(x)| > t\}) \geq C_2\nu(\{x \in [0,2\pi) \mid |T_{\phi|_\mathbb{Z}}F(x)| > te\})$$

for sufficiently small $\delta > 0$, we have for $t > a$,

$$\begin{aligned} & C_1 t\nu(\{x \in [0,2\pi) \mid |T_{\phi|_\mathbb{Z}}F(x)| > te\}) \\ \leq\ & \sup_{t>a}(\delta^{\frac{1}{2}})^{\frac{1}{p}} t\mu(\{x \in \mathbb{R} \mid |\omega_\delta(x)T_{\phi|_\mathbb{Z}}F(x)| > t\})^{\frac{1}{p}} \\ \leq\ & C_2(\delta^{\frac{1}{2}})^{\frac{1}{p}} \parallel T_\phi \parallel_{L^p(\mu)\to L^{p,q}(\mu)} \parallel \omega_\delta F \parallel_{L^p(\mu)}, \end{aligned}$$

for some constants C_1 and C_2. By Lemma 1(ii) we have that

$$C_1 t\nu(\{x \in [0,2\pi) \mid |T_{\phi|_\mathbb{Z}}F(x)| > te\}) \leq C_2 \parallel T_\phi \parallel_{L^p(\mu)\to L^{p,q}(\mu)} \parallel F \parallel_{L^p(\mu)}$$

for $t > a$. Hence, when $a \downarrow 0$, we obtain that

$$t\nu(\{x \in [0,2\pi) \mid |T_{\phi|_\mathbb{Z}}F(x)| > t\}) \leq C_3 \parallel T_\phi \parallel_{L^p(\mu)\to L^{p,q}(\mu)} \parallel F \parallel_{L^p(\nu)}$$

for $t > 0$ and

$$||T_{\phi|_\mathbb{Z}}F||_{L^{p,\infty}(\nu)} \leq C_3||T_\phi||_{L^p(\mu)\to L^{p,q}(\mu)}||F||_{L^p(\nu)}.$$

Therefore, we get the desired result. □

3. The converse of Theorem 1

We shall consider the converse of Theorem 1 by the method of Igari [5].

Definition 3. *For $\varepsilon > 0$, let $\phi(x)$ be a bounded continuous function on $\mathbb{R}$, and*

$$\tilde{T}_\varepsilon F(x) = \sum_m \phi(\varepsilon m)\hat{F}(m)e^{imx} \qquad (F \in L^2(\mathbb{T})).$$

Also let $U(x)$ be a nonnegative function on $\mathbb{R}$ with homogeneous of degree $\gamma \in \mathbb{R}$ i.e. $U(\varepsilon x) = \varepsilon^\gamma U(x)$ ($\varepsilon > 0$), and $u(x)$ the 2π periodic function on $\mathbb{R}$ such that $u(x)$ is the restriction of $U(x)$ on $[-\pi,\pi)$. For example, we give $U(x) = |x|^\gamma$ ($\gamma \in \mathbb{R}$).

Then we obtain the result which is the converse result of Theorem 1.

Theorem 2. *Let $1 \leq p < \infty$. Under the above notation, we assume that there exists a constant C independent of $\varepsilon > 0$ such that $||\tilde{T}_\varepsilon F||_{L^p_u(\mathbb{T})} \leq C||F||_{L^p_u(\mathbb{T})}$ for all $F \in C^\infty(\mathbb{T})$, where $C^\infty(\mathbb{T})$ is the set of all infinitely differentiable functions on $\mathbb{T}$. Then we obtain that*

$$||T_\phi f||_{L^p_U(\mathbb{R})} \leq C||f||_{L^p_U(\mathbb{R})} \qquad (f \in C_c^\infty(\mathbb{R})).$$

Proof. For $f \in C_c^\infty(\mathbb{R})$, we define $f_\varepsilon(x) = f(\frac{x}{\varepsilon})$. Since supp $f_\varepsilon \subset (-\pi,\pi)$ for sufficiently small $\varepsilon > 0$, we may assume $f_\varepsilon \in C^\infty(\mathbb{T})$. Then, we have

$$\tilde{T}_\varepsilon f_\varepsilon(x) = \sum_n \phi(\varepsilon n)\hat{f}_\varepsilon(n)e^{inx}.$$

On the other hand, we get that $\hat{f}_\varepsilon(n) = \varepsilon\hat{f}(\varepsilon n)$, and $\tilde{T}_\varepsilon f_\varepsilon(x) = \sum_n \phi(\varepsilon n)\varepsilon\hat{f}(\varepsilon n)e^{inx}$. Since by the assumption $\parallel \tilde{T}_\varepsilon f_\varepsilon \parallel_{L^p_u(\mathbb{T})} \leq C \parallel f_\varepsilon \parallel_{L^p_u(\mathbb{T})}$, we have that $||f_\varepsilon||^p_{L^p_u(\mathbb{T})} =$

$\varepsilon^{\gamma+1}||f||^p_{L^p_U(\mathbb{R})}$. On the other hand, we calculate

$$\begin{aligned} ||\tilde{T}_\varepsilon f_\varepsilon||^p_{L^p_u(\mathbb{T})} &= \left|\left|\sum_n \phi(\varepsilon n)\varepsilon\hat{f}(\varepsilon n)e^{inx}\right|\right|^p_{L^p_u(\mathbb{T})} \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}}\left|\sum_n \phi(\varepsilon n)\hat{f}(\varepsilon n)e^{in\varepsilon t}\varepsilon\right|^p U(t)dt\cdot\varepsilon^{\gamma+1}, \end{aligned}$$

and we have that

$$\frac{1}{2\pi}\int_{\mathbb{R}}\chi_{(-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon})}(t)\left|\sum_n \phi(\varepsilon n)\hat{f}(\varepsilon n)e^{in\varepsilon t}\varepsilon\right|^p U(t)dt\cdot\varepsilon^{\gamma+1} \le C\varepsilon^{\gamma+1}||f||^p_{L^p_U(\mathbb{R})}.$$

Moreover, by the definition of the Riemann integral, we have that

$$\lim_{\varepsilon\to 0}\chi_{(-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon})}(t)\sum_n \phi(\varepsilon n)\hat{f}(\varepsilon n)e^{in\varepsilon t}\varepsilon = T_\phi f(t),$$

and

$$\begin{aligned} ||T_\phi f||^p_{L^p_U(\mathbb{R})} &= \frac{1}{2\pi}\int_{\mathbb{R}}\liminf_{\varepsilon\to 0}\chi_{(-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon})}(t)\left|\sum_n \phi(\varepsilon n)\hat{f}(\varepsilon n)e^{in\varepsilon t}\varepsilon\right|^p U(t)dt \\ &\le \liminf_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\mathbb{R}}\chi_{(-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon})}(t)\left|\sum_n \phi(\varepsilon n)\hat{f}(\varepsilon n)e^{in\varepsilon t}\varepsilon\right|^p U(t)dt \\ &\le C||f||^p_{L^p_U(\mathbb{R})}. \end{aligned}$$

Therefore, we obtain that

$$||T_\phi f||_{L^p_U(\mathbb{R})} \le C||f||_{L^p_U(\mathbb{R})} \quad (f\in C_c^\infty(\mathbb{R})). \qquad \square$$

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Mathematics Section, Division of Innovative Technology and Science, Graduate School of Natural Science and Technology, Kanazawa University Kanazawa 920-1192, Japan
E-mail address: `kanjin@staff.kanazawa-u.ac.jp`

Chofu Kita High School Chofu,182-0011,Japan

Department of Mathematical Sciences, Faculty of Science, Yamagata University, Yamagata, 990-8560, Japan
E-mail address: `esato@sci.kj.yamagata-u.ac.jp`