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On a Refraction Problem

Nenosuke FUNAYAMA*, Shinkichi HIRABUKI,**
Josuke HAKEDA* and Kunio SATO*

*Department of Basic Technology, Faculty of Engineering
**The Technical College, Yamagata University

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Abstract

A condition on the point of refraction is stated in connection with the rank of matrices. (Main Theorem) As an application a condition on the points of refraction of light is given.

1. Main Theorem

In this paper we are concerned with a refraction problem which is stated as follows :

Let V^{n+1} be an (n+1)-dimensional real vector space whose points are denoted as (x, y), where y is a vector in V^n.

Let π be the intersection of m hypersurfaces $\varphi_i(x,y)=0$ $(i=1,2,\cdots\cdots, m)$, and let $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1<x_2$, be two fixed points in V^{n+1}, and $g(x, y, y')$ and $h(x, y, y')$ are functions on $V\times V^n\times V^n$ having continuous derivatives up to the third order.

An addmissible curve $y(x)$ has a piecewise continuous first derivative y' and joins A and B.

Let $P(\xi, \eta)$ be any point in π, with $x_1<\xi<x_2$, and let I be the integral defined by

$$I=\int_{x_1}^{\xi} g(x,y,y')\,dx+\int_{\xi}^{x_2} h(x,y,y')\,dx$$

A refraction problem is the problem to find both the point P_0 in π and the addmissible function $y(x)$ which make I minimum.

The following assumptions are made :

(A) For any point P (ξ,η) in π, where $x_1 < \xi < x_2$, Euler equations for g : $g_{yi} - \frac{d}{dx} g'_{yi} = 0$ $(i=1,2,\cdots,m)$ satisfy two points boundary conditions, that is, the set of solutions of the equations which pass through A (x_1,y_1) and P (ξ,η) is uniquely determined, and the solution minimizes

$$I_1 = \int_{x_1}^{\xi} g\ (x,y,y')\ dx$$

(B) Likewise the same conditions for $h(x,y,y')$ and $I_2 = \int_{\xi}^{x_2} h(x,y,y')dx$ as for g and I_1 in (A) are satisfied :

Euler equations $h_{yi} - \frac{d}{dx} h'_{yi} = 0$ $(i=1,2,\cdots,m)$ satisfy two points boundary conditions and the solution minimizes I_2.

Define $I(P) = \int_{x_1}^{\xi} g\ (x,y,y')\ dx + \int_{\xi}^{x_2} h(x,y,y')dx$, where y is determined by the assumptions (A) and (B).

Let Q $(\xi+\delta\xi, \eta+\delta\eta)$ be any neighboring point in π, and let the corresponding y be $y+\delta y$. Then δy satisfies $\delta y(x_1)=0$, $\delta y(x_2)=0$ and $\delta y(\xi+\delta\xi)=\eta+\delta\eta$.

$$\begin{aligned}\Delta I &= I(Q) - I(P)\\ &= \int_{x_1}^{\xi+\delta\xi} g\ (x,\ y+\delta y,\ y'+\delta y')\ dx\\ &+ \int_{\xi+\delta\xi}^{x_2} h(x,y+\delta y,y'+\delta y')dx\\ &- \int_{x_1}^{\xi} g(x,y,y')dx - \int_{\xi}^{x_2} h\ (x,y,y')\ dx\end{aligned}$$

Let δI be the first variation of I (linear principal part of ΔI) :

$$\delta I = \{(g-\Sigma y'_i g_{y'i})_P - (h-\Sigma y'_i h_{y'i})_P\}\ \delta\xi + \Sigma\ \{(g_{y'i})_P - (h_{y'i})_P\}\ \delta\eta_i = 0 \qquad (1)$$

For the proof of the equation (1) and the other related facts see the references 1, 2, 4 and 5.

Note that $(y'_i)_P$ takes two values since a refraction takes place at P.

P and Q are in π and so $(\delta\xi,\ \delta\eta_1,\cdots\cdots,\delta\eta_n)$ is not arbitrary. It satisfies the following conditions

$$\varphi_{ix}\delta\xi + \sum_{j=1}^{n} \varphi_{iy_j}\ \delta\eta_j = 0 \qquad (i=1,2,\cdots\cdots,m) \qquad (2)$$

(1) must be satisfied by any $(\delta\xi,\ \delta\eta_1,\cdots\cdots,\delta\eta_n)$ satisfying (2), that is (2) imply (1).

Let us define the following fundamental matrix of φ :

$$\Phi = \begin{pmatrix} \varphi_{1x} & \varphi_{1y_1} & \cdots\cdots\cdots\cdots & \varphi_{1y_n} \\ \varphi_{2x} & \varphi_{2y_1} & \cdots\cdots\cdots\cdots & \varphi_{2y_n} \\ \varphi_{mx} & \varphi_{my_1} & \cdots\cdots\cdots\cdots & \varphi_{my_n} \end{pmatrix}$$

Let $\kappa = \Big((g-\Sigma y'_i g_{y'_i})_P - (h-\Sigma y'_i h_{y'_i})_P, (g_{y'_1})_P - (h_{y'_1})_P, \cdots\cdots, (g_{y'_n})_P - (h_{y'_n})_P\Big)$

Lemma. (2) imply (1) if and only if

$$\text{rank}\begin{pmatrix}\Phi\\ \kappa\end{pmatrix} = \text{rank}\ \Phi$$

Then we have the following theorem.

Theorem Under the suitable differentiability conditions made in the precedings and under the assumptions (A) and (B), the point of refraction, the point P in π which minimizes I, satisfies the condition

$$\textbf{rank}\begin{pmatrix}\Phi\\ \kappa\end{pmatrix} = \textbf{rank}\ \Phi$$

2. An Application

Let n = 2, m = 1 and $g=\dfrac{\sqrt{1+y'^2+z'^2}}{v_1}$, $h=\dfrac{\sqrt{1+y'^2+z'^2}}{v_2}$

The problem of refraction of light is in the case. g and h satisfy the necessary conditions. Put $\alpha=y'(\xi)$, $\beta=z'(\xi)$ with respect to g, and $\gamma=y'(\xi)$, $\delta=z'(\xi)$ with respect to h.

$$\text{Put } L=\frac{1}{v_1\sqrt{1+\alpha^2+\beta^2}},\quad M=\frac{1}{v_2\sqrt{1+\gamma^2+\delta^2}},$$

then the rank condition in the Theorem is :

$$\text{rank}\begin{pmatrix} L\text{-}M & \alpha L\text{-}\gamma M & \beta L\text{-}\delta M\\ (\varphi_x)_P & (\varphi_y)_P & (\varphi_z)_P\end{pmatrix} = 1,$$

or equivalently (L-M, αL-γM, βL-δM) is proportional to (φ_x, φ_y, φ_z) at P :

$$\begin{cases} L-M=k\varphi_x\\ \alpha L-\gamma M=k\varphi_y\\ \beta L-\delta M=k\varphi_z\end{cases} \tag{3}$$

If k = 0, then L=M, $\alpha L=\gamma M$, $\beta L=\delta M$, aud so $\alpha=\gamma$ dnd $\beta=\delta$, and then $v_1=v_2$.

Thus we have, if $v_1 \neq v_2$

$$\begin{vmatrix} 1 & 1 & \varphi_x\\ \alpha & \gamma & \varphi_y\\ \beta & \delta & \varphi_z\end{vmatrix} = 0 \tag{4}$$

(4) means that the vectors $\mathbf{p}=(1, \alpha, \beta)$, $\mathbf{q}=(1, \gamma, \delta)$ and $\mathbf{n}=(\varphi_x, \varphi_y, \varphi_z)$ are coplanar.

The other condition implied by (3) is

$$\frac{\sin\ \widehat{(\mathbf{p},\mathbf{n})}}{v_1}=\frac{\sin\ \widehat{(\mathbf{q},\mathbf{n})}}{v_2}$$

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1つの屈折問題

船山 子之助・ 平吹 慎吉・羽毛田 穣祐・佐藤 邦夫

工学部 共通講座
*工業短期大学部

n＋1次元の実ベクトル空間で f(x,y,y′) は m 個の超曲面の交わり π を境として不連続とし，その両側では必要な条件を満足するものとする。双方の側の2定点A，Bがあるとき π 上の1点Pを求めて，A，P，Bを通る y(x) を求めて

$$\int_a^b f(x, y, y')\,dx$$

を極小とする。Pの満足すべき条件として行列の階数についての条件を求める。(主定理)
応用として光の屈折問題について考察する。