

Negative Examples in Mathematical Analysis Related to Convexity Properties of Functions

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Abstract.

We give negative examples to the following plausible statements:

- I. Pointwise product of two positive, strictly convex functions is convex.
- II. If $\omega(\delta)$, $\omega(0)=0$, is a monotone increasing function which satisfies the " Δ_2 -condition" ($\omega(2\delta)=O(\omega(\delta))$), then, as $\delta>0$ tends to 0, $\omega(\delta)\neq O(\delta)$ implies $\omega(\delta)/\delta\rightarrow\infty$.

Additional conditions validating these statements are also shown.

Introduction.

Recall that a real-valued function f defined over an interval in the real line is said to be convex (resp. concave) if for every pair λ, μ , $\lambda>0, \mu>0$, $\lambda+\mu=1$, we have

$$f(\lambda x_1 + \mu x_2) \leq \lambda f(x_1) + \mu f(x_2) \quad (\text{resp. } \geq \lambda f(x_1) + \mu f(x_2)).$$

The property is strict if the equality " $=$ " is excluded. It is well known that a convex (resp. concave) function is differentiable almost everywhere and its derivative is non-decreasing (resp. non-increasing).

The paper consists of two parts.

I. Pointwise product of Positive Convex Functions.

Let f and g be two positive convex functions. It is plausible that their pointwise product $h(x)=f(x)g(x)$ also is convex; this is actually the case if f and g are of the same monotonicity (increasing together or decreasing together). In fact, putting $x_0=\lambda x_1+\mu x_2$ and observing

$$\begin{aligned} & (f(x_1)-f(x_2))(g(x_1)-g(x_2)) \geq 0 && \text{we have} \\ h(x_0) &= f(x_0)g(x_0) \leq (\lambda f(x_1) + \mu f(x_2))(\lambda g(x_1) + \mu g(x_2)) \\ &= \lambda^2 f(x_1)g(x_1) + \lambda\mu(f(x_1)g(x_2) + f(x_2)g(x_1)) + \mu^2 f(x_2)g(x_2) \\ &\leq \lambda^2 f(x_1)g(x_1) + \lambda\mu(f(x_1)g(x_1) + f(x_2)g(x_2)) + \mu^2 f(x_2)g(x_2) \end{aligned}$$

$$= \lambda h(x_1) + \mu h(x_2), \quad \text{as was to be proved.}$$

A little more generally, the product h is convex over an interval (a, b) if, for some $c \in (a, b)$, f and g are of the same monotonicity over (a, c) as well as over (c, b) (see B. page 55, Exercise 17.) For, take x_1 and x_2 so that $a < x_1 < c < x_2 < b$. Since f and g are convex, the only admissible change in monotonicity is from-decrease-to-increase, so that the product h must assume its minimum at c , and consequently, as to the graph $y=h(x)$, the chord joining the two points $(x_1, h(x_1))$ and $(x_2, h(x_2))$ must lie above the corresponding arc.

On the other hand, without the assumption of the monotonicity, the product $h(x)$ need not be convex. We give two examples.

Example 1.

Take, for $0 \leq x < \infty$, $f_1(x) = (x+1)^{1/2} e^x$, $g_1(x) = e^{-x}$. It is readily seen that these functions are strictly convex over the positive real line, but their product $h_1(x) = (x+1)^{1/2}$ is strictly concave there.

Example 2.

Take $f_2(x) = \exp(-x)$ for $x < 0$, $= \exp(-x - x^2/2)$ for $x \geq 0$
 $g_2(x) = \exp(x - x^2/2)$ for $x < 0$, $= \exp x$ for $x \geq 0$.

They both have increasing derivatives and are strictly convex over the whole real line, but their product $h_2(x) = \exp(-x^2/2)$ is strictly concave for $-1 \leq x \leq 1$.

Remark.

In Example 2, the product loses convexity only locally. One can somewhat widen the interval of concavity by some means (for example by a change of scale), but, since no positive function can be concave over the whole real line, (except the trivial case of constant functions), the above example is essentially best possible.

II. Concerning Growth Property of Continuity.

Let $\omega(\delta)$ be an abstract modulus of continuity: $\omega(\delta)$ is defined for $0 \leq \delta \leq \delta_0$ with some $\delta_0 > 0$, $\omega(0) = 0$, increasing (thus positive), continuous and subadditive.

In (T, page 74), the author makes use of the following plausible but unproved statement.

Proposition.

If $\omega(\delta)$ is not $O(\delta)$ as $\delta \rightarrow +0$, then $\lim_{\delta \rightarrow +0} (\omega(\delta)/\delta) = \infty$.

Since the negation of the statement " $\omega(\delta)=O(\delta)$ " gives only $\limsup_{\delta \rightarrow +0} (\omega(\delta)/\delta) = \infty$, the above proposition is not evident. Fortunately, the author makes recovery by later assuming the concavity of ω , which implies $\omega(x) \geq x\omega'(x)$ and the monotonicity of $\omega(x)/x$.

The following example shows that if one replaces the subadditivity of ω by (weaker) Δ_2 -condition, the above proposition is disproved.

Example 3.

For every α , $0 < \alpha \leq 1$, there exists a continuous function (say ω_α) defined on the positive real line, $\omega_\alpha(0)=0$, increasing, and satisfying Δ_2 -condition, that is, $\omega_\alpha(2\delta) \leq C_\alpha \omega_\alpha(\delta)$ for some $C_\alpha > 0$ independent of δ , and, moreover

$$\limsup_{\delta \rightarrow +0} (\omega_\alpha(\delta)/\delta) = \infty, \quad \liminf_{\delta \rightarrow +0} (\omega_\alpha(\delta)/\delta) = 0.$$

It is convenient to make a change of variable $x=1/\delta$ and to consider the behavior as $x \rightarrow \infty$. We need several auxiliary functions and their derivatives.

$$p_0(x) = \pi \log_2 \log_2 x = (\pi/\log 2) (\log \log x - \log \log 2)$$

$$p'_0(x) = (\pi/\log 2) (1/x \log x) \leq 5/x \log x$$

$$p_1(x) = 1 + \cos p_0(x) = 1 + \cos (\pi \log_2 \log_2 x)$$

$$p'_1(x) = -\sin p_0(x) (\pi/\log 2) (1/x \log x)$$

$$p_2(x) = \log \log x, \quad p'_2(x) = 1/x \log x$$

$$p_3(x) = 1/\log \log x, \quad p'_3(x) = -1/(x \log x (\log \log x)^2).$$

Now we define $\omega_\alpha(\delta) = \Omega_\alpha(x)$ ($x=1/\delta$) as follows:

$$\Omega_\alpha(x) = x^{-\alpha} (p_1(x)p_2(x) + (1/\alpha)p_3(x)).$$

Of course, we may modify the values of $\omega_\alpha(\delta)$ away from the origin, that is, in dealing with the function $\Omega_\alpha(x)$, we may confine our attention to large values of x ($x > \exp e^8$ for example).

1°. $\Omega_\alpha(x)$ is a decreasing function.

In fact, take the derivative, multiply by $x^{\alpha+1}$, and discard two negative terms p'_3 and $-p_1p_2$. There remain three terms $x p'_1 p_2$, $x p_1 p'_2$ and $-p_3$, of which the last is certainly negative. Now, the first two do not exceed respectively $5 \log \log x / \log x$ and $2 / \log x$, and since the function $\log \log x$ grows far more slowly than $\log x$, the contribution of $-p_3$ determines the signature, so that the sum of the three terms is negative.

2°. $\omega_\alpha(\delta)$ satisfies the Δ_2 -condition.

What we have to prove is the existence of a constant C_α such that

$$p_1(2x)p_2(2x) + p_3(2x)/\alpha \geq C_\alpha (p_1(x)p_2(x) + p_3(x) + p_3(x)/\alpha).$$

Apply the mean-value theorem to the differences $p_1(2x)-p_1(x)$ and $p_3(2x)-p_3(x)$ and recall the estimates of the respective derivatives, obtaining

$$|p_1(2x)-p_1(x)| < 5/\log x \text{ and } |p_3(2x)-p_3(x)| < 1/\log x.$$

Multiplying both sides of the inequality $p_1(2x) - p_1(x) > -5/\log x$ by $p_2(2x)$ we have $p_1(2x)p_2(2x) > p_1(x)p_2(2x) - 5p_2(2x)/\log x$

$$\geq p_1(x)p_2(x) - 5p_2(2x)/\log x.$$

On the other hand, multiplying both sides of $p_3(2x) > p_3(x) - 1/\log x$ by $(1/\alpha)$ and adding to the above inequality side by side, we obtain

$$p_1(2x)p_2(2x) + p_3(2x)/\alpha \geq p_1(x)p_2(x) + p_3(x)/\alpha - N(x)$$

where $N(x) = 5p_2(2x)/\log x + 1/\alpha \log x$ does not exceed, for x large enough, $1/2$ times $p_1(x)p_2(x) + p_3(x)/\alpha$. Thus we may take $C_\alpha = 1/2$.

3°. $\limsup_{x \rightarrow \infty} x^\alpha \Omega_\alpha(x) = \infty$.

Take $x_n = 2^{2^{2n}}$, $n=1, 2, 3, \dots$. We have $p_0(x_n) = 2n$, $p_1(x_n) = 2$, $p_2(x_n) = 2n \log 2 + \log \log 2$, so that $x_n^\alpha \Omega_\alpha(x_n) \rightarrow \infty$.

4°. $\liminf_{x \rightarrow \infty} x^\alpha \Omega_\alpha(x) = 0$.

Take $w_n = 2^{2^{2n+1}}$, $n=1, 2, 3, \dots$. We have in turn $p_0(w_n) = (2n+1)\pi$, $p_1(w_n) = 0$, $n=1, 2, 3, \dots$. The assertion now follows from the property of p_3 .

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References

- B N. Bourbaki, Fonctions d'une variable réelle, Hermann, Paris, 1951.
 T V. Totik, On the strong approximation by the (C, α) -means of Fourier series, Analysis Mathematica, **6** (1980), 57–85.

解析学における反例 — 関数の凸性に関連して

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次の 2 つの命題に対して反例を与える。

- I 正值凸関数の点ごとの積は凸関数である。
- II $\omega(\delta)$ が、原点で 0 となり、 Δ_2 条件をみたす増加関数であるとき、 $\delta > 0$ が 0 に近づくに際して $\omega(\delta) \neq O(\delta)$ であれば、 $\omega(\delta)/\delta \rightarrow \infty$ である。

これらの命題が正当になる付帯条件も述べる。