Critical Case in the Global Approximation Theorem

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Abstract

We give a unified proof of the global approximation theorem in the critical case $\alpha=2$, which have been treated individually by several authors. Our proof applies to all exponential-type operators with $\phi(x)$ of degree not exceeding 2.

1. Notations and Theorems

Let $f(x)$ be a continuous real-valued function on the interval $D:=[A,B]\cap(-\infty,\infty)$, in symbols: $f\in C[A,B]$. We use the following notations:

$$\Delta_h^2 f(x)=f(x+h)-2f(x)+f(x-h) \qquad (x\in D_h),$$

$$\omega_2(f;\delta)=\sup_{0<h\leqq\delta}\sup_{x\in D_h}|\Delta_h^2 f(x)|,$$

$$\mathrm{Lip}_2\alpha=\{f\in C[A,B];\ \omega_2(f;\delta)=O(\delta^\alpha),\ \delta\to 0_+\},$$

where $D_h:=[A+h,B-h]\cap(-\infty,\infty)$.

C.P.May [7] introduced the exponential-type operators

$$L_n(f;x)=\int_A^B W(n,x,u)f(u)\,du \qquad (n\geqq 1),$$

where $W(n,x,u)\geqq 0$ is a function on D such that

$$\int_A^B W(n,x,u)\,du=1 \tag{1}$$

and

$$\frac{\partial}{\partial x}W(n,x,u)=\frac{n}{\phi(x)}W(n,x,u)(u-x) \tag{2}$$

are satisfied with a polynomial $\phi(x)$ of degree $\leqq 2$, $\phi(x)>0$ on (A,B) and $\phi(A)=0$, $\phi(B)=0$ if $A,B\neq\pm\infty$. In view of (1) and (2), $L_n(f;x)$ preserves linear functions, and we have by simple calculations (cf. [8])

$$L_n((t-x)^2;x)=\frac{\phi(x)}{n}. \tag{3}$$

In [8] we have proved the following theorem for those $W(n,x,u)$'s which satisfy some additional conditions. Let us call them tame exponential-type operators for the sake of brevity.

Theorem 1. For tame $L_n(f;x)$ and for $0<\alpha<2$ the following statements are equivalent:

$(\mathrm{I})_\alpha$ $\qquad f\in\mathrm{Lip}_2\alpha$,

$(\mathrm{II})_\alpha$ $\qquad |L_n(f;x)-f(x)|\leqq M\left[\frac{\phi(x)}{n}\right]^{\alpha/2}$ $\quad(n\geqq 1,\ x\in D)$.

In this paper we consider the critical case $\alpha=2$ for $f\in C[A,B]$ with $f(x)=O(e^{N|x|})$ for some $N>0$.

Theorem 2. The following statements are equivalent:

$$(\mathrm{I})_2 \qquad f \in \mathrm{Lip}_2\, 2,$$

$$(\mathrm{II})_2 \qquad |L_n(f;x) - f(x)| \leqq M\left[\frac{\phi(x)}{n}\right] \quad (n \geqq 1,\ x \in D),$$

Though we needed some conditions upon $W(n,x,u)$ for Theorem 1 and had to exclude the Post–Widder operators corresponding to $\phi(x) = x^2$, we do not need special conditions and can include the Post–Widder operators in Theorem 2. The proof of the direct part $(\mathrm{I})_2 \Rightarrow (\mathrm{II})_2$ is a standard procedure and is carried out in the same way as Theorem 1, so we refer the reader to the proof of the direct part in [8]. For the proof of the inverse part $(\mathrm{II})_2 \Rightarrow (\mathrm{I})_2$ we use two lemmas and an idea of Grundmann. The author learned from M.Becker a particular (Bernstein) case of Lemma 2 as well as the reference [6].

2. Proof of Theorem 2, the Inverse Part

We use the following two lemmas.

Lemma 1 (Voronovskaja–type relation). If $f(x) = O(e^{N|x|})$ for some $N > 0$ and $f \in C^2[A_0,B_0]$ $(A < A_0 < B_0 < B)$, then

$$\lim_{n\to\infty} n[L_n(f;x) - f(x)] = \frac{1}{2}\phi(x) f''(x) \quad (x \in [A_0,B_0]). \tag{4}$$

Proof. We sketch only the outline of the lengthy proof. Firstly, from Corollary 2.7. in M.Ismail and C.P.May [5] we have

$$\int_{|u-x|\geqq\delta} W(n,x,u)e^{N|u|}du = O(n^{-k}) \quad (\text{as } n\to\infty). \tag{5}$$

$$(\delta, k > 0,\ x \in [A_0, B_0])$$

Now we apply Taylor's formula to $L_n(f;x) - f(x)$, to obtain the equality (4) from (1), (2), (3) and (5). A concrete calculation is left to the reader.

Lemma 2. In order that $f(x)$ is convex on D, it is necessary and sufficient that

$$L_n(f;x) \geqq f(x) \quad (n \geqq 1,\ x \in D).$$

Proof. The necessity is easily verified by the same method in L.Kosmák [6] concerning the Bernstein polynomials. Thus we prove only the sufficiency of this lemma. For this purpose we use the so-called parabola technique (cf. B.Bajšanski and R.Bojanic [1]).

From the assmption of the lemma we have

$$\limsup_{n\to\infty} n[L_n(f;x) - f(x)] \geqq 0 \quad (x \in D). \tag{6}$$

Next we suppose that $f(x)$ is not convex, then there exists an interval $[A_1,B_1] \subset (A,B)$ with midpoint x_0 and a linear function $l(x)$ such that

$$f(A_1) = l(A_1),\ f(B_1) = l(B_1)$$

and $f(x_0) > l(x_0)$. If we take

$$g(x) = \begin{cases} f(x) - l(x) & (x \in [A_1,B_1]) \\ 0 & (\text{otherwise}), \end{cases}$$

then we can find the parabola $q(x) = \alpha x^2 + \beta x + r$ $(\alpha < 0)$ and $x_1 \in (A_1,B_1)$ with the properties:

$$q(x) \geqq g(x) \ (x \in [A_1,B_1]) \text{ and } q(x_1) = g(x_1).$$

Taking $Q(x) = \max\{q(x), 0\}$, we have

$$Q(x) \geqq g(x) \ (x \in D) \text{ and } Q(x_1) = g(x_1).$$

Thus by Lemma 1 and $L_n(l;x) = l(x)$ we obtain

$$\limsup_{n\to\infty} n[L_n(f;x_1)-f(x_1)]$$
$$=\limsup_{n\to\infty} n[L_n(g;x_1)-g(x_1)]$$
$$\leqq \limsup_{n\to\infty} n[L_n(Q;x_1)=Q(x_1)]=\alpha\phi(x_1)<0.$$

Since $x_1 \in D$ this contradicts to (6), and the proof of Lemma 2 is complete.

Proof of Theorem 2. We show only the inverse part by the comment in Section 1. In view of the assumption $(\mathrm{II})_2$ and the relation (3) we have

$$|L_n(f;x)-f(x)|\leqq M[\frac{\phi(x)}{n}]$$
$$=M[L_n((t-x)^2;x)]$$
$$=M[L_n(t^2;x)-x^2].$$

Thus we obtain for $x \in D$

$$Mx^2 \pm f(x) \leqq L_n(Mt^2 \pm f(t);x).$$

The above inequality implies, by Lemma 2, that $Mx^2 \pm f(x)$ are convex functions. Hence there follows

$$\varDelta_h^2[Mt^2 \pm f(t)](x) \geqq 0 \qquad (x \in D)$$

and

$$|\varDelta_h^2 f(x)| \leqq M\varDelta_h^2[t^2](x)=O(h^2).$$

This proves $f \in \mathrm{Lip}_2\, 2$.

3, Remarks

Theorem 2 applies to the six normalized types of operators which was determined by M.Ismail and C.P.May [5] as the general form of exponential-type operators. Their operators include Bernstein polynomials and Gauss–Weierstrass, Szász–Mirakjan, Baskakov and other operators. As to related results on operators which are not of exponential–type, see for example M.Becker and R.J.Nessel [2].

The condition $f(x)=O(e^{N|x|})$ for some $N>0$ was introduced in C.P.May [7] as growth–test functions. Evidently we dispense with this condition for a finite interval [A,B]. And for an infinite interval, by the example in [5], this condition may be best possible concerning the existence of exponential–type operators, therefore this condition seems reasonable, apart from the coincidence with the qualification "exponential".

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